In Defense of Euclidean Proof

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Chapter 1

Introduction

One finds the following declaration of love in Bertrand Russell's autobiography: "At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world" [11]. It is unclear whether or not young master Russell's edition of the *Elements* included actual diagrammatic figures, but it is clear that later in life, Russell had little sympathy for diagrammatic proof: "In the best books there are no figures at all" [10]. Apparently, by 1901 the nearly 30-year-old Russell had outgrown his childish proclivities. As one of the foremost early progenitors of modern logic, Russell's mature aversion to diagram-based reasoning can be seen as the "party line" of late-19$^{\text{th}}$ and early-20$^{\text{th}}$ century logic.

Let us consider a typical proof from Euclid's *Elements*, that of Proposition 12 from Book I. The goal: *given a line L and a point p not on L, to construct line N through p perpendicular to L.* First pick a point q on the opposite side of L from p. Then draw the circle α through q which is centered at p. Take the intersection points a, b of L and γ. By Proposition 9, construct a line N that bisects $\angle apb$. (The resulting diagram is given by Figure 1.1.) Using earlier Propositions, namely those involving SAS and SSS, Euclid can conclude that N is indeed perpendicular to L.

Certain diagrammatic information is crucial to the above proof; for instance, q must lie on the given side of L in order for the intersection points a, b to exist. But what is it about such diagrammatic proofs that leads figures such as Russell to discount them? The primary concerns are these:

1. Inferences drawn from diagrams are sometimes based on spatial intuition rather than precisely formulated logical rules.
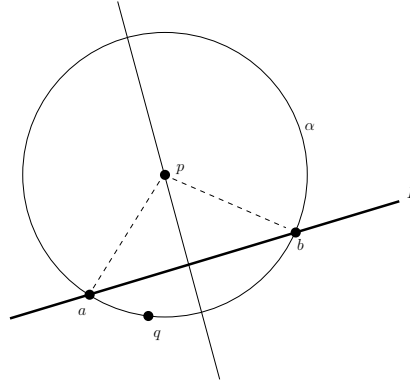
Figure 1.1: Book I, Proposition 12

2. General conclusions are drawn, without basis, from particular diagrams.

These are certainly valid concerns, as we tend to prefer to believe that we have firmer epistemological grounds for mathematical conclusions than what would be provided by methods which are open to these objections.

But Manders [3] has argued that common objections to Euclid's methods of proof are overblown. Concerning objection (1), for instance, the appearance of such impropriety is often a case of simple omission due to a lack of *formality* rather than of *rigor*; let us explain what we mean by this. Consider Euclid's proof of Proposition 1 of Book I. Given a segment from point a to point b, the construction of an equilateral triangle on the segment is sought. Euclid constructs two circles, one centered at a and passing through b, and vice versa; he then takes an intersection point c of the two circles. (See Figure 1.2.) But if one examines Euclid's postulates, there is nothing
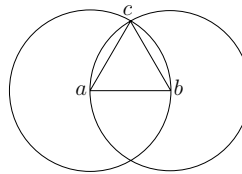

Figure 1.2: Book I, Proposition 1

that guarantees that there is such an intersection point to take; there is a gap

in the argument. This is seen as a very minor omission, however, when one considers the usual practice of informal mathematics, in which one is not as attentive to such matters as when dealing with formalizations. In any case, one could certainly imagine a time-travelling interlocutor with a knowledge of Cartesian analytic geometry objecting to Euclid, "But your plane might consist only of points with rational coordinates!" and a brought-up-to-speed Euclid replying, "But of course that's not the kind of plane I have in mind!" Bottom line, such omissions are real, but they can be closed without much ado.

The second kind of objection is more telling. A useful distinction Manders makes in this regard is between *exact* and *co-exact* properties of a diagram. Manders observes that information which is recorded diagrammatically in Euclidean proofs is only of a topological or regional sort, e.g. the incidence of points, lines and circles, or betweenness facts, or which side of a line a point lies in. Manders terms this kind of information "co-exact." In contrast, "exact" metric information, e.g. the congruence of two segments or angles, must be made explicit textutally.

With this observation Manders can explain how Euclid manages to avoid drawing improper inferences from a diagram. But something which Manders' analysis does not capture is how a diagrammatic proof involving *constructions* of new objects can be general. That is, he does not answer how Euclid rules out improper construction that rely on particular features of a given diagram. Now we seek to craft a formal system E for ruler-and-compass constructions that faithfully captures Euclidean diagrammatic reasoning, and in particular we aim to codify Euclid's methodology both for drawing inferences from diagrams and for constructing new diagrams; in Manders' terminology, Euclid's "diagram discipline." With our formalization come explicit preconditions for performing constructions and drawing inferences, by which we can answer objection (2); our preconditions are the means by which we guarantee that general conclusions drawn from a diagram do not improperly depend on particular features of the diagram.

Before getting to our system E, let us consider two previous formalizations of diagrammatic proof. Nathaniel Miller, in his doctoral dissertation [4], created a formal system called FG. This system bears little relation to our E in form, and no relation in its origins. We will not comment on FG here, but instead give the interested reader a pointer: Miller has a book [5] which is based on his dissertation, and Mumma has written a thoroughgoing review [7] which sheds light on differences between the approaches embodied by FG

and the kind of approach we take. All we will say here is that (1) FG is not all that faithful to Euclidean methods of proof, despite its diagrammatic nature, and (2) FG is not as amenable as one would like to metamathematical investigations (e.g. the question of completeness, which we address for E below).

Mumma himself, in his own doctoral dissertation [6], was inspired by Manders' take on the Euclidean diagram and set about creating a formal system Eu for Euclidean diagrammatic proof; the system E expounded below is in many respects an outgrowth of Eu due to Avigad, Mumma and the author. While it is rather different in form, it was created in the same spirit and with the same goal of extending Manders' analysis.

The present thesis is but a piece of a larger paper forthcoming from Avigad, Mumma and the author [1]. The particular goal of this paper is a metamathematical investigation of E; namely, we will prove that it is sound and complete with respect to a certain fragment of the ruler-and-compass theorems.

Chapter 2

The Diagrammatic Proof System E

In this second chapter we will detail a formal system E which is intended to codify the diagram discipline Euclid implicitly maintains in the *Elements*. The idea is to craft E so that it truly is faithful to the manner of Euclidean proof, and then ultimately to show that it is sound and complete with respect to ruler-and-compass constructions. Those matters are addressed in later parts of the paper; for now we simply want to lay the system down.

2.1   Syntax and Structure of Proofs

Our language is many-sorted, and we use differing notations for differing sorts:

Points:	$p, q, r, \ldots$
Lines:	$L, M, N, \ldots$
Circles:	$\alpha, \beta, \gamma, \ldots$
Segment Lengths:	$\overline{pq}, \overline{qr}, \ldots$
Angle Magnitudes:	$\angle pqr, \angle abc, \ldots$

There is also a constant $\mathfrak{r}$ intended to denote the magnitude of a right angle. We have an assortment of atomic predicates, listed here with their intended meanings (though they are mostly self-explanatory):

$\mathrm{on}(p, L)$	Point p is on line L.
$\mathrm{bet}(p, q, r)$	Point q lies (strictly) between p, r.
$\mathrm{same}(p, q, L)$	Points p, q are on the same side of L.
$\mathrm{on}(p, \gamma)$	Point p is on circle γ.
$\mathrm{in}(p, \gamma)$	Point p lies within γ.
$\mathrm{intersect}(L, M)$	Lines L, M intersect.
$\mathrm{intersect}(L, \alpha)$	Line L intersects circle α (twice).
$\mathrm{intersect}(\alpha, \beta)$	Circles α, β intersect (twice).
$\overline{ab} = \overline{pq}$	The segment lengths are the same.
$\overline{ab} < \overline{pq}$	The first segment is shorter.
$\angle abc = \angle pqr$	The angles are the same.
$\angle abc < \angle pqr$	The first angle is lesser.

Statements in E are written as "sequents"

$$\Gamma(\vec{p}, \vec{L}, \vec{\alpha}) \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta}) \Delta(\vec{p}, \vec{q}, \vec{L}, \vec{M}, \vec{\alpha}, \vec{\beta}),$$

where Γ and Δ are just lists of literals (atomic or negated atomic assertions), or $\perp$. Unlike sequents found in standard sequent calculi for first-order logic, *both* of the lists Γ and Δ are intended to be read as conjunctions of their components, rather than Δ being read disjunctively. Given the intended meaning of our sequents, there is a natural first-order sentence in the many-sorted language of E that corresponds to any E-sequent as above, namely

$$\forall \vec{p}, \vec{L}, \vec{\alpha} \left[\bigwedge \Gamma(\vec{p}, \vec{L}, \vec{\alpha}) \ \rightarrow \ \exists \vec{q}, \vec{M}, \vec{\beta} \bigwedge \Delta(\vec{p}, \vec{q}, \vec{L}, \vec{M}, \vec{\alpha}, \vec{\beta}) \right].$$

This is the logical form taken by Euclid's Propositions. NB: Given an E-sequent $\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta}) \Delta$ we will, in the sequel, often abuse notation and also use $\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta}) \Delta$ to refer to this first-order sentence; the context will always be sufficient to avoid confusion.

Now proofs in E are simply trees whose nodes are such sequents, and whose edges correspond to various kinds of inference rules:

1. Logical Rules

2. Construction Rules

3. Demonstration Rules

   - Diagrammatic Inferences

- Metric Inferences
- Transfer Inferences

We explain each of these sorts of rules in turn.

2.2 Logical Rules

2.2.1 Theorem Application

We would like to have an inference rule that corresponds to the practice of applying a previously established result. Supposing that said result takes the sequent form $\Lambda \rightsquigarrow (\exists \vec{r}, \vec{N}, \vec{\gamma})\Theta$, then the *theorem application* inference rule is

$$\frac{\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Delta \quad \Lambda \rightsquigarrow (\exists \vec{r}, \vec{N}, \vec{\gamma})\Theta}{\Gamma \rightsquigarrow (\exists \vec{x}, \vec{q}\vec{M}, \vec{\beta})\Delta, \Delta'}$$

subject to several restrictions. (1) The sequent $\Lambda \rightsquigarrow (\exists \vec{r}, \vec{N}, \vec{\gamma})\Theta$ can be relettered, obtaining $\Lambda' \rightsquigarrow (\exists \vec{r}, \vec{N}, \vec{\gamma})\Theta'$ whose free variables are a subset of those of $\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Delta$, and such that every element of Λ' is an immediate consequence of Γ, Δ. (2) $\Delta' \subseteq \Theta'$. (3) $\vec{x}$ consists of those variables among $\vec{r}, \vec{N}, \vec{\gamma}$ that appear in Δ'.

Some bit of explanation is in order. The idea is that we arrive at the sequent $\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Delta$, and so have a configuration of objects with properties Γ, Δ. We also have an established result $\Lambda \rightsquigarrow (\exists \vec{r}, \vec{N}, \vec{\gamma})\Theta$. By condition (1), the objects with properties Γ, Δ also have the properties Λ', and so we may conclude the existence of some subset of the objects guaranteed to exist by the theorem.

2.2.2 Case Splits

We allow ourselves case splits on atomic formulas. This is in keeping with Euclid's practice; see Proposition 8 from Book I, for instance. Formally, our case splitting rule takes the following form:

$$\frac{\Gamma \rightsquigarrow (\exists \vec{p}, \vec{L}, \vec{\alpha})\Delta \quad \Gamma, \Delta, \varphi \rightsquigarrow \psi \quad \Gamma, \Delta, \neg\varphi \rightsquigarrow \psi}{\Gamma \rightsquigarrow (\exists \vec{p}, \vec{L}, \vec{\alpha})\Delta, \psi}$$

where φ is atomic and ψ is any literal.

2.3   Construction Rules

By *construction rules* we mean inference rules of E which, given some configuration of objects, asserts the existence of new objects with certain properties. To wit, they take the general form

$$\frac{\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Delta}{\Gamma \rightsquigarrow (\exists \vec{x}, \vec{q}, \vec{M}, \vec{\beta})\Delta, \Theta}$$

where $\vec{x}$ is a sequence of new point, line and/or circle variables with the properties expressed in Θ.

Such rules are conceptually no different than theorem application. That is, each can be seen as the application of a theorem $\Lambda \rightsquigarrow (\exists \vec{x})\Theta$, in the sense of the previous section. The $\vec{x}$ are the objects constructed by the rule, and Λ consists of the preconditions for applying it. We present our construction rules below by presenting sequents of the form $\Lambda \rightsquigarrow (\exists \vec{x})\Theta$, and the corresponding rule is the corresponding theorem application.

