Review of Solomon Feferman's *In the Light of Logic*

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In December of 1998, a conference was held at Stanford in honor of Solomon Feferman's seventieth birthday. For the event, mathematicians, philosophers, computer scientists, historians, and even a linguist or two gathered from about a dozen countries to celebrate Feferman's work in logic. Surveying this work, one detects a reassuring uniformity of method, and a general template for research in the field: first, find an issue relevant to the philosophy or history of mathematics that calls for clarification; next, design formal representations of the relevant mathematical phenomena, and apply the methods of logic and proof theory to their analysis; and finally, interpret the results of the inquiry with respect to the original issues. Feferman's contributions to mathematical logic are substantial, but his work is perhaps best appreciated in this broader context. And it is precisely this interplay between mathematical logic and the philosophy of mathematics that constitutes the central theme of *In the Light of Logic,* a collection of reflective essays spanning the last two decades.

These writings do not offer a grand unified theory of mathematics, but, rather, focus on more sharply delimited issues. For example, a number of essays, including "Infinity in Mathematics," "Weyl Vindicated," and "Why a Little Bit Goes a Long Way," address the notion of *predicativity*. In an informal sense, the word is used to describe approaches to mathematics in which concepts are introduced "from the bottom up," in accordance with the Poincaré-Russell Vicious Circle Principle. From such a standpoint, one may be willing to accept quantification over the natural numbers as unproblematic, but, taking *sets* of natural numbers to be given by explicitly defined properties, one may reject definitions which treat the collection of all such sets as a completed totality. This informal stance forms the basis of Hermann Weyl's landmark 1918 work, *Das Kontinuum,* in which a significant portion of real analysis is developed on such a foundation. Since then, a good deal of effort has gone towards providing a formal analysis of predicativity, but no one has done more than Feferman to shape our current understanding of the term. In the 1960's, he identified a constellation of formal theories that seem to exhaust the predicativist's means, and using methods of Gerhard Gentzen, Kurt Schütte, and the "Munich School" of proof theory, he determined the ordinal, Γ_0, that represents the least upper bound to their proof-theoretic strength. In more recent years, a number of theories have been shown to be proof-theoretically reducible to Feferman's, and most interested parties are willing to use this criterion to obtain a working definition of predicativity. In other words, the general feeling is that Feferman has gotten it right, both in providing a formal analysis of a certain foundational stance, and in identifying an interesting and significant segment of mathematical reasoning.

Ongoing work in proof theory has shown that a good deal of mathematics can be carried out in predicative theories. In fact, many mathematical arguments can be formalized in theories that are weaker; and sometimes one can even get away with theories conservative over primitive recursive arithmetic, which many take to represent David Hilbert's notion of *finitistic* reasoning. One route to obtaining results like these, dating back to Hilbert and Paul Bernays and vigorously pursued by Harvey Friedman, Stephen Simpson, and others, is to use restricted subsystems of second-order arithmetic to carry out the formalization. A drawback of this approach is that in these theories mathematical objects (functions, fields, metric spaces, measures, etc.) have to be coded as sets of numbers, which explicitly restricts the discourse to mathematical objects which are countably presented. Feferman has developed a more flexible approach with his theories of explicit mathematics, where the countable interpretations lie, in a sense, behind the scenes. This difference aside, however, the two approaches paint pictures that are essentially in agreement as to "where" everyday mathematical reasoning takes place.

Some of the conclusions that Feferman wishes to draw from this analysis are controversial. While Friedman and others have discovered examples of combinatorial theorems (like Kruskal's theorem, and finitizations thereof) whose proofs are necessarily impredicative, Feferman has conjectured, on the basis of the evidence just described, that at least those branches of mathematics that have direct scientific applications can be developed on a predicative basis. He takes this to weigh against arguments offered by W. V. Quine and Hilary Putnam in favor of a realist, set-theoretic ontology, which maintain that such a mathematical ontology is indispensable to the sciences. Feferman laments the fact that arguments like these have, to a large extent, supplanted alternative philosophies for the foundations of mathematics, like constructivism or predicativism. In an essay provocatively titled "Infinity in Mathematics: Is Cantor Necessary?" he quotes Weyl in characterizing the platonism which underlies Cantorian set theory as the "medieval metaphysics of mathematics," and pronounces it "utterly unsatisfactory as a philosophy of our subject."

