Category Theory

Final

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Question 1:

Let $U : \mathbf{Mon} \to \mathbf{Sets}$ be the forgetful functor.
So $U(M, u, \cdot) = M$ and for $h : (M, u, \cdot) \to (N, e, *)$, $U(h) = h$.
Claim: U is representable.
Let $C = F\{a\}$ be the free monoid on one generator. We will show that $U \cong \mathrm{Hom}_{\mathbf{Mon}}(C, -)$
So we define $\phi : U \to \mathrm{Hom}_{\mathbf{Mon}}(C, -)$ and show ϕ is a natural iso.
Then the components $\phi_M : UM \to \mathrm{Hom}_{\mathbf{Mon}}(C, M)$.
For each $x \in Um$ we will assign an element of $\mathrm{Hom}_{\mathbf{Mon}}(C, M)$, a function from $C = F\{a\}$ to M.
By the UMP of free monoids it suffices to specify a mapping from the generators ($\{a\}$) to $U(M)$.
So let $\phi_M\ x\ a = x$.
Then let the inverse component $\psi_M : \mathrm{Hom}_{\mathbf{Mon}}(C, M) \to M$ by $\psi_M\ f = fa$.
Clearly $\psi_M(\phi_M\ x) = \phi_M\ x\ a = x$ and $\phi_M(\psi_M\ f)\ a = \phi_M\ (f\ a)\ a = f\ a$
We showed in class since each component is an iso, if ϕ is natural then ψ is as well.
Thus it now remains to show ϕ is natural. We check the following diagram commutes.

$$\begin{array}{ccc} U(M, \cdot, u) & \xrightarrow{\phi_M} & \mathrm{Hom}_{\mathbf{Mon}}(C, M) \\ {\scriptstyle Uh}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Mon}}(C, h)} \\ U(N, *, e) & \xrightarrow[\phi_N]{} & \mathrm{Hom}_{\mathbf{Mon}}(C, N) \end{array}$$

Let $x \in U\ M$.
Then $\mathrm{Hom}_{\mathbf{Mon}}(C, h)(\phi_M x) = h \circ (\phi_M\ x)$ and $\phi_N((U\ h)\ x) = \phi_N\ (h\ x)$.
To check if these two arrows are the same, since C is a free monoid on one generator, it suffices to check where both arrows map the generator a.
$h(\phi_M\ x\ a) = h\ x$ and $\phi_N\ (h\ x)\ a = h\ x$, so the two arrows are the same.
Thus the diagram commutes and ϕ is natural.
Therefore $U \cong \mathrm{Hom}_{\mathbf{Mon}}(C, -)$ and thus representable.

Let $U \times U \to \textbf{Sets}$ be given by $(U \times U)(M, \cdot, u) = UM \times UM$ and $(U \times U)h = \langle U\ h, U\ h \rangle$.
Claim: $U \times U$ is representable.
We will show $U \times U \cong \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, -)$.
Again we define $\phi : U \times U \to \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, -)$.
For each $M : \mathbf{Mon}$, $\phi_M : UM \times UM \to \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, M)$.
So $\phi_M\ \langle x, y \rangle : F\{a,b\} \to M$.
As before, it suffices to specify a map on the generators.
Thus let $\phi_M\ \langle x, y \rangle\ a = x$ and $\phi_M\ \langle x, y \rangle\ b = y$.
Then we let the inverse component $\psi_M\ f = \langle f\ a, f\ b \rangle$.
So $\psi_M(\phi_M\ \langle x, y \rangle) = \langle \phi_M\ \langle x, y \rangle\ a, \phi_M\ \langle x, y \rangle\ b \rangle = \langle x, y \rangle$
and $\phi_M(\psi_M\ f)\ a = \phi_M\ (\langle f\ a, f\ b \rangle)\ a = f\ a$, $\phi_M(\psi_M\ f)\ b = \phi_M(\langle f\ a, f\ b \rangle)\ b = f\ b$.
To check naturality we verify the following diagram commutes:

$$\begin{array}{ccc} UM \times UM & \overset{\phi_M}{\longrightarrow} & \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, M) \\ {\scriptstyle (U \times U)h} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, h)} \\ UN \times UN & \underset{\phi_N}{\longrightarrow} & \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, N) \end{array}$$