2.3.1   Adding Points

In this section we will present the sequents $\Lambda \rightsquigarrow (\exists \vec{x})\Theta$ in words. In fact, we will present the consequent in the main column, and any preconditions (the Λ) will be presented in parentheses. We use the locution "new" to mean that the point p being added can, in addition to what is stated, be asserted to be distinct from elements already in the diagram. For instance, supposing we have a line L with points a, b already on it, rule (2.2) can be read as allowing us to construct the point p on L and furthermore assert that p is distinct from a and/or b.

$$
\begin{array}{lcr}
\text{Let } p \text{ be a "new" point} & & (2.1) \\
\text{Let } p \text{ be a "new" point on } L & & (2.2) \\
\text{Let } p \text{ be a "new" point between } q \text{ and } r \text{ on } L & (q, r \text{ distinct points on } L) & (2.3) \\
\text{Let } p \text{ be a "new" point extending } qr \text{ on } L & & (2.4) \\
\text{Let } p \text{ "new" be opposite } L \text{ from } q & (q \text{ not on } L) & (2.5) \\
\text{Let } p \text{ "new" be on the same side of } L \text{ as } q & (q \text{ not on } L) & (2.6)
\end{array}
$$

$$\text{Let } p \text{ be a ``new'' point on } \gamma \qquad (2.7)$$
$$\text{Let } p \text{ be a ``new'' point inside } \gamma \qquad (2.8)$$
$$\text{Let } p \text{ be a ``new'' point outside } \gamma \qquad (2.9)$$
$$\text{Let } p \text{ be the center of } \gamma \qquad (2.10)$$

2.3.2 Adding Lines and Circles

$$\text{Let } L \text{ be a line through } p, q \quad (p \neq q) \qquad (2.11)$$
$$\text{Let } \gamma \text{ be the circle through } q \text{ with center } p \quad (p \neq q) \qquad (2.12)$$

2.3.3 Adding Intersections

Here, a blanket precondition (which is not stated in the list) for each rule, is that the objects in question do indeed intersect, in the sense of the diagrammatic intersection rules found below among the demonstration rules.

Let p be the intersection of L, M		(2.13)
Let p be an intersection of α and L		(2.14)
Let $p \neq q$ be two intersections of α and L		(2.15)
Let p be an intersection of L, α between q, r	$(\mathrm{on}(r, L), \neg\mathrm{in}(r, \alpha), \neg\mathrm{on}(r, \alpha))$	(2.16)
Let p be an intersection of L, α extending rq	$(\mathrm{on}(q, L), \mathrm{on}(r, L), r \neq q, \mathrm{in}(q, \alpha))$	(2.17)
Let p be an intersection of α, β		(2.18)
Let $p \neq q$ be intersections of α, β		(2.19)
Let p be the intersection of α, β on the same side of L as q	$(\mathrm{center}(r, \alpha), \mathrm{center}(s, \beta), \mathrm{on}(r, L), \mathrm{on}(s, L), \neg\mathrm{on}(q, L))$	(2.20)
Let p be the intersection of α, β on the opposite side of L from q	$(\mathrm{center}(r, \alpha), \mathrm{center}(s, \beta), \mathrm{on}(r, L), \mathrm{on}(s, L), \neg\mathrm{on}(q, L))$	(2.21)

2.4 Demonstration Rules

In contrast to construction rules, our *demonstration rules* do not introduce new objects; that is, they take the general form

$$\frac{\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Delta}{\Gamma \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Delta, \Theta}$$

As with construction rules, these are again instances of theorem application. So again we present these rules by presenting the appropriate sequent/theorem which is applied by the rule.

2.4.1   Diagrammatic Inferences

With the consideration of what we term diagrammatic inferences, we will briefly interrupt our simple listing of rules for a bit of discussion. As we said, below we indicate a plethora of inference rules in the form of initial theorems that can be applied. But the one-step diagrammatic inferences our system E is to license are not merely the indicated instances of theorem application; rather, we want our notion of a one-step diagrammatic inference to correspond more closely with the kinds of inferences that Euclid immediately "reads off" of a diagram.

To better illustrate the point we are getting at, let us suppose we have distinct points a, b, c, d, e, and we know that c is between a and e, b is between a and c, and d is between c and e. Well, we have a series of several rules below concerning betweenness, using which we could construct a chain of inferences and conclude that we know all of the betweenness relations encapsulated in Figure 2.1. But Euclid would engage in no such involved chain of reasoning
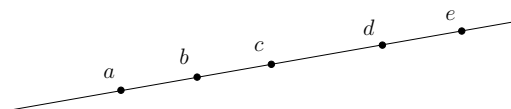

Figure 2.1: Points on a line

in order to reach this conclusion; to the contrary, if one constructs a diagram by placing c between a and e, and then b between a and c, and then d between c and e, one gets exactly Figure 2.1, and the conclusions are immediate! This is exactly the kind of situation in which, according to the modern logical party line, Euclid is doing something like appealing to intuition and has left a logical gap. To this way of thinking, Euclid's move is improper, and we should prefer instead a chain of reasoning as described. For instance, this is the way a proof would proceed in Tarski's typical formal system for plane geometry.[1] But we claim that Euclid's one-step inference is not at all

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[1]See Chapter 4 if you are unfamiliar with Tarski's system.

improper, and we want E to reflect this ability to make such inferences in one step.

To this end, the official diagrammatic inference rule for E is as follows:

> A literal A may be concluded as an *immediate diagrammatic consequence* of a set of diagram assertions Γ, provided that A is in the smallest set of assertions which contains Γ and is closed under the rules listed below.[2]

The crucial point is that there is nothing untoward about this definition; if we are given Γ and A, it is *decidable* whether or not A is an immediate diagrammatic consequence in the defined sense. All undecidability in E is pushed into the construction rules.[3]

Now, on with the lists.

Generalities

The following give the uniqueness of a line through two points, the uniqueness of a circle's center, and the fact that the center lies inside the circle.

$$p \neq q, \text{on}(p, L), \text{on}(q, L), \text{on}(p, M), \text{on}(q, M) \quad \rightsquigarrow \quad L = M \qquad (2.22)$$

$$\text{center}(p, \gamma), \text{center}(q, \gamma) \quad \rightsquigarrow \quad p = q \qquad (2.23)$$

$$\text{center}(p, \gamma) \quad \rightsquigarrow \quad \text{in}(p, \gamma) \qquad (2.24)$$

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[2]For clarity we note that as part of this definition of consequence we take it to be the case that if $\perp$ is among Γ, then anything counts as a diagrammatic consequence.

Moreover, each of the rules is presented as an implication

$$A_1, \ldots, A_n \rightsquigarrow B,$$

where the A_i's and B are literals in the language of E. But the rules we intend to close under for our notion of diagrammatic consequence do not include just the implications presented; rather, each such stands also for the various contrapositive formulations

$$A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, \neg B \rightsquigarrow \neg A_j$$

as well, which we intend to be included in the list of rules under which we close.

[3]As discussed further in a footnote below, Ziegler [15] has shown that any finitely axiomatized theory which is includes all ruler-and-compass theorems is undecidable. We suspect (though have not settled) that the same holds for the fragment of ruler-and-compass theorems that our E completely finitely axiomatizes. The point here is just that, while the system as a whole might be undecidable, the notion of immediate diagrammatic consequence given here is not.

We also have the following equality rules; here x refers to a variable of any of our sorts, and φ to any atomic formula.

$$\rightsquigarrow \quad x = x \tag{2.25}$$

$$x = y, \varphi(x) \quad \rightsquigarrow \quad \varphi(y) \tag{2.26}$$

$$x = y, \varphi(y) \quad \rightsquigarrow \quad \varphi(x) \tag{2.27}$$

Betweenness Rules

$$\mathrm{bet}(r, q, p) \quad \rightsquigarrow \quad \mathrm{bet}(p, q, r), p \neq q, p \neq r, \neg\mathrm{bet}(q, p, r) \tag{2.28}$$

$$\mathrm{bet}(p, q, r), \mathrm{on}(p, L), \mathrm{on}(r, L) \quad \rightsquigarrow \quad \mathrm{on}(q, L) \tag{2.29}$$

$$\mathrm{bet}(p, q, r), \mathrm{on}(p, L), \mathrm{on}(q, L) \quad \rightsquigarrow \quad \mathrm{on}(r, L) \tag{2.30}$$

$$\mathrm{bet}(p, q, r), \mathrm{bet}(p, s, q) \quad \rightsquigarrow \quad \mathrm{bet}(p, s, r), \mathrm{bet}(s, q, r) \tag{2.31}$$

$$\mathrm{bet}(p, q, r), \mathrm{bet}(p, s, q) \quad \rightsquigarrow \quad \mathrm{bet}(s, q, r) \tag{2.32}$$

$$\mathrm{bet}(p, q, r), \mathrm{bet}(q, r, s) \quad \rightsquigarrow \quad \mathrm{bet}(p, r, s), \mathrm{bet}(p, q, s) \tag{2.33}$$

$$\mathrm{on}(p, L), \mathrm{on}(q, L), \mathrm{on}(r, L), p \neq q, p \neq r, q \neq r, \neg\mathrm{bet}(p, q, r), \neg\mathrm{bet}(q, p, r) \rightsquigarrow \mathrm{bet}(p, r, q) \tag{2.34}$$

Same-side Rules

$$\neg\mathrm{on}(p, L) \quad \rightsquigarrow \quad \mathrm{same}(p, p, L) \tag{2.35}$$

$$\mathrm{same}(p, q, L) \quad \rightsquigarrow \quad \neg\mathrm{on}(p, L), \mathrm{same}(q, p, L) \tag{2.36}$$

$$\mathrm{same}(p, q, L), \mathrm{same}(p, r, L) \quad \rightsquigarrow \quad \mathrm{same}(q, r, L) \tag{2.37}$$

$$\neg\mathrm{on}(p, L), \neg\mathrm{on}(q, L), \neg\mathrm{on}(r, L), \neg\mathrm{same}(p, q, L), \neg\mathrm{same}(p, r, L) \rightsquigarrow \mathrm{same}(q, r, L) \tag{2.38}$$

Pasch Rules

$$\mathrm{bet}(p, q, r), \mathrm{same}(p, r, L) \quad \rightsquigarrow \quad \mathrm{same}(p, q, L) \tag{2.39}$$

$$\mathrm{bet}(p, q, r), \mathrm{on}(p, L), \neg\mathrm{on}(q, L) \quad \rightsquigarrow \quad \mathrm{same}(q, r, L) \tag{2.40}$$

$$\mathrm{bet}(p, q, r), \mathrm{on}(q, L) \quad \rightsquigarrow \quad \neg\mathrm{same}(p, r, L) \tag{2.41}$$

$$p \neq r, q \neq r, p \neq q, L \neq M, \mathrm{on}(p, M), \mathrm{on}(q, M), \neg\mathrm{same}(p, q, L) \rightsquigarrow \mathrm{bet}(p, r, q) \tag{2.42}$$

Triple-Incidence

$$\begin{array}{c} \text{on}(r, M), \text{on}(r, N), \text{on}(r, L), \text{on}(q, M), \text{on}(p, N), \\ \text{same}(s, p, M), \text{same}(s, q, N), \text{on}(s, L) \end{array} \leadsto \neg\text{same}(p, q, L) \qquad (2.43)$$

$$\begin{array}{c} \text{on}(r, M), \text{on}(r, N), \text{on}(r, L), \text{on}(t, M), t \neq r, \text{on}(p, N), \\ \text{same}(s, p, M), \neg\text{same}(s, t, N), \text{on}(s, L), \neg\text{on}(s, N) \end{array} \leadsto \text{same}(p, t, L) \qquad (2.44)$$

$$\begin{array}{c} \text{on}(r, M), \text{on}(r, N), \text{on}(r, L), \text{on}(t, M), \text{on}(p, N), \text{same}(s, p, M), \\ \neg\text{same}(s, t, N), \text{on}(s, L), \neg\text{on}(s, N), \text{same}(u, t, N), \text{same}(u, p, M) \end{array} \leadsto \text{same}(p, u, L) \qquad (2.45)$$

Circle Rules

$$\text{in}(p, \gamma), \text{on}(q, \gamma), \text{on}(r, \gamma), \text{on}(p, L), \text{on}(q, L), \text{on}(r, L) \leadsto \text{bet}(p, q, r) \qquad (2.46)$$

$$\text{in}(p, \gamma), \text{on}(q, \gamma), \neg\text{in}(r, \gamma), \neg\text{on}(r, \gamma), \text{on}(p, L), \text{on}(r, L) \leadsto \text{bet}(p, r, q) \qquad (2.47)$$

$$\text{in}(p, \gamma), \neg\text{in}(q, \gamma), \text{bet}(p, q, r), \text{on}(p, L), \text{on}(q, L) \leadsto \neg\text{in}(r, \gamma), \neg\text{on}(r, \gamma) \qquad (2.48)$$

$$\begin{array}{c} \text{in}(p, \gamma), \text{in}(q, \delta), \text{on}(p, \delta), \neg\text{in}(q, \gamma), \text{on}(p, L), \\ \text{on}(q, L), \text{on}(r, \gamma), \text{on}(r, \delta), \text{on}(s, \gamma), \text{on}(s, \delta), r \neq s \end{array} \leadsto \neg\text{same}(r, s, L) \qquad (2.49)$$

Intersection Rules

Here we give conditions under which various pairs of objects intersect. Note that we mean intersection in a strong sense; namely when we say that two circles intersect, or that a line and a circle do, we mean that there are *two* intersection points, rather than tangency.