Feferman does not, however, offer much detail as to the form of platonism that he takes to underlie set theory, nor does he discuss a number of philosophical views, realist or otherwise, that can reasonably be associated with the subject. His efforts are directed, more specifically, towards a critique of the indispensability arguments mentioned above. Suggesting that the Cantorian world-view is dispensable is a sure-fire way of raising set theorists' hackles, and so it is important to keep clear what is at stake. Feferman is not primarily concerned with offering a normative account of mathematical practice, but, rather, determining the appropriate logical and philosophical analysis thereof. A set theorist could respond to the proof-theoretic account in a number of ways. First, she could argue that this account has something of a revisionist tinge: mathematicians use "large" sets and impredicative definitions freely, and though (after the fact, and in principle) we can often eliminate these features from a given theory, the impredicative conception may have been essential to the theory's development, and may still be essential to a full understanding of its terms. As far as the outer reaches of set theory are concerned, she could defend the subject as a search for principles that might, in the future, play a central role in everyday mathematics. After all, large cardinal axioms have consequences for the continuum, and Friedman has long been searching for interesting finitary consequences as well. If our scientific theories do not currently warrant granting large cardinals full ontological status, one day they might; and a liberal and open-minded view of mathematics should prevent us from adopting a philosophical stance that precludes this possibility. Along these lines, our set theorist might even question the extent to which a minimal mathematical ontology is desirable. If, as Georg Cantor and Hilbert maintained, the essence of mathematics lies in its freedom of abstraction, then the proof-theorist's drive to tether the subject to a restricted ontology misses the point; the resulting account of mathematics may soothe the apprehensive philosopher, but it may have little to do with mathematics itself.

Feferman would, no doubt, have much to say in reply. But I think it is important to emphasize that there is more to the proof-theoretic program than the negative task of reining in profligate ontologies. A pair of essays titled "Foundational Ways" and "Working Foundations" offer a thoughtful and spirited discussion of research in the foundations of mathematics, surveying a range of methodological approaches, emphasizing the positive gains of such research, and illustrating these gains with well-chosen examples. The benefits discussed include the clarification of fundamental concepts, the resolution of conceptual problems and paradoxes, the systematic organization of mathematical ideas, and the avenues towards generalization made possible by such a presentation. From this point of view, the proof-theoretic formalizations and reductions discussed above are to be valued for the additional information they provide, with regard to the structure of the mathematical theories under consideration. For example, the development of real analysis in a restricted framework helps us determine which features of the mathematical universe are essential to the subject, and how they are used; it shows us that the theorems of analysis hold true not only in the set-theorist's well-stocked universe, but in restricted models as well; and it helps us locate the "concrete" content of the theory's abstract concepts. As to the broader philosophical implications, Feferman writes:

> It seems to me that the information provided by the kind of case studies described here involving the formalization of considerable tracts of everyday mathematics in appropriate systems, in combination with the results of their proof-theoretic reductions... must be taken into account in the continuing effort to develop a relevant and sustainable philosophy of mathematics. ("What rests on what?" p. 207).

But with characteristic caution, he adds:

> Though I personally believe that the kinds of results described here on the whole strengthen the case for a nonplatonistic philosophy of mathematics and further undermine the case for set-theoretical realism, they do not speak for themselves to that extent, and it is at that point that a well-informed philosophical discussion must take over. (Ibid.)

In other words, the results of the proof-theoretic inquiry seem to tell us *something* about mathematics, and should be taken into account; but the philosophical mileage one obtains from these results may vary.

In the Light of Logic is divided into five parts. The first part focuses on historical topics, and surveys some of the major lines of thought that have shaped the development of mathematical logic in this century. The second part provides a general discussion of the foundational enterprise, and includes a balanced critique of Imre Lakatos' views on mathematics; with regard to the latter, Feferman finds "much to agree with in both the general approach and detailed analysis," but ultimately concludes that the theory taken in isolation fails to account for the distinctive character of mathematics. The third part examines the work of Kurt Gödel, and weaves together aspects of his life, mathematics, and philosophy, in a moving way. The fourth part outlines the proof-theoretic view of mathematics, embodied in the kinds of results discussed above; and the fifth part explores the consequences of these results for the philosophy of mathematics, largely vis-à-vis indispensability arguments. The book is essential reading for anyone interested in these topics, with prose that is crisp and precise yet accessible to a general audience.

For the most part, the research discussed in these essays is foundational, in the traditional sense: the logician's task is seen to be that of devising formal accounts that characterize the basic stuff of which the mathematical universe is composed, and the philosopher's task is seen to be that of providing epistemological and ontological justifications of mathematics, as portrayed by these accounts. Today there is a growing sense that these are not the only issues in the philosophy of mathematics that are worthy of sustained inquiry. One may seek, for example, accounts of mathematics which illuminate the structural view of the subject that has been on the rise since the Cantor-Dedekind revolution of the last century; theories of meaning and rationality that explain how mathematical theories grow and evolve over time; or theories of discovery and explanation in mathematics, akin to those that have been developed for the sciences. But the tone of Feferman's writing suggests that he would not be averse to investigations of this sort, and traditionalists among us like to think that the general methodological approach exhibited in these essays has much to offer in that regard. In the end, the extent to which mathematical approaches can inform philosophical inquiry is subject to debate; but the rigor and clarity of Feferman's essays make a strong case on their behalf.