Let $\langle x, y \rangle \in UM \times UM$.
To see the diagram commutes we need to show the final results are the same arrow from $F\{a,b\}$ to N.
So it suffices to check the result of each side of the diagram applied to a and b.

$$\begin{aligned}
(\mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, h) \circ \phi_M)\ \langle x, y \rangle\ a &= \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, h)(\phi_M\ \langle x, y \rangle)\ a && [\text{Simplify } \circ] \\
&= (h \circ (\phi_M\ \langle x, y \rangle))\ a && [\text{Defn of } \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, -)] \\
&= h(\phi_M\ \langle x, y \rangle\ a) && [\text{Simplify } \circ] \\
&= h\ x && [\text{Defn of } \phi_M] \\
&= \phi_N\ \langle h\ x, h\ y \rangle\ a && [\text{Defn of } \phi_N] \\
&= \phi_N\ ((U \times U)\ h\ \langle x, y \rangle)\ a && [\text{Defn of } U \times U] \\
&= (\phi_N \circ (U \times U\ h))\ \langle x, y \rangle\ a && [\text{Expand } \circ]
\end{aligned}$$

$$\begin{aligned}
(\mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, h) \circ \phi_M)\ \langle x, y \rangle\ b &= \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, h)(\phi_M\ \langle x, y \rangle)\ b && [\text{Simplify } \circ] \\
&= (h \circ (\phi_M\ \langle x, y \rangle))\ b && [\text{Defn of } \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, -)] \\
&= h(\phi_M\ \langle x, y \rangle\ b) && [\text{Simplify } \circ] \\
&= h\ y && [\text{Defn of } \phi_M] \\
&= \phi_N\ \langle h\ x, h\ y \rangle\ b && [\text{Defn of } \phi_N] \\
&= \phi_N\ ((U \times U)\ h\ \langle x, y \rangle)\ b && [\text{Defn of } U \times U] \\
&= (\phi_N \circ (U \times U\ h))\ \langle x, y \rangle\ b && [\text{Expand } \circ]
\end{aligned}$$

So the naturality square commutes and ϕ is natural.
Thus $U \times U \cong \mathrm{Hom}_{\mathbf{Mon}}(F\{a,b\}, -)$ as needed to show $U \times U$ representable.

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Problem 2:

Claim: The multiplication map $m_M : UM \times UM \to UM$ are the components of a natural transformation $m : U \times U \to U$.
So for any $h : (M, m_M, u_M) \to (N, m_N, u_N)$ an arrow in **Mon**,
we need the following square to commute:

$$\begin{array}{ccc} UM \times UM & \xrightarrow{m_M} & UM \\ {\scriptstyle (U \times U)h} \downarrow & & \downarrow {\scriptstyle Uh} \\ UN \times UN & \xrightarrow[m_N]{} & UN \end{array}$$

Let $\langle x, y \rangle \in UM \times UM$. Then:

$$\begin{aligned} (Uh \circ m_M)\langle x, y \rangle =& Uh(m_M \langle x, y \rangle) \\ =& h(m_M \langle x, y \rangle) && \text{[Defn of U]} \\ =& m_N \langle h\ x, h\ y \rangle && [h \text{ is arrow in } \mathbf{Mon}] \\ =& m_N((U \times U)\ h\ \langle x, y \rangle) && [\text{Defn of } U \times U] \\ =& (m_N \circ (U \times U)h)\langle x, y \rangle \end{aligned}$$

So the diagram commutes as needed.

Claim: The unit of monoids $u_M : \{\star\} \to UM$ are the components of a natural transformation $u : 1 \to U$ where $1 : \mathbf{Mon} \to \mathbf{Sets}$ by $1(M, m_M, u_M) = \{\star\}$.
So we need to show for any arrow $h : (M, m_M, u_M) \to (N, m_N, u_N)$, the following commutes:

$$\begin{array}{ccc} 1M & \xrightarrow{u_M} & UM \\ {\scriptstyle 1h} \downarrow & & \downarrow {\scriptstyle Uh} \\ 1N & \xrightarrow[u_N]{} & UN \end{array}$$

$$\begin{aligned} (Uh \circ u_M) =& h \circ u_M && [\text{Defn of } U] \\ =& u_N && [h \text{ is an arrow in } \mathbf{Mon}] \\ =& u_N \circ 1h && [1h \text{ is the identity }] \end{aligned}$$

Thus the square commutes and u is natural as claimed.