$$\text{on}(p, M), \text{on}(q, M), \neg\text{on}(p, L), \neg\text{on}(q, L), \neg\text{same}(p, q, L) \leadsto \text{intersect}(L, M) \qquad (2.50)$$
$$\text{on}(p, L), \text{in}(p, \alpha) \leadsto \text{intersect}(L, \alpha) \qquad (2.51)$$
$$\text{in}(p, \alpha), \text{on}(p, \beta), \text{in}(q, \beta), \not\text{in}(q, \alpha) \leadsto \text{intersect}(\alpha, \beta) \qquad (2.52)$$
$$\text{in}(p, \alpha), \neg\text{in}(p, \beta), \text{in}(q, \beta), \neg\text{in}(q, \alpha), \text{in}(r, \alpha), \text{in}(r, \beta) \leadsto \text{intersect}(\alpha, \beta) \qquad (2.53)$$

2.4.2   Metric Inferences

Segment Magnitudes

We include inference rules for the binary operation $+$ and the binary relation $<$ on segment magnitudes that amount to:

$$+ \text{ is associative and commutative, with identity } 0 \qquad (2.54)$$

$$< \text{ is a linear order with least element } 0 \qquad (2.55)$$

$$x < y \rightsquigarrow x + z < y + z \qquad (2.56)$$

With the exception that Euclid had in mind only positive magnitudes (we include 0 merely for convenience), these are exactly the general properties which Euclid assumes of magnitudes. The further property which we include it seems that Euclid takes to be clear by definition:

$$\rightsquigarrow \quad \overline{pq} = \overline{qp} \qquad (2.57)$$

Angle Magnitudes

As with segment magnitudes, we have

$$+ \text{ is associative and commutative, with identity } 0 \qquad (2.58)$$

$$< \text{ is a linear order with least element } 0 \qquad (2.59)$$

$$x < y \rightsquigarrow x + z < y + z \qquad (2.60)$$

And our further property is just

$$p \neq q, p \neq r \rightsquigarrow \angle pqr = \angle rqp \qquad (2.61)$$

2.4.3   Transfer Inferences

Diagram-Segment Rules

$$p = q \quad \rightsquigarrow \quad \overline{pq} = 0 \qquad (2.62)$$

$$\overline{pq} = 0 \quad \rightsquigarrow \quad p = q \qquad (2.63)$$

$$\text{bet}(p,q,r) \quad \rightsquigarrow \quad \overline{pq} + \overline{qr} = \overline{pr} \qquad (2.64)$$

$$\text{center}(p,\alpha), \text{center}(p,\beta), \text{on}(q,\alpha), \text{on}(r,\beta), \overline{pq} = \overline{pr} \quad \rightsquigarrow \quad \alpha = \beta \qquad (2.65)$$

$$\text{center}(p,\alpha), \text{on}(q,\alpha), \overline{pr} = \overline{pq} \quad \rightsquigarrow \quad \text{on}(r,\alpha) \qquad (2.66)$$

$$\text{center}(p,\alpha), \text{on}(q,\alpha), \text{on}(r,\alpha) \quad \rightsquigarrow \quad \overline{pr} = \overline{pq} \qquad (2.67)$$

$$\text{center}(p,\alpha), \text{on}(q,\alpha), \overline{pr} < \overline{pq} \quad \rightsquigarrow \quad \text{in}(r,\alpha) \qquad (2.68)$$

$$\text{center}(p,\alpha), \text{on}(q,\alpha), \text{in}(r,\alpha) \quad \rightsquigarrow \quad \overline{pr} < \overline{pq} \qquad (2.69)$$

Diagram-Angle Rules

$$p \neq q, p \neq r, \text{on}(p, L), \text{on}(q, L), \text{on}(r, L), \neg\text{bet}(q, p, r) \quad \rightsquigarrow \quad \angle qpr = 0 \qquad (2.70)$$

$$p \neq q, p \neq r, \text{on}(p, L), \text{on}(q, L), \angle qpr = 0 \quad \rightsquigarrow \quad \text{on}(r, L), \neg\text{bet}(q, p, r) \quad (2.71)$$

$$\text{on}(p, L), \text{on}(q, L), \text{bet}(p, r, q), \neg\text{on}(s, L), \angle prs = \angle srq \quad \rightsquigarrow \quad \angle prs = \mathfrak{r} \qquad (2.72)$$

$$\text{on}(p, L), \text{on}(q, L), \text{bet}(p, r, q), \neg\text{on}(s, L), \angle prs = \mathfrak{r} \quad \rightsquigarrow \quad \angle prs = \angle srq \qquad (2.73)$$

$$\begin{array}{l} L \neq M, \text{on}(p, L), \text{on}(p, M), \text{on}(q, L), \text{on}(r, M), p \neq q, p \neq r, \\ \neg\text{on}(s, L), \neg\text{on}(s, M), \angle qpr = \angle qps + \angle spq \end{array} \rightsquigarrow \text{same}(q, s, M), \text{same}(r, s, L)$$

$$(2.74)$$

$$\begin{array}{l} L \neq M, \text{on}(p, L), \text{on}(p, M), \text{on}(q, L), \text{on}(r, M), p \neq q, p \neq r, \\ \neg\text{on}(s, L), \neg\text{on}(s, M), \text{same}(q, s, M), \text{same}(r, s, L) \end{array} \rightsquigarrow \angle qpr = \angle qps + \angle spq \quad (2.75)$$

$$\begin{array}{l} \text{on}(a, L), \text{on}(b, L), \text{on}(b, M), \text{on}(c, M), \text{on}(c, N), \text{on}(d, N), \\ b \neq c, \text{same}(a, d, N), (\angle abc + \angle bcd < \mathfrak{r} + \mathfrak{r}) \end{array} \rightsquigarrow \text{intersect}(L, N) \qquad (2.76)$$

2.5 Derived Rules

We should expect to be able to make the following inference,

$$\frac{\Gamma \rightsquigarrow (\exists \vec{p}, \vec{L}, \vec{\alpha})\Delta \quad \Lambda \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Theta}{\Gamma, \Lambda \rightsquigarrow (\exists \vec{p}, \vec{q}, \vec{L}, \vec{M}, \vec{\alpha}, \vec{\beta})\Delta, \Theta}$$

a sort of "conjunction" rule. We note here that this is indeed a derived rule of our system. If we have a proof of $\Gamma \rightsquigarrow (\exists \vec{p}, \vec{L}, \vec{\alpha})\Delta$, we can obtain a proof of $\Gamma, \Lambda \rightsquigarrow (\exists \vec{p}, \vec{L}, \vec{\alpha})\Delta$ simply by adding in Λ to the antecedent of every line of the proof. Similarly, we obtain a proof of $\Gamma, \Lambda \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Theta$. But then we can conclude

$$\frac{\Gamma, \Lambda \rightsquigarrow (\exists \vec{p}, \vec{L}, \vec{\alpha})\Delta \quad \Gamma, \Lambda \rightsquigarrow (\exists \vec{q}, \vec{M}, \vec{\beta})\Theta}{\Gamma, \Lambda \rightsquigarrow (\exists \vec{p}, \vec{q}, \vec{L}, \vec{M}, \vec{\alpha}, \vec{\beta})\Delta, \Theta}$$

as an instance of theorem application.

We note further that proof by contradiction is a derived rule of E, given our apparatus for case splits. Supposing that $\bot$ follows from $\neg A$, then A certainly does as well (recalling that our notion of diagrammatic consequence allows us to infer anything from a collection of diagram assertions that includes $\bot$). As A follows from A, of course, a case split allows us to conclude A outright.

2.6   A Conservative Extension

For the purpose of translating between formal systems in our completeness proof, we will want to work with a conservative extension of the above system. We add some new constants and some new axioms. The constants we add are

$$\{\mathfrak{c}_1^L \mid L \text{ a line variable}\} \cup \{\mathfrak{c}_2^L \mid L \text{ a line variable}\},$$

$$\{\mathfrak{c}_1^\gamma \mid \gamma \text{ a circle variable}\} \cup \{\mathfrak{c}_2^\gamma \mid L \text{ a circle variable}\}.$$

The idea is that the constants $\mathfrak{c}_1^L, \mathfrak{c}_2^L$ denote two arbitrary, distinct points on L, and $\mathfrak{c}_1^\gamma, \mathfrak{c}_2^\gamma$ denote, repsectively, the center of γ and an arbitrary point on γ. Thus we add the following axiom schemes:

$$\rightsquigarrow \mathfrak{c}_1^L \neq \mathfrak{c}_2^L, \mathrm{on}(\mathfrak{c}_1^L, L), \mathrm{on}(\mathfrak{c}_2^L, L),$$

$$\rightsquigarrow \mathrm{center}(\mathfrak{c}_1^\gamma, \gamma), \mathrm{on}(\mathfrak{c}_2^\gamma, \gamma).$$

It is clear that this is indeed a conservative extension of E. For instance, any use of $\mathfrak{c}_1^L, \mathfrak{c}_2^L$ in a proof can be replaced by a construction of two distinct points $p \neq q$ on L (where the variables p, q do not already appear in the proof); then use p, q in place of $\mathfrak{c}_1^L, \mathfrak{c}_2^L$ in the proof and you still have a proof. For the rest of the paper, we assume for the sake of our completeness proof that these constants and axiom schemes are a part of E.

Chapter 3

Illustrative Constructions in E

3.1 Some *Elements*

Here we will present semi-formal E proofs of some of the early propositions from the *Elements*. Note that though the proofs are indeed less than formal, in another sense they in fact go overboard as compared to what is allowed in E. Specifically, our notion of immediate diagrammatic consequence would allow us to make certain inferences below in one go, whereas we choose to indicate the chain of reasoning involving the demonstration rules of E. This is done simply in order to give the reader a feel for the adequacy of the lists of rules given in the previous chapter. The reader will find it instructive to construct a diagram as he goes, and also verify our use of various inference rules.

E-Proposition 3.1 (Proposition I.1)**.** *Given distinct points a, b, construct a point c such that triangle abc is equilateral.*

Proof. As $a \neq b$, we can construct circle α through b centered at a, and also circle β through a centered at b (2.12). By (2.24), a is inside α and b is inside β. Thus, since a is inside α and on β, and vice versa for b, we know that α, β intersect (2.52). So we take point c on both α and β (2.18). Now using (2.67) we can arrive at the fact that triangle abc is equilateral. $\qquad\square$

We note that our rules allow us to prove a variant of the above, where we take the two distinct intersection points of the circles (i.e. using rule (2.19), and thus obtaining two equilateral triangles abc and abd. We will reference this fact below.

E-Proposition 3.2 (Proposition I.2). *Let $a \neq b$ on L. Let c be any other point. Construct a point f such that $\overline{cf} = \overline{ab}$.*

Proof. First we apply Proposition 1, obtaining an equilateral triangle acd. Construct circle α through b, centered at a (2.12). Draw line M through a, d (2.11).[1] We know that M and α intersect (2.24, 2.51). So take point e on α such that $\mathrm{bet}(d, a, e)$ (2.17).

Now construct circle δ through e, centered at d (2.12). Draw line N through d, c (2.11). We know that N and δ intersect (2.24, 2.51). Furthermore, we know that $\overline{dc} = \overline{da} < \overline{de}$ since a is between d and e (2.64). Thus c is inside δ (2.68). Therefore we may take f on δ such that $\mathrm{bet}(d, c, f)$ (2.17).

From here straightforward uses of metric inferences, transfer inferences, and equality rules yields
$$\overline{cf} = \overline{ab}.$$
We leave it to the reader to check this. □

E-Proposition 3.3 (Proposition I.3). *Let $a \neq b$. Let c be any other point, on some line L. Construct a point e on L such that $\overline{ab} = \overline{ce}$.*

Proof. Apply 3.2 to obtain a point d such that $\overline{ab} = \overline{cd}$. As $c \neq d$, construct circle γ through d centered at c. As L and γ intersect (2.24, 2.51), we can take e on both L and γ (2.14). It is easy to derive that $\overline{ce} = \overline{ab}$. □

We skip ahead a few propositions and see how a proof of Proposition 12 in E compares and contrasts with what we indicated in the introduction. As there, we will assume the earlier propositions from the *Elements*.

E-Proposition 3.4 (Proposition I.12). *Suppose point p lies off of line L. Then there is a line M through p such that L and M form a right angle.*

Proof. Let q be a point on the other side of L from p. Construct circle α with center p, passing through q (2.12). As p, q are on opposite sides of L, they are distinct. So draw the line M through p, q. M and L intersect (2.50), and so we can take the intersection point r (2.13). By (2.42), r is between p and q. Thus $\overline{pr} + \overline{rq} = \overline{pq}$ (2.64). As $\overline{rq} \neq 0$, $\overline{pr} < \overline{pq}$. Thus r is inside α (2.68).

---

[1]Even here we have left a small gap, though one which our rules do close. Specifically, in order to apply this line construction rule, we need $a \neq d$. But this follows from the facts that $a \neq c$, $\overline{ac} = \overline{ad}$, and our metric transfer rules. We will leave similar gaps at times; thus the *semi*-formal nature of the presented proofs.

Thus L and α intersect (2.51). So we may take points $a \neq b$ on L which are also on α (2.15). (See Figure 3.1.)[2]
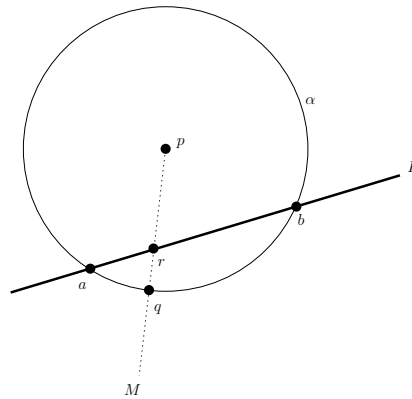

Figure 3.1: Book I, Proposition 12

And here he would proceed using earlier Propositions from the *Elements.* $\square$

3.2 Some More Technical Constructions

In this section we prove some results in E that are not all that inherently interesting as geometric constructions; they are presented both to offer further illustration of the workings of E and, in fact, to prove steps in the coming completeness proof.

First, here is a slight variant of Proposition I.3 above, which we present because it ties in directly with one of Tarski's first-order axioms for plane geometry.

E-Proposition 3.5. *Let a, b, c, d be distinct points, with c, d on line L. Construct point e on L such that* (1) $\overline{ab} = \overline{de}$, (2) $\mathrm{bet}(c, d, e)$.

Proof. We proceed as before, but apply a different construction rule at the end when taking our intersection point. Namely:

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[2]Note that our proof looks like what was done in the introduction, but now we have used our explicitly formulated rules in order to justify taking the intersection points a, b.

Apply 3.2 to get f such that $\overline{df} = \overline{ab}$. Construct δ through f centered at d. As before, L, δ intersect (2.24, 2.51). But now we have a rule that allows us to take e with precisely the properties of the conclusion (2.17).   □

Lemma 3.6 (Uniqueness of Perpendiculars). *Suppose point p is on line L. Suppose p is also on lines M, N, and that each of these is perpendicular to L. Then $M = N$.*

Proof. Suppose for contradiction that $M \neq N$. Let $q \neq p$ be on M (2.2). We leave it to the reader to construct a point $r \neq p$ on N such that $\text{same}(q, r, L)$. The same can be repeated in order to obtain $s \neq p$ on L such that $\text{same}(r, s, M)$. But in this scenario, we can apply (2.75) to conclude that

$$\angle qpr + \angle rps = \angle qps.$$

By hypothesis, then, $\angle qpr + \mathfrak{r} = \mathfrak{r}$, so that $\angle qpr = 0$. But now we have that r is on M (2.71). Thus $M = N$ (2.22); contradiction.   □

E-Proposition 3.7. *Suppose $p \neq q$ are on the same side of L. Then there are points r, s, t such that: (1) s, t are on L, (2) r is the intersection of the line through p, s and the line through q, t.*

Proof. Using 3.4, drop a perpendicular N from p to e on L, and M from q to f on L.

CASE: $e \neq f$. Then we set $s := f$ and $t := e$. First of all, p, t are on the same side of M.

Then draw line O through p, s. Since p, t are on the same side of M and p, q are on the same side of L, we know by (2.43) that q, t are on opposite sides of O. (See Figure 3.2.)
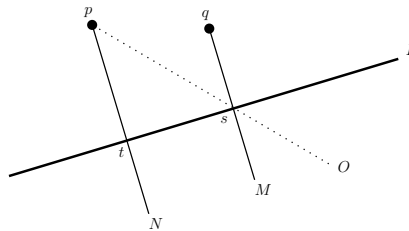

Figure 3.2: E-Proposition 3.7

Thus when we draw line P through q, t, we get our desired intersection point r.

CASE: $e = f$. In this case, the uniqueness of perpendiculars yields $M = N$. We now have two subcases; how we handle them can be pictured thus as in Figure 3.3.

Figure 3.3: E-Proposition 3.7

SUBCASE: q is between p and e. Let t be a point on L distinct from e. Connect line P through q, t. Now choose a point s on L that is between t and e. We know that e, s are on the same side of P (2.40), and that e, p are not on the same side of P (2.41). Thus p and s are on opposite sides of P (2.37). So when we draw line O through p, s we get our desired intersection point r.

SUBCASE: p is between q and e. Proceed symmetrically. □

E-Proposition 3.8. *Suppose we have a line L, and points p, q, r, s, t satisfying* (1) *and* (2) *from the previous theorem. Then p, q are on the same side of L.*

Proof. This is immediate from a couple uses of (2.40). □

Chapter 4

The Adequacy of E

We can judge the success of E, given the intentions behind its creation, by answering the following questions:

(i)   Is it really faithful to Euclid's diagrammatic approach?
(ii) Does it suffice for *all* (and only) ruler-and-compass constructions of the form of those in the *Elements*?

Question (i) is partly logical and partly philological, depending on a close reading of Euclid's *Elements*; we have indicated above that we feel it can be answered in the affirmative, but we wish to discuss the matter no further here.[1] Our present interest is in answering the purely mathematical question (ii).

4.1   What Form Completeness?

There are two well-known, equivalent ways to characterize the ruler-and-compass theorems. One is semantic, involving Cartesian plane structures which are built on Euclidean fields; the other is syntactic, an axiomatization (in a more restricted language than that of E) of the ruler-and-compass theorems which is due to Tarski. Before presenting these characterizations, let us outline the coming completeness proof.

   We use the locution "$\Gamma \rightsquigarrow (\exists \vec{x})\Delta$ is valid" as shorthand for $\Gamma \rightsquigarrow (\exists \vec{x})\Delta$ being valid in the class of structures marked out in the semantic characterization below. Later on we will craft a modification T of Tarski's original

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[1]A more probing account of the degree to which E is faithful to the *Elements* will be a part of [1].

system that still characterizes the body of theorems in question, but which also has some nice formal properties not shared by the original. We will define a translation $\pi : \mathcal{L}(\mathsf{E}) \to \mathcal{L}(\mathsf{T})$ that maps E-sequents to *regular* sentences (defined below), and a re-translation $\rho : \mathcal{L}(\mathsf{T}) \to \mathcal{L}(\mathsf{E})$. Ultimately, we will show that the systems and translations involved have the following properties:

1. $\Gamma \rightsquigarrow (\exists \vec{x})\Delta$ is valid if and only if $\mathsf{T} \vdash \pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$.

2. If $T \vdash \varphi$ where φ is regular, then $\mathsf{E} \vdash \rho(\varphi)$.

3. If $\mathsf{E} \vdash \rho(\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta))$, then $\mathsf{E} \vdash (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$.

Establishing those facts is sufficient; it immediately follows that $\mathsf{E} \vdash (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$ for any valid $\Gamma \rightsquigarrow (\exists \vec{x})\Delta$. First things first, let us now describe the semantic picture of things.

4.1.1   Arithmetic Cartesian Planes

We seek a semantics for ruler-and-compass constructions, by which we can judge the strength of E. Since the time of Descartes, it has been common practice to arithmetize geometry. For instance, in standard Euclidean plane geometry we can think of ourselves as working in a Cartesian plane structure that is based on a *real closed field* $\mathbb{F}$:

(1) $\langle \mathbb{F}, + \cdot, 0, 1 \leq \rangle$ is an ordered field,
(2) $\mathbb{F} \models \forall x (x > 0 \to \exists y (y^2 = x))$
         (positive elements have square roots),
(3) $\mathbb{F}$ satisfies an axiom stating that any
         odd-degree polynomial has a root.

Given an ordered field $\mathbb{F}$, we now spell out exactly what we mean, formally, by the *Cartesian plane structure on* $\mathbb{F}$ *in the language of* E. It will look something like this:

$$\mathscr{F} := \langle \mathbb{F}^2, L^{\mathbb{F}}, C^{\mathbb{F}}, A^{\mathbb{F}}, M^{\mathbb{F}}, \ldots \rangle.$$

Here the set $\mathbb{F}^2$ of all ordered pairs from $\mathbb{F}$ is the universe of points. $L^{\mathscr{F}}$, $C^{\mathscr{F}}$ $A^{\mathscr{F}}$ and $M^{\mathscr{F}}$ denote the universes of lines, circles, angles and magnitudes, respectively. The ... elide over the relations that are the interpretations in $\mathscr{F}$

of the various relation symbols of $\mathcal{L}(\mathsf{E})$. We will not go into the details here of how these interpretations are defined in terms of elements of the structure. Instead we will just point out that they are defined exactly as you (probably) remember from grade-school arithmetic geometry; for the gist of things, see the simplified semantics below, for which we do spell out the details.

We can now give a formal criterion for when some $\Gamma \rightsquigarrow (\exists \vec{x})\Delta$ is a theorem of Euclidean plane geometry:

$$\mathscr{F} \models (\Gamma \rightsquigarrow (\exists \vec{x})\Delta) \quad \text{for all real closed fields } \mathbb{F} \text{ ?}$$

In fact, this criterion can be simplified a bit. Tarski [12] showed, via quantifier elimination, that the theory of real closed fields (RCF) is complete and decidable. The completeness result in particular implies that all models of RCF are elementarily equivalent; thus we can simplify the criterion above by stating it in terms of a particular real closed field. Take, say, $\mathscr{R}$, built upon the standard real numbers $\mathbb{R}$ (which is certainly a real closed field). Then we can restate the criterion above:

$$\mathscr{R} \models (\Gamma \rightsquigarrow (\exists \vec{x})\Delta) \quad ?$$

Now that is all well and good for plane geometry as we often think of it in, say, physics; that is, when we want to deal with the *complete* ordered field $\mathbb{R}$. But we are interested in Euclid's mathematics in particular; namely, our focus is on ruler-and-compass constructions. We want to live in a plane consisting only of points that can be pinpointed (or, "constructed") from a given starting point using ruler-and-compass. A contrast: in $\mathscr{R}$, it is true that for any points a, b, c, there are points d, e such that the lines through ad and ae trisect the angle $\angle bac$. (This is a consequence of the compeleteness of $\mathbb{R}$.) But it is well-known that there are angles which cannot be trisected using ruler-and-compass (e.g. $\pi/3$); the problem is that the points d, e which exist in $\mathbb{R}^2$ cannot be pinpointed with the restricted method of ruler-and-compass.

So what sort of ordered field gives rise to this sparser, more limited Cartesian plane? Real closed fields are no longer the answer; rather, the pertinent notion is that of a *Euclidean field* $\mathbb{K}$: an ordered field in which every $x > 0$ has a square root (so, a real closed field minus condition (3)). Essentially, condition (2) gives us enough points so that circles intersect when they should, while the omission of (3) corresponds to the fact that we lose out on points such as those needed to trisect $\pi/3$. The question of whether

some $\Gamma \leadsto (\exists \vec{x})\Delta$ is a theorem in the ruler-and-compass setting reduces to the following:

$$\mathscr{K} \models (\Gamma \leadsto (\exists \vec{x})\Delta) \quad \text{for all Euclidean fields } \mathbb{K} \ ?^2$$

In light of the foregoing discussion, our ultimate goal is this: for any E-proposition $\Gamma \leadsto (\exists \vec{x})\Delta$ we claim that the following hold.

SOUNDNESS: If $E \vdash (\Gamma \leadsto (\exists \vec{x})\Delta)$, then $\mathscr{K} \models (\Gamma \leadsto (\exists \vec{x})\Delta)$ for all Euclidean fields $\mathbb{K}$.

COMPLETENESS: If $\mathscr{K} \models (\Gamma \leadsto (\exists \vec{x})\Delta)$ for all Euclidean fields $\mathbb{K}$, then $E \vdash (\Gamma \leadsto (\exists \vec{x})\Delta)$.

4.1.2   A Simplified Semantics

In demonstrating the completeness of E, we will be making use of Tarski's completeness results for his first-order axiomatization of plane geometry. The language $\mathcal{L}$ of Tarski's system T^-, defined below, has only a single sort and (in addition to the ubiquitous $=$) only predicates $B(xyz)$ and $xy \equiv vu$, which are intended to mean "y is between[3] x and z" and "the distance from x to y is the same as that from v to u." Just as we defined, given an ordered field $\mathbb{F}$, the $\mathcal{L}(E)$-structure $\mathscr{F}$, we can define a Cartesian plane structure for this language; namely,

$$\mathscr{F}_{\mathcal{L}} := \langle \mathbb{F}^2, B^{\mathbb{F}}, \equiv^{\mathbb{F}} \rangle,$$

where

$$B^{\mathbb{F}} := \left\{ (x, y, z) \in (\mathbb{F}^2)^3 \ \middle| \ \begin{array}{l} (x_1 - y_1) \cdot (y_2 - z_2) = (x_2 - y_2) \cdot (y_1 - z_1), \\ (x_1 - y_1) \cdot (y_1 - z_1) \geq 0, \\ (x_2 - y_2) \cdot (y_2 - z_2) \geq 0. \end{array} \right\}$$

and

$$\equiv^{\mathbb{F}} := \ \{(x, y, u, v) \in (\mathbb{F}^2)^4 \mid (x_1 - y_1)^2 + (x_2 - y_2)^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2\}.$$

Note that, as promised above, these defining conditions are exactly the familiar ones seen in the earliest of algebra classes . Again, we are interested in

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[2]We remark that this criterion cannot be simplified as in the case above. The reason for this is that the theory in question here is not complete. See footnote 4 for more detail.

[3]In Tarski's system this is a *non-strict* betweenness (i.e. it could be that $x = y$ or $y = z$), in contrast to the strict betweenness of E.

this simpler kind of structure because it enjoys a tight connection (exposited below) with Tarski's formal system. As it turns out, we will in fact be interested in a modification of Tarski's T^-. We will call this system T, and it is also defined below. At some point we will want to indicate a semantics for T that is closely related to that for T^-; but we will postpone our discussion of this until after we have introduced the Tarskian systems.

4.1.3 Tarskian Axiomatizations of Geometry

As indicated above, we will now turn to Tarski's axiomatization of plane geometry, which continued the march toward formalized geometry initiated by Hilbert's (still informal) *Grundlagen der Geometrie* [2], for a syntactic characterization of the class of theorems that we are interested in. As indicated above, Tarski's original formal system for plane geometry is in the language $\mathcal{L}$, whose only non-logical predicates are B and $\equiv$. Its axiomatization consists of (the universal closures of) the following (see, e.g. [14]):

EQUIDISTANCE AXIOMS

(E1) $ab \equiv ba$

(E2) $(ab \equiv pq) \wedge (ab \equiv rs) \rightarrow (pq \equiv rs)$

(E3) $(ab \equiv cc) \rightarrow a = b$

BETWEENNESS AXIOM

(B) $B(abd) \wedge B(bcd) \rightarrow B(abc)$

SEGMENT CONSTRUCTION AXIOM

(SC) $\exists x (B(qax) \wedge (ax \equiv bc))$

FIVE-SEGMENT AXIOM

(5S) $[\neg(a = b) \wedge B(abc) \wedge B(pqr) \wedge (ab \equiv pq)$
$(bc \equiv qr) \wedge (ad \equiv ps) \wedge (bd \equiv qs)] \rightarrow (cd \equiv rs)$

PASCH AXIOM

(P) $B(apc) \wedge B(qcb) \rightarrow \exists x (B(axq) \wedge B(bpx))$

2-DIMENSION AXIOMS

(2L) $\exists a, b, c [\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab)]$

(2U) $\neg(a = b) \wedge \bigwedge_{i=1}^{3} x_i a \equiv x_i b$
$\rightarrow (B(x_1 x_2 x_3) \vee B(x_2 x_3 x_1) \vee B(x_3 x_1 x_2))$

EUCLID'S AXIOM (PARALLEL POSTULATE)

(PP) $B(adt) \wedge B(bdc) \wedge \neg(a = d)$
    $\rightarrow \exists x, y(B(abx) \wedge B(acy) \wedge B(ytx))$

CONTINUITY AXIOM SCHEME

(C)$_\varphi$ $\exists a \forall x, y(\varphi(x) \wedge \psi(y) \rightarrow B(axy)) \rightarrow \exists b \forall x, y(\varphi(x) \wedge \psi(y) \rightarrow B(xby))$

(In this last, φ contains no free a, b, y, and ψ contains no free a, b, x.) We call the above theory T^+. Tarski proved the following:

Fact 4.1 (Tarski [13]). T^+ *is sound and complete for real closed fields:*

$$\mathsf{T}^+ \vdash \varphi \iff \mathscr{R}_\mathcal{L} \models \varphi.$$

So T^+ is sufficient to prove *any* theorem of Euclidean plane geometry; this is greater strength than E is meant to have, as discussed above. We would like to judge the strength of E against a weakening of T^+. In fact, Tarski already established how to appropriately weaken T^+ so that the result proves exactly the ruler-and-compass theorems in the language $\mathcal{L}$. By T^- we mean the theory obtained from T^+ by doing away with the full Continuity scheme (which is what makes T^+ jibe with the completeness of $\mathbb{R}$), and to have in its stead the following:

INTERSECTION AXIOM

(Int) $(ax \equiv ax') \wedge (az \equiv az') \wedge B(axz) \wedge B(xyz)$
    $\rightarrow \exists y'((ay \equiv ay') \wedge B(x'y'z'))$

Intuitively, this axiom says that any line through a point lying inside a circle intersects the circle (see Figure 4.1).

Fact 4.2 (Tarski [13]). *The weakened system* T^- *is sound and complete for ruler-and-compass theorems:*

$$\mathsf{T}^- \vdash \varphi \iff \mathscr{K}_\mathcal{L} \models \varphi \text{ for all Euclidean fields } \mathbb{K}.[4]$$

---

[4]Note that T^- is finitely axiomatized; for we have replaced the lone axiom scheme of T^+ with a single axiom. Ziegler [15] proved that any finitely axiomatizable theory of fields that has among its models an algebraically closed field, a real closed field or a field of p-adic numbers, is an undecidable theory. It is clear from the present result that T^- has a real closed field among its models (since a real closed field is, *a fortiori*, Euclidean). Thus T^- is undecidable, hence incomplete. In light of this and the characterization given by Fact 4.2, it is clear that the criterion

$$\mathscr{K} \models (\Gamma \rightsquigarrow (\exists \vec{x})\Delta) \text{ for all Euclidean fields } \mathbb{K} ?$$

discussed above cannot be replaced by a criterion referencing satisfaction in a single structure.
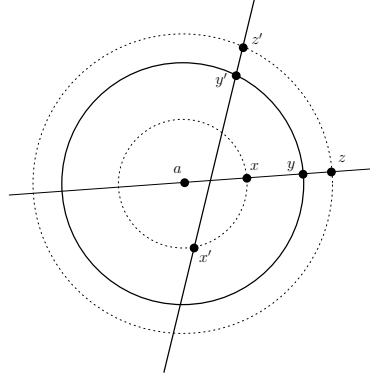
Figure 4.1: Intersection Axiom

4.1.4 A Useful Reformulation of T^-

A theory is called *geometric* if all of its axioms are sentences of the following form:

$$(\star) \quad \forall \mathbf{x} \left[\bigwedge_{i=1}^{m} B_i(\mathbf{x}) \;\rightarrow\; \bigvee_{j=1}^{n} \left(\exists \mathbf{y}_j \bigwedge_{k=1}^{\ell_j} A_{j,k}(\mathbf{x}, \mathbf{y}_j) \right) \right],$$

where the A's and B's are atomic formulas (including $\top$ and $\bot$), and each of $\mathbf{x}$, $\mathbf{y}$ or the antecedent of the conditional could be empty. Formulas of the form $(\star)$ are called *geometric*; note that this class includes the regular formulas as a special case (namely, where there is only a single disjunct). Sara Negri [8], building on earlier joint work with Jan von Plato [9], has established a cut-elimination theorem for geometric theories that we can put to use in our completeness proof. Suppose we have a geometric theory G, formulated in a standard two-sided sequent calculus.[5] Then G can be recast equivalently by replacing each of its geometric axioms like the one above with a corresponding inference rule, called a *geometric rule scheme* (GRS):

$$\frac{\vec{A_{1,\cdot}}(\mathbf{x}, \mathbf{y}_1), \Gamma \;\Rightarrow\; \Delta \quad \cdots \quad \vec{A_{n,\cdot}}(\mathbf{x}, \mathbf{y}_n), \Gamma \;\Rightarrow\; \Delta}{\vec{B}(\mathbf{x}), \Gamma \;\Rightarrow\; \Delta}$$

where the variables in the $\mathbf{y}_j$'s do not appear free in $\vec{B}$, Γ or Δ.[6]

---

[5]For concreteness, we fix one such sequent calculus; see the appendix.

[6]It is a straightforward exercise to verify that this inference rule has the same strength as assuming the corresponding geometric formula as a non-logical axiom. We also note

Henceforth, when we speak of the geometric theory G, we refer to its GRS formulation. Negri's principal result is the following theorem, whose corollary we will apply later.

Theorem 4.3. *Any geometric theory* G *has cut-elimination.*

Corollary 4.4 (Weak Subformula Property)**.** *If* G ⊢ φ, *then there is a proof of φ that mentions only subformulas of φ, and possibly some other* atomic *formulas.*

Proof. Among the logical rules, only cut removes formulas. And inspection of the GRS above reveals that a GRS can only remove atomic formulas.   ☐

Expanding the Language of T

Tarski's axiomatization for T^- is *nearly* geometric. The only stumbling block is that in $(\star)$ the conjunctions are required to be conjunctions of atomic formulas, *not literals.* Thus, for instance, the lower 2-dimensional axiom

$$\exists a, b, c(\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab))$$

is not geometric. We remedy this situation by introducing explicit predicates for the negations of $=$ and B and $\equiv$; that is, we expand our language to one called $\mathcal{L}(T)$ by adding predicates $\neq$ and $\overline{B}$ and $\not\equiv$; and we add the (geometric) axioms

$$\forall x, y[(x = y) \vee (x \neq y)], \qquad \forall x, y[(x = y) \wedge (x \neq y) \rightarrow \perp]$$

(as well as analogous ones for $B, \overline{B}$ and $\equiv, \not\equiv$) to T. We will call these "negativity axioms" below. Also, we replace any negated instances of $=$ or B (there are no such negated instances of $\equiv$) from the previous axiomatization of T^- with the new corresponding predicate, thus obtaining a geometrically axiomatized theory. Finally, for technical reasons seen below, we will add constants $\mathfrak{c}_1^N, \mathfrak{c}_2^N$ and $\mathfrak{c}_1^\gamma, \mathfrak{c}_2^\gamma$ for all line and circle variables of $\mathcal{L}(E)$, as in

---

that the form given here is not quite that which appears in [8]; the reason is that the rules must be presented with the $\vec{B}(\mathbf{x})$ repeated in the premises in order for Negri to prove the admissibility of the structural rules of contraction and weakening, along with cut-elimination. The variant we are using is notationally simpler, and for our purposes just as good; using the original formulation would not alter our completeness proof in any essential way.

Chapter 1. We call the resulting theory simply T, and denote its language by $\mathcal{L}(\mathsf{T})$.

Before going further we make one remark. Let $\sigma : \mathcal{L}(\mathsf{T}) \to \underline{\mathcal{L}}$ be the obvious, natural translation (which maps, e.g., occurrences of $\overline{B}(xyz)$ to $\neg B(xyz)$, and so on). Then we have:

Fact 4.5. *Let $\varphi \in \mathcal{L}(\mathsf{T})$. Then*

$$\mathsf{T} \vdash \varphi \iff \mathsf{T}^- \vdash \sigma(\varphi).$$

Proof. This is just true by virtue of the negativity axioms of the expanded T, which guarantee that the new predicates really behave like the negations; that's why we have added those axioms after all. $\square$

GRS's for Non-Logical Axioms

Now we go further and put the non-logical axioms of T into the form of geometric rule schemes. First of all, the negativity axioms look like this:

$$\frac{(x = y), \Gamma \Rightarrow \Delta \quad (x \neq y), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ Neg} \qquad \frac{\bot, (x = y), (x \neq y), \Gamma \Rightarrow \Delta}{(x = y), (x \neq y), \Gamma \Rightarrow \Delta} \text{ Neg}$$

And similarly for the other predicates. The remaining axioms are as follows:

$$\underline{\text{EQUIDISTANCE GRS's}}$$

$$\frac{ab \equiv ba, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ E1}$$

$$\frac{(pq \equiv rs), (ab \equiv pq), (ab \equiv rs), \Gamma \Rightarrow \Delta}{(ab \equiv pq), (ab \equiv rs), \Gamma \Rightarrow \Delta} \text{ E2}$$

$$\frac{(a = b), (ab \equiv cc), \Gamma \Rightarrow \Delta}{(ab \equiv cc), \Gamma \Rightarrow \Delta} \text{ E3}$$

$$\underline{\text{BETWEENNESS GRS}}$$

$$\frac{B(abc), \Gamma \Rightarrow \Delta}{B(abd), B(bcd), \Gamma \Rightarrow \Delta} \text{ B}$$

$$\text{SEGMENT CONSTRUCTION GRS}$$

$$\frac{B(qax), (ax \equiv bc), \Gamma \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; \Delta} \; \text{SC}(x)$$

$$\text{FIVE-SEGMENT GRS}$$

$$\frac{(cd \equiv rs), \Gamma \Rightarrow \Delta}{a \neq b, B(abc), B(pqr), (ab \equiv pq), (bc \equiv qr), (ad \equiv ps), (bd \equiv qs), \Gamma \Rightarrow \Delta} \; \text{5S}$$

$$\text{PASCH GRS}$$

$$\frac{B(axq), B(bpx), B(apc), B(qcb), \Gamma \;\Rightarrow\; \Delta}{B(apc), B(qcb), \Gamma \;\Rightarrow\; \Delta} \; \text{P}(x)$$

$$\text{2-DIMENSION GRS'S}$$

$$\frac{\neg B(abc), \neg B(bca), \neg B(cab), \Gamma \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; \Delta} \; \text{2L}(a, b, c)$$

$$\frac{B(x_1x_2x_3), \vec{P}, \Gamma \;\Rightarrow\; \Delta \qquad B(x_2x_3x_1), \vec{P}, \Gamma \;\Rightarrow\; \Delta \qquad B(x_3x_1x_2), \vec{P}, \Gamma \;\Rightarrow\; \Delta}{\vec{P}, \Gamma \;\Rightarrow\; \Delta} \; \text{2U}$$

Note that $\vec{P} := \{a \neq b, (x_1a \equiv x_1b), (x_2a \equiv x_2b), (x_3a \equiv x_3b)\}$ in the above.

$$\text{PARALLEL POSTULATE GRS}$$

$$\frac{B(abx), B(acy), B(ytx), \Gamma \Rightarrow \Delta}{B(adt), B(bdc), a \neq d, \Gamma \Rightarrow \Delta} \; \text{PP}(x, y)$$

$$\text{INTERSECTION GRS}$$

$$\frac{(ay \equiv ay'), B(x'y'z'), \Gamma \Rightarrow \Delta}{(ax \equiv ax'), (az \equiv az'), B(axz), B(xyz), \Gamma \Rightarrow \Delta} \; \text{Int}(y')$$

Semantics for T

Recall our definition of $\mathscr{F}_{\mathcal{L}}$ for an ordered field $\mathbb{F}$. The analogous structure for the expanded language $\mathcal{L}(\mathsf{T}) \supseteq \mathcal{L}$ is the following expansion of $\mathscr{F}_{\mathcal{L}}$:

$$\mathscr{F}_{\mathsf{T}} := \langle \mathbb{F}^2, B^{\mathbb{F}}, \equiv^{\mathbb{F}}, \overline{B}^{\mathbb{F}}, \not\equiv^{\mathbb{F}}, \neq^{\mathbb{F}} \rangle,$$

where

$$\begin{aligned}
\overline{B}^{\mathbb{F}} &:= \mathbb{F}^2 \setminus B^{\mathbb{F}}, \\
\not\equiv^{\mathbb{F}} &:= \mathbb{F}^2 \setminus (\equiv^{\mathbb{F}}), \\
\neq^{\mathbb{F}} &:= \mathbb{F}^2 \setminus \{(x, x) \mid x \in \mathbb{F}\}.
\end{aligned}$$

Note that these last definitions are appropriate because $\overline{B}$, $\not\equiv$ and $\neq$ are intended to be interpreted as the negations of the corresponding predicates.

Now a quick note for later use. Let $\sigma : \mathcal{L}(\mathsf{T}) \to \mathcal{L}$ be the natural translation from before.

Fact 4.6. *Let $\mathbb{F}$ be an ordered field, and let $\varphi \in \mathcal{L}(\mathsf{T})$. Then*

$$\mathscr{F}_{\mathsf{T}} \models \varphi \iff \mathscr{F}_{\mathcal{L}} \models \sigma(\varphi).$$

Proof. Straightforward from the definitions of $\mathscr{F}_{\mathsf{T}}$, $\mathscr{F}_{\mathcal{L}}$ and σ. □

Corollary 4.7 (Tarski's Fact Reformulated). *For any $\varphi \in \mathcal{L}(\mathsf{T})$ we have:*

$$\mathsf{T} \vdash \varphi \iff \mathscr{K}_{\mathsf{T}} \models \varphi \text{ for all Euclidean fields } \mathbb{K}.$$

Proof. Immediate from established results:

$$\begin{aligned}
\mathsf{T} \vdash \varphi &\iff \mathsf{T}^- \vdash \sigma(\varphi) \\
&\iff \text{all } \mathscr{K}_{\mathcal{L}} \models \sigma(\varphi) \\
&\iff \text{all } \mathscr{K}_{\mathsf{T}} \models \varphi.
\end{aligned}$$

These hold, respectively, by the earlier Fact 4.5, Tarski's Fact 4.2, and the preceding fact. □

A Technical Lemma

Here we will give a result indicating a particular relationship between the structures $\mathscr{K}$ and $\mathscr{K}_{\mathsf{T}}$ when $\mathbb{K}$ is a Euclidean field.

For a fixed first-order language, we call a formula of the form

$$\forall \mathbf{x} \left[\bigwedge_i P_i(\mathbf{x}) \to \exists \mathbf{y} \bigwedge_j Q_j(\mathbf{x}, \mathbf{y}) \right],$$

where the P_i's and Q_j's are atomic, *regular*; a formula of the same form, but where the P_i's and Q_j's are allowed to be literals, we will call *pseudo-regular*.[7] Note that any $\Gamma \rightsquigarrow \Delta$ is a pseudo-regular $\mathcal{L}(\mathsf{E})$-sentence.

We will now define a translation $\pi : \mathcal{L}(\mathsf{E}) \to \mathcal{L}(\mathsf{T})$ in such a way that the range of π is included in the regular sentences; the idea is that we want to represent (in a ruler-and-compass-equivalent way) the pseudo-regular $\mathcal{L}(\mathsf{E})$-sentences $\Gamma \rightsquigarrow (\exists \vec{x})\Delta$ as regular $\mathcal{L}(\mathsf{T})$-sentences. Again, we want to do this for purely technical proof-theoretic reasons that become clear below.

For the purpose of getting a mapping π to regular formulas, we first want to determine ways to express the E-literals in T using only formulas of an even more restricted form. Specifically, for each E-literal A we will define a corresponding $\mathcal{L}(\mathsf{T})$-formula $\overline{\pi}(A)$ of the following form:

$$\exists \mathbf{z} \left(\bigwedge_k M_k(\mathbf{z}) \right)$$

where the M_k's are atomic. (Formulas of this form are sometimes referred to as *positive primitive* formulas.) Without further ado, here is the listing for each E-literal.

- $\mathrm{on}(p, N) \;\mapsto\; \exists a, b(\underbrace{\mathfrak{c}_1^N a \equiv \mathfrak{c}_1^N b \;\wedge\; \mathfrak{c}_2^N a \equiv \mathfrak{c}_2^N b \;\wedge\; pa \equiv pb}_{=: \; \zeta(\mathfrak{c}_1^N, \mathfrak{c}_2^N, p, a, b)}).$

- $\neg\mathrm{on}(p, N) \;\mapsto\; \underbrace{\overline{B}(\mathfrak{c}_1^N \mathfrak{c}_2^N p) \wedge \overline{B}(\mathfrak{c}_1^N p \mathfrak{c}_2^N) \wedge \overline{B}(p \mathfrak{c}_1^N \mathfrak{c}_2^N)}_{=: \; \chi(\mathfrak{c}_1^N, \mathfrak{c}_2^N, p)}.$

- $\mathrm{same}(p, q, N) \;\mapsto\;$

    $\exists r, s, t, a, b(\zeta(\mathfrak{c}_1^N, \mathfrak{c}_2^N, s, a, b) \wedge \zeta(\mathfrak{c}_1^N, \mathfrak{c}_2^N, t, a, b) \wedge \chi(\mathfrak{c}_1^N, \mathfrak{c}_2^N, r) \wedge B(prs) \wedge B(qrt)).$

---

[7]A note on usage: the term regular is employed in this way in the literature; as far as we know, pseudo-regular is not used by anyone.

- $\neg\text{same}(p, q, N) \mapsto \exists r, a, b(\zeta(\mathfrak{c}_1^N, \mathfrak{c}_2^N, r, a, b) \wedge B(prq)).$

- $\text{bet}(p, q, r) \mapsto B(pqr) \wedge p \neq q \wedge q \neq r \wedge p \neq r.$

- $\neg\text{bet}(p, q, r) \mapsto$

$$\exists a, b, f, g, h, x, y, z \left[\begin{array}{l} \chi(a, b, q) \wedge a \neq p \wedge a \neq q \wedge a \neq r \wedge b \neq p \wedge b \neq q \wedge b \neq r \wedge \\ B(apx) \wedge B(aqy) \wedge B(arz) \wedge p \neq x \wedge q \neq y \wedge r \neq z \wedge \\ \underline{B}(bpf) \wedge \underline{B}(bqg) \wedge B(brh) \wedge p \neq f \wedge q \neq g \wedge r \neq h \wedge \\ \overline{B}(xyz) \wedge \overline{B}(fgh) \end{array} \right]$$

- $\overline{xy} = \overline{zw} \mapsto xy \equiv zw.$

- $\overline{xy} \neq \overline{zw} \mapsto xy \not\equiv zw.$

- $\overline{xy} < \overline{zw} \mapsto \exists a(a \neq w \wedge z \neq w \wedge B(zaw) \wedge xy \equiv za).$

- $\overline{xy} \not< \overline{zw} \mapsto \exists a(B(xay) \wedge xa \equiv zw).$

- $\text{on}(p, \gamma) \mapsto \mathfrak{c}_1^\gamma p \equiv \mathfrak{c}_1^\gamma \mathfrak{c}_2^\gamma.$

- $\neg\text{on}(p, \gamma) \mapsto \mathfrak{c}_1^\gamma p \not\equiv \mathfrak{c}_1^\gamma \mathfrak{c}_2^\gamma.$

- $\text{in}(p, \gamma) \mapsto \exists x(B(\mathfrak{c}_1^\gamma px) \wedge p \neq x \wedge (\mathfrak{c}_1^\gamma x \equiv \mathfrak{c}_1^\gamma \mathfrak{c}_2^\gamma)).$

- $\neg\text{in}(p, \gamma) \mapsto \exists x(B(\mathfrak{c}_1^\gamma xp) \wedge (\mathfrak{c}_1^\gamma x \equiv \mathfrak{c}_1^\gamma \mathfrak{c}_2^\gamma)).$

- $\angle xyz = \angle x'y'z' \mapsto$

  $\exists u, v, u', v'(B(xuy) \wedge B(yvz) \wedge B(x'u'y') \wedge B(y'v'z') \wedge (uy \equiv u'y') \wedge (yv \equiv y'v') \wedge (uv \equiv u'v')).$

- $\angle xyz \neq \angle x'y'z' \mapsto$

  $\exists u, v, u', v'(B(xuy) \wedge B(yvz) \wedge B(x'u'y') \wedge B(y'v'z') \wedge (uy \equiv u'y') \wedge (yv \equiv y'v') \wedge (uv \not\equiv u'v')).$

We have not yet indicated the $\overline{\pi}$-images for literals involving the intersect predicate. The positive literals in this regard are straightforwardly expressed in positive primitive manners by combining the $\overline{\pi}$-images of things like $\text{on}(p, L)$

and on(p, α), and we omit the details.. The negative literals, which assert *non*-intersection, require something more roundabout. For instance,

$$\neg\mathrm{intersect}(\alpha, \beta) \ \mapsto \ \exists p, a, b \begin{bmatrix} \mathfrak{c}_1^\alpha \mathfrak{c}_2^\alpha \equiv \mathfrak{c}_1^\alpha a \ \wedge \ \mathfrak{c}_1^\beta \mathfrak{c}_2^\beta \equiv \mathfrak{c}_1^\beta b \ \wedge \\ a \neq p \ \wedge \ a \neq b \ \wedge \ b \neq p \ \wedge \\ B(\mathfrak{c}_1^\alpha a p) \ \wedge \ B(\mathfrak{c}_1^\beta b p) \ \wedge \ B(a p b) \end{bmatrix}$$

Appropriate positive primitive $\overline{\pi}$-images for the literals $\neg\mathrm{intersect}(L, \alpha)$ and $\neg\mathrm{intersect}(L, M)$ can be found using $\overline{\pi}$-images from above, notably that for $\angle xyz = \mathfrak{r}$. For instance, to say that $\neg\mathrm{intersect}(L, \alpha)$, we assert the existence of points a, b, c, where a is on α, $b \neq c$ are on L, a is strictly between $\mathfrak{c}_1^\alpha$, and $\angle abc = \mathfrak{r}$. Similarly, $\neg\mathrm{intersect}(L, M)$ can be expressed by asserting the existence of a, b, c, d, where $a \neq b$ are on L, $c \neq d$ are on M, and the angles $\angle abc$, $\angle bcd$, $\angle cda$, $\angle cab$ are all right.

We now extend $\overline{\pi}$ to a translation $\pi : \mathcal{L}(\mathsf{E}) \to \mathcal{L}(\mathsf{T})$ that maps every $\Gamma \rightsquigarrow (\exists\vec{x})\Delta$ to a regular sentence. Suppose our $\Gamma \rightsquigarrow (\exists\vec{x})\Delta$ is of the form

$$A_1, \ldots, A_k \rightsquigarrow (\exists\vec{x}) B_1, \ldots, B_m,$$

where we have

$$\overline{\pi}(A_i) = \exists \mathbf{z}_i \left(\bigwedge_{q=1}^{n_i} M_q \right), \quad \overline{\pi}(B_i) = \exists \mathbf{y}_i \left(\bigwedge_{q=1}^{p_i} N_q \right).$$

Then we define

$$\pi(\Gamma \rightsquigarrow (\exists\vec{x})\Delta) = \forall \mathbf{z}_1, \ldots, \mathbf{z}_k \left[\bigwedge_{i=1}^{k} \left(\bigwedge_{q=1}^{n_i} M_q \right) \to \exists \mathbf{y}_1, \ldots, \mathbf{y}_m \bigwedge_{i=1}^{m} \left(\bigwedge_{q=1}^{p_i} N_q \right) \right].$$

Note that this is a regular sentence. The following lemma captures all that we need to know about π.

Lemma 4.8. *For any Euclidean field $\mathbb{K}$*

$$\mathscr{K} \models (\Gamma \rightsquigarrow (\exists\vec{x})\Delta) \iff \mathscr{K}_\mathsf{T} \models \pi(\Gamma \rightsquigarrow (\exists\vec{x})\Delta).$$

Proof. In first-order logic, $\pi(\Gamma \rightsquigarrow (\exists\vec{x})\Delta)$ is equivalent to $\bigwedge_i \overline{\pi}(A_i) \to \bigwedge_j \overline{\pi}(B_j)$, where the A_i's are the literals from Γ and the B_j's those from Δ. So it suffices to verify that for every E-literal A in the table above,

$$\mathscr{K} \models A \iff \mathscr{K}_\mathsf{T} \models \overline{\pi}(A).$$

Many of the cases are immediate, and at base verifying any of the cases just comes down to doing some algebra involving the coordinates $(x_1, x_2) \in \mathbb{K}^2$ of the collections of points involved. We will only indicate a couple of cases.

- $\text{on}(p, N)$: Suppose that p is on N. Certainly $\mathfrak{c}_1^N$ and $\mathfrak{c}_2^N$ are as well. Construct the equilateral triangles $a\mathfrak{c}_1^N\mathfrak{c}_2^N$ and $b\mathfrak{c}_1^N\mathfrak{c}_2^N$ à la Proposition 1. We need only check that $\overline{ap} = \overline{bp}$. This is just a matter of picking coordinates for the points and checking the algebra.

  Conversely, suppose that points a and b are equidistant from each of $\mathfrak{c}_1^N$, $\mathfrak{c}_2^N$ and p. By Tarski's Fact, we can appeal to the upper 2-dimensional axiom and conclude that the three points all lie on one line. And this line clearly must be N.

- $\text{same}(p, q, N)$: For this case we have done the work in the previous chapter. See E-Propositions 3.7 and 3.8.

$\square$

4.2 Completeness Proof

The reason we are interested in the above reformulation of Tarski's system is that, being a geometric theory cast in that manner, we can apply Negri's cut-elimination theorem (and the weak subformula property it implies) to it. The overall strategy for what remains of the completeness proof is as follows:

- Suppose $\mathsf{T} \vdash \pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$.

- (Reduction Lemma) Use Negri's theorem to obtain a "nice" proof of $\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$ in T.

- (Main Lemma 1). Read off a proof of an appropriate E-sequent

$$\rho(\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta))$$

  in E from the nice proof in T.

- (Main Lemma 2). Show that, from the preceding E-proof, we can in fact recover a proof of $\Gamma \rightsquigarrow (\exists \vec{x})\Delta$ itself.

First we define the translation ρ taking us from regular sentences of $\mathcal{L}(\mathsf{T})$ to E-sequents. For the base cases, where we just have a single atomic formula, we map as follows:

$$
\begin{array}{rcl}
B(pqr) & \mapsto & (\exists L, a, b).[a \neq b, a \neq p, a \neq q, a \neq r, b \neq p, b \neq q, b \neq r, \\
       &   & \mathrm{on}(a, L), \mathrm{on}(b, L), \mathrm{on}(p, L), \mathrm{on}(q, L), \mathrm{on}(r, L), \mathrm{bet}(a, q, b), \\
       &   & \neg\mathrm{bet}(a, q, p), \neg\mathrm{bet}(p, a, q), \neg\mathrm{bet}(q, b, r), \neg\mathrm{bet}(r, q, b)] \\
\overline{B}(pqr) & \mapsto & \neg\mathrm{bet}(p, q, r), p \neq q, q \neq r \\
p = q & \mapsto & p = q \\
p \neq q & \mapsto & \neg(p = q) \\
xy \equiv vu & \mapsto & \overline{xy} = \overline{vu} \\
xy \not\equiv vu & \mapsto & \overline{xy} \neq \overline{vu}
\end{array}
$$

Why the first two are appropriate should be clear upon reflection (remembering that $\mathrm{bet}(p, q, r)$ is meant to be strict, while $B(pqr)$ is not), and the others are obvious. For the inductive step, suppose that the regular formula

$$
\forall \mathbf{x} \left[\bigwedge_i P_i(\mathbf{x}) \;\rightarrow\; \exists \mathbf{y} \left(\bigwedge_j Q_j(\mathbf{x}, \mathbf{y}) \right) \right]
$$

maps to $\Gamma \rightsquigarrow \Delta$ under ρ. Then we just make the obvious definitions, mapping

$$
\forall \mathbf{x}, \mathbf{z} \left[M(\mathbf{x}, \mathbf{z}) \wedge \bigwedge_i P_i(\mathbf{x}) \;\rightarrow\; \exists \mathbf{y} \left(\bigwedge_j Q_j(\mathbf{x}, \mathbf{y}) \right) \right]
$$

to $\Gamma, \rho(M(\mathbf{x}, \mathbf{z})) \rightsquigarrow \Delta$, and mapping

$$
\forall \mathbf{x} \left[\bigwedge_i P_i(\mathbf{x}) \;\rightarrow\; \exists \mathbf{y}, \mathbf{z} \left(M(\mathbf{x}, \mathbf{y}, \mathbf{z}) \wedge \bigwedge_j Q_j(\mathbf{x}, \mathbf{y}) \right) \right]
$$

to $\Gamma \rightsquigarrow \rho(M(\mathbf{x}, \mathbf{y}, \mathbf{z})), \Delta$. (A remark: when adding $B(pqr)$ clauses inductively, we use a new line variable for the ρ image each time.)

Our next main step is to analyze proofs of regular theorems φ in T, and to construct corresponding proofs of $\rho(\varphi)$ in E. But first of all, we need to make a preparatory reduction. As will be indicated below (and as might already be clear to the reader), our system E essentially has much of the same logical machinery as does T; for instance, the case-splitting rules built into E give us the strength of negativity axioms. But there are items that can appear in regular formulas, for which E has no corresponding apparatus—specifically $\rightarrow$ and $\forall$.

In order to show that we can read off proofs in E from proofs of regular theorems in T, then, we will show that sequent calculus rules governing

$\rightarrow$ and $\forall$ need not play any meaningful role in the T proofs. To wit, the following lemma shows that we can (rather literally) think of such rule uses as afterthoughts tacked onto the end of a proof of a regular theorem. (As a side note, we remark that the lemma is proved in the general setting of an arbitrary geometric theory G and its regular theorems.)

Lemma 4.9 (Reduction Lemma). *Suppose a geometric theory G proves the regular formula*

$$\varphi := \forall \mathbf{x} \left[\bigwedge_i P_i(\mathbf{x}) \; \rightarrow \; \exists \mathbf{y} \left(\bigwedge_j Q_j(\mathbf{x}, \mathbf{y}) \right) \right].$$
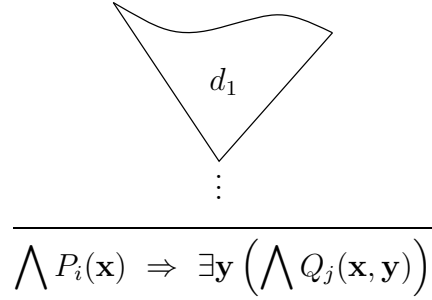
Then there is a proof which ends thus:

$$\cfrac{\cfrac{\bigwedge P_i(\mathbf{x}) \; \Rightarrow \; \exists \mathbf{y} \left(\bigwedge Q_j(\mathbf{x}, \mathbf{y}) \right)}{\Rightarrow \; \bigwedge P_i(\mathbf{x}) \; \rightarrow \; \exists \mathbf{y} \left(\bigwedge Q_j(\mathbf{x}, \mathbf{y}) \right)} \text{R} \rightarrow}{\vdots} \text{R}\forall$$

$$\cfrac{\vdots}{\Rightarrow \; \forall \mathbf{x} \left(\bigwedge P_i(\mathbf{x}) \; \rightarrow \; \exists \mathbf{y} \left(\bigwedge Q_j(\mathbf{x}, \mathbf{y}) \right) \right)} \text{R}\forall$$

And the only rule uses in d_1 are R$\wedge$, L$\wedge$, L$\bot$, R$\exists$, *as well as GRS instances.*

Proof. Given that φ is provable, it is clear that

$$\bigwedge P_i(\mathbf{x}) \; \Rightarrow \; \exists \mathbf{y} \left(\bigwedge Q_j(\mathbf{x}, \mathbf{y}) \right)$$

is provable as well. By Negri's theorem, this sequent has a cut-free proof d_1, which is as stated in the conclusion of the lemma. $\qquad\square$

The foregoing lemma shows, in particular, that any regular theorem φ of T has a very nice cut-free proof in which the R $\rightarrow$ and R$\forall$ rules do not interweave with the "axioms" (i.e. geometric rule schemes) in any way in proving the regular theorem. In essence, the contentful part of the proof is just the

$$\bigwedge P_i(\mathbf{x}) \;\Rightarrow\; \exists \mathbf{y}\left(\bigwedge Q_j(\mathbf{x}, \mathbf{y})\right)$$

Our ultimate claim, embodied in the lemmas below, is that derivations of the form d_1 can be closely mirrored by proofs in E. From here on out, we focus not on the regular sentences themselves which are theorems, but rather the sequents of the form

$$\bigwedge_i P_i(\mathbf{x}) \;\Rightarrow\; \exists \mathbf{y}\left(\bigwedge_j Q_j(\mathbf{x}, \mathbf{y})\right)$$

that are the conclusions of derivations like d_1. Note that we can just as well think of ρ as acting on such sequents, rather than the corresponding regular sentence; this is how we think of ρ below.

With the foregoing machinery in place, the following two lemmas are largely straightforward dealings with the formal apparatus of E. The first of them shows that the ρ-image of any regular theorem from T has a proof in E.

Lemma 4.10 (Main Lemma 1). *If* $\mathsf{T} \vdash \varphi$*, where* φ *is regular, then* $\mathsf{E} \vdash \rho(\varphi)$*.*

Proof Outline.

- We consider a cut-free proof of φ of the form indicated in the previous lemma, and focus on the proof d_1 of the sequent

$$\bigwedge_i P_i(\mathbf{x}) \;\Rightarrow\; \exists \mathbf{y} \bigwedge_j Q_j(\mathbf{x}, \mathbf{y}).$$

  Inducting on the structure of the proof d_1, our task reduces to showing that every step carried out in d_1 can be mirrored by a proof in E.

- Essentially, logical axioms and the logical rules which can appear in d_1 are already incorporated into the machinery of the formal system E; ditto for the Negativity axioms.

- With one exception, the GRS's we consider are of the form

$$\frac{A_1, \ldots, A_n, \Gamma \ \Rightarrow \ \Delta}{B_1, \ldots, B_m, \Gamma \ \Rightarrow \ \Delta} \text{ GRS}$$

(That is, these GRS's correspond to the Tarskian axioms which are *regular*.) In these cases, it suffices by the induction hypothesis to show that

$$\rho(B_1), \ldots, \rho(B_m) \rightsquigarrow \rho(A_1), \ldots, \rho(A_n).$$

The details for certain of these cases are given below.

- The remaining case is the sole GRS which is not regular (the Upper 2D axiom). Our task in this case is not really all that different from the previous cases; there is just one wrinkle.

□

Proof Details: Logical.

- $(L\wedge, R\wedge)$: We note that we do not have the symbol $\wedge$ in the language of E; instances of it get unpacked via the translation ρ. The $L\wedge$ rule becomes essentially vacuous, and the $R\wedge$ rule is mirrored by the derived conjunction rule of Chapter 2.

- $(R\exists, L\bot)$: These rules are subsumed under the notion of diagrammatic consequence detailed in Chapter 2.

□

Proof Details: Regular GRS.

- $(E1, E2, E3)$. Given the trivial nature of ρ for $\equiv$, it is easy to see that these cases are handled by our metric rules.

- $(2L)$. Let a be a point. Construct point $b \neq a$. Construct line L through a, b. Construct point c that is not on L. Each of $\text{bet}(a, b, c)$ or $\text{bet}(b, a, c)$ or $\text{bet}(a, c, b)$ leads to $\text{on}(c, L)$, hence a contradiction. Thus we can conclude $\neg\text{bet}$ for each, which is what we need.

- (SC). One can check that E-Propositions 3.3 and 3.5 provide the constructions that we need here to supplement the inductive hypothesis.

- (Int). By inductive hypothesis, we need to prove

$$\overline{ax} = \overline{ax'}, \overline{az} = \overline{az'}, \mathrm{on}(a, L), \mathrm{on}(x, L), \mathrm{on}(z, L), \neg\mathrm{bet}(a, z, x), \neg\mathrm{bet}(x, a, z),$$
$$\mathrm{on}(x, M), \mathrm{on}(y, M), \mathrm{on}(z, M), \neg\mathrm{bet}(y, x, z), \neg\mathrm{bet}(x, z, y)$$
$$\rightsquigarrow$$
$$(\exists y', N).\overline{ay} = \overline{ay'}, \mathrm{on}(x', N), \mathrm{on}(y', N), \mathrm{on}(z', N), \neg\mathrm{bet}(x', z', y'), \neg\mathrm{bet}(y', x', z')$$

Given the antecedent, we first note that uniqueness of lines yields $L = M$. We now reason by cases; as with other times, we present only the case in which all points concerned are distinct, as the other cases are only easier.

Draw line N through x' and z'. We thus seek a y' between x', z' such that $\overline{ay} = \overline{ay'}$. So we draw the circle γ through y centered at a. Since $\mathrm{bet}(a, x, y)$, we know (2.48) that x is inside γ; as $\overline{ax} = \overline{ax'}$, x' is also inside. Similarly, z' is outside; thus (2.16) we can construct y' on γ (hence $\overline{ay} = \overline{ay'}$ as well) that is also between x' and z'.

- We omit the remaining cases for the axioms (5S), (P), (PP) and (B), leaving the details to the interested reader. Like the previous cases, all that is involved are fairly straightforward uses of the rules of E.

□

Proof Details: Upper 2-Dimensional. Suppose we have $a \neq b$, and $\overline{x_i a} = \overline{x_i b}$ $(i = 1, 2, 3)$. We argue by cases concerning equalities among the x_i's. But we only present the case in which all are distinct; the other cases are only easier.

For each i, construct cricle γ_i with center x_i, passing through b. Construct line L through a, b. By the reasoning used earlier in E-Proposition 3.4, each x_i is on the line perpendicular to L, call it M.

Now we reason by cases, considering each parity for each $\mathrm{bet}(x_i, x_j, x_k)$; there are eight cases (omitting symmetry in the bet arguments). In the four for which two positive bet relations hold, we get contradictions. In the other four cases, we have two negative instances holding. But that is precisely what we need, given the inductive hypothesis. □

Given the previous lemma, we are halfway home. For suppose we have some E-sequent $(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$. This has its corresponding regular $\mathcal{L}(\mathsf{T})$-formula $\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$, and we now know that when we map this back to E via ρ, the image has a proof. This is almost enough; the trouble of course is that $\rho(\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta))$ is not quite the same thing as $(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$;

typically the former features extra structure in both the antecedent and consequent. The following lemmas demonstrate that, from the E proof of the more "populated" proposition, we can in fact recover a proof of the original $(\Gamma \leadsto (\exists \vec{x})\Delta)$.

Lemma 4.11. *Let M be any literal of* E*. Then*

1. $\mathsf{E} \vdash (M \leadsto \rho(\pi(M)))$,

2. $\mathsf{E} \vdash (\rho(\pi(M)) \leadsto M)$.

Proof. In an effort to avoid some needless tedium, we will not give the details for every such M. And as we will note, some of the work has already been done in the previous chapter.

- $\neg \text{on}(p, L)$: We need to prove

$$\neg \text{on}(p,L) \leadsto \mathfrak{c}_1^N \neq \mathfrak{c}_2^N, \mathfrak{c}_1^N \neq p, \mathfrak{c}_2^N \neq p, \neg \text{bet}(\mathfrak{c}_1^N, \mathfrak{c}_2^N, p), \neg \text{bet}(\mathfrak{c}_1^N, p, \mathfrak{c}_2^N), \neg \text{bet}(\mathfrak{c}_2^N, \mathfrak{c}_1^N, p)$$

  and the converse. The first direction is immediate. Supposing any of those on the RHS were to hold positively, our betweenness rules yield $\text{on}(p, N)$, contradiction.

  Conversely, suppose we have the RHS above. Suppose for contradiction that $\text{on}(p, N)$. Since p, $\mathfrak{c}_1^N$ and $\mathfrak{c}_2^N$ are all distinct, one of the betweenness conditions on the RHS will hold positively; contradiction.

- $\text{bet}(p, q, r)$: We note that $\rho\pi(\text{bet}(p, q, r))$ comes to

$$(\exists L).[\text{on}(p, L), \text{on}(q, L), \text{on}(r, L), \neg \text{bet}(p, r, q), \neg \text{bet}(q, p, r), p \neq q, q \neq r, p \neq r].$$

  Suppose $\text{bet}(p, q, r)$. Then a couple applications of (2.28) yield $p \neq q$, $q \neq r$, $p \neq r$ $\neg \text{bet}(q, p, r)$ and $\neg \text{bet}(p, r, q)$. Then construct line L through p, q; by (2.29), r is on L too.

  Conversely, suppose the above $\rho(\pi(\text{bet}(p, q, r)))$ holds. We are done by (2.33).

- $\neg \text{bet}(p, q, r)$: Here $\rho(\pi(\neg \text{bet}(p, q, r)))$ comes to

$$(\exists a, b, f, g, h, x, y, z, X). \quad \begin{aligned} &\text{on}(a, Y), \text{on}(q, Y), \neg \text{on}(b, Y), \\ &\text{bet}(a, p, x), \text{bet}(a, q, y), \text{bet}(a, r, z), \\ &\text{bet}(b, p, f), \text{bet}(b, q, g), \text{bet}(b, r, h), \\ &\neg \text{bet}(x, y, z), x \neq y, y \neq z, \\ &\neg \text{bet}(f, g, h), f \neq g, g \neq h] \end{aligned}$$

after a bit of simplification.

Suppose $\rho(\pi(\neg\text{bet}(p, q, r)))$ and assume for contradiction that we have
$\text{bet}(p, q, r)$. Then we can draw line M through p, q, r. From the fact
that a, b, q are not collinear, we know that one of a, b (WLOG, say
a), is not on M. It now follows straightforwardly from our Pasch and
triple-incidence rules that $\text{bet}(x, y, z)$ (draw a picture for guidance);
contradiction.

Conversely, suppose $\neg\text{bet}(p, q, r)$. There are several cases based on
whether p, q, r are distinct from one another or not, and whether p, q, r
are collinear or not. We present the most involved construction case,
where all three points are distinct, and they are not collinear.

Draw line L through p, q. Take a point on u on L between p and q
(2.3). Draw line Z through u, r, and take point a on Z extending ur
(2.4). Now just draw lines X through a, p and Y through a, q, and take
points x extending ap, y extending aq and z extending au. As above,
the Pasch and triple-incidence rules yield $\text{bet}(x, z, y)$ and the necessary
properties follow.

We then do the analogous construction involving b, this time starting
with a point v between q and r. We are done.

- $\text{on}(p, \gamma)$ or $\neg\text{on}(p, \gamma)$: Each is immediate using (2.66, 2.67).

- $\overline{xy} = \overline{zw}$ or $\overline{xy} \neq \overline{zw}$: Similarly straightforward.

- $\overline{xy} < \overline{zw}$: In this case $\rho(\pi(\overline{xy} < \overline{zw}))$ is

$$(\exists a, L).\text{on}(z, L), \text{on}(a, L), \text{on}(w, L), a \neq w, z \neq w, \neg\text{bet}(a, z, w), \neg\text{bet}(z, w, a), \overline{xy} = \overline{za}.$$

  Suppose $\overline{xy} < \overline{zw}$. From this, $\overline{zw} > 0$, hence $z \neq w$. So construct the
line L through z and w. In case $x = y$, then z itself will be our a. Let's
work in case $x \neq y$ then. We apply E-Proposition 3.2 to get b such
that $\overline{xy} = \overline{zb}$. Draw circle β through b centered at z. As z is inside β
and on L, we know that β and L intersect (2.51). Since $\overline{zb} = \overline{xy} < \overline{zw}$,
we know that w lies outside β (2.67, 2.69). Thus we may take the
intersection point a of β and w such that $\text{bet}(z, a, w)$ (2.16). It is clear
that this is the a we need.

Conversely, suppose that we have the above $\rho(\pi(\overline{xy} < \overline{zw}))$. In case
$z \neq a$, it follows that $\text{bet}(z, a, w)$ (2.33). Then $\overline{za} + \overline{aw} = \overline{zw}$ (2.64).

As $a \neq w$, $\overline{aw} > 0$ (2.62, 2.55). By the metric rules, then, $\overline{zw} > \overline{xy}$. In case $z = a$, then $\overline{xy} = \overline{za} = 0$ (2.62) and $\overline{zw} = \overline{aw}$. Again, as $a \neq w$, $\overline{aw} > 0$. So $\overline{zw} > \overline{xy}$ as desired.

- $\overline{xy} \not< \overline{zw}$: Similar to the previous.

- We leave the verification for literals involving angles to the interested reader.

$\square$

Lemma 4.12 (Main Lemma 2). *Consider an* E*-proposition* $(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$, *and suppose that we have* $\mathsf{E} \vdash \rho(\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta))$. *Then* $\mathsf{E} \vdash (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$.

Proof. Let $\Gamma = A_1, \ldots, A_m$ and $\Delta = B_1, \ldots, B_n$. Furthermore, let us write $A_i' := \rho(\pi(A_i))$ and $B_i' := \rho(\pi(B_i))$. It is easy to verify that $\rho \circ \pi$ acts "componentwise" in the sense that

$$\rho(\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)) = A_1', \ldots, A_m' \rightsquigarrow B_1', \ldots, B_n'.$$

So our supposition amounts to $\mathsf{E} \vdash A_1', \ldots, A_m' \rightsquigarrow B_1', \ldots, B_n'$.

But the previous lemma tells us that $\mathsf{E} \vdash (\Gamma \rightsquigarrow A_1', \ldots, A_m')$, and also that $\mathsf{E} \vdash (B_1', \ldots, B_n' \rightsquigarrow \Delta)$. Simply stringing these facts together yields the desired $\mathsf{E} \vdash (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$. $\square$

With the foregoing lemmas in hand, the completeness we seek is just a stone's throw away. Putting everything from this chapter together, we get:

Theorem 4.13 (Completeness). *If* $\mathscr{K} \models (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$ *for every Euclidean field* $\mathbb{K}$, *then* $\mathsf{E} \vdash (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$.

Proof. Suppose that every $\mathscr{K} \models (\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$. By Lemma 4.8, every $\mathscr{K}_\mathsf{T} \models \pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$. By Corollary 4.7 (the reformulation of Tarski's Fact),

$$\mathsf{T} \vdash \pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta).$$

As we have seen, $\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta)$ is regular, and so Lemma 4.10 tells us that $\mathsf{E} \vdash \rho(\pi(\Gamma \rightsquigarrow (\exists \vec{x})\Delta))$. Finally, then, Lemma 4.12 yields

$$\mathsf{E} \vdash (\Gamma \rightsquigarrow (\exists \vec{x})\Delta).$$

$\square$

4.3   Conclusions

4.3.1   Soundness

We have focused above on the completeness of E; let us say a few words about its soundness. One could give a direct argument involving the semantics, but the framework erected for our completeness proof includes most of the work needed for a proof-theoretic demonstration of soundness.

Specifically, the same argument of Main Lemma 2 reverses to show the converse: that $E \vdash (\Gamma \rightsquigarrow (\exists\vec{x})\Delta)$ implies $E \vdash \rho(\pi(\Gamma \rightsquigarrow (\exists\vec{x})\Delta))$. Linking that with the biconditional facts appealed to in Theorem 4.13, the only missing ingredient for a soundness proof is a converse to Main Lemma 1: $E \vdash \rho(\varphi)$ implies $T \vdash \varphi$. The proof would look much like our Main Lemma 1, except mirroring E-proofs with T-proofs; this task is easier than the other direction given the logical strength of T.

4.3.2   The Upshot

In a rough sense, our result shows that E is much like a cut-free regular fragment of T. One might then object that there is not much new about E. We remind the reader that in devising E, we were not aiming for something brand new. Rather, we sought a straightforward formalization of some *very old* mathematics, that of Euclid's *Elements*. Our claim was that it enjoys more solid logical underpinnings than is often suggested; the fact that our E closely resembles a natural fragment of (a slight reworking of) Tarski's 20th-century axiomatization is merely a reflection of the fact that Euclid's informal methods of proof, diagram-based though they are, really are not all that far removed from those of modern logic. The crux of the difference between a formalization like Tarski's and our more faithfully Euclidean one is not that the former enjoys a more precise and rigorous foundation than the latter; rather, the difference lies primarily in the type of inferences one which one considers to be immediate.

Appendix A

Sequent Calculus

For concreteness we have fixed a standard two-sided sequent calculus in which our formalization of T resides. Its logical axioms are sequents of the form $P, \Gamma \Rightarrow \Delta, P$ with P atomic, and its inference rules are as follows:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \; \mathrm{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \; \mathrm{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \; \mathrm{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \; \mathrm{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \; \mathrm{L}\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \; \mathrm{R}\rightarrow$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta} \; \mathrm{L}\bot$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \; \mathrm{L}\forall \qquad \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A} \; \mathrm{R}\forall$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \; \mathrm{L}\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A} \; \mathrm{R}\exists$$

Here y cannot appear free in $\Gamma, \Delta, \forall x A$ in rule $\mathrm{R}\forall$, nor can it appear free in $\exists x A, \Gamma, \Delta$ in rule $\mathrm{L}\exists$.

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