Problem 3:

Claim: $\mathcal{U} = (U, u, m)$ is a monoid in $\mathbf{Sets^{Mon}}$.
Recall that $(\mathbf{Sets^{Mon}}, 1, \times)$ is a monoidal category where:
$(F \times G)M = FM \times GM$ and 1 is the evident constant functor from problem 3.
The associator is $\alpha_{F,G,H} : (FM \times GM) \times HM \to FM \times (GM \times HM)$
given by $(\alpha_{F,G,H})_M((x, y), z) = (x, (y, z))$
and the unitors, $\lambda_F : 1 \times F \to F$ given by $(\lambda_F)_M(\star, x) = x$
and $\rho_F : F \times 1 \to F$ given by $(\rho_F)_M(x, \star) = x$
So for $\mathcal{U}$ to be a monoid in $\mathbf{Sets^{Mon}}$ we need that:

$$m \circ (m \times 1_U) = m \circ (1_U \times m) \circ \alpha_{U,U,U}$$

$$m \circ (u \times 1_U) \circ \lambda_U^{-1} = 1_U = m \circ (1_U \times u) \circ \rho_U^{-1}$$

It suffices to check these identities at an arbitrary monoid M.

$$\begin{aligned}
(m \circ (m \times 1_U))_M((x, y), z) =& m_M \circ (m_M \circ 1_{UM})((x, y), z) && \text{[Compose by components]}\\
=& m_M(m_M(x, y), z) && \text{[Simplify } \circ\text{]}\\
=& m_M(x, m_M(y, z)) && \text{[Monoid multiplication associates.]}\\
=& m_M((1_{UM} \times m_M)(x, (y, z))) && \text{[Expand function application]}\\
=& m_M((1_{UM} \times m_M)((\alpha_{U,U,U})_M((x, y), z))) && \text{[Re–associate]}\\
=& (m_M \circ (1_{UM} \times m_M) \circ (\alpha_{U,U,U})_M)((x, y), z) && \text{[Expand } \circ\text{]}\\
=& (m \circ (1_U \times m) \circ \alpha_{U,U,U})_M((x, y), z) && \text{[Compose by components]}
\end{aligned}$$

$$\begin{aligned}
(m \circ (u \times 1_U) \circ \lambda_U^{-1})_M\ x =& (m_M \circ (u_M \times 1_{UM}) \circ (\lambda_U^{-1})_M)\ x\\
=& m_M((u_M \times 1_{UM})((\lambda_U^{-1})_M\ x))\\
=& m_M((u_M \times 1_{UM})\langle \star, x\rangle)\\
=& m_M\langle u_M, x\rangle\\
=& x && \text{[Defn of monoid unit]}\\
(1_U)_M\ x =& 1_{UM}\ x\\
=& x\\
(m \circ (1_U \circ u) \circ \rho_U^{-1})_M =& (m_M \circ (1_{UM} \times u_M) \circ (\rho_U^{-1})_M)\ x\\
=& m_M((1_{UM} \times u_M)((\rho_U^{-1})_M\ x))\\
=& m_M((1_{UM} \times u_M)\langle x, \star\rangle)\\
=& m_M\langle x, u_M\rangle\\
=& x && \text{[Defn of monoid unit]}
\end{aligned}$$

Thus $\mathcal{U}$ satisfies the properties of a monoid object in $\mathbf{Sets^{Mon}}$

Let $M : \mathbf{Mon}$. Let $M^* : \mathbf{Sets}^{\mathbf{Mon}}$ by $M^*(A) = A\ M$ and $M^*(\phi) = \phi_M$.
Claim: M^* preserves monoid objects.
Consider a monoid $\mathcal{A} = (A, \eta, \mu)$.
Then $M^*(\mathcal{A}) = M^*(A, \eta, \mu) = (M^*A, M^*\eta, M^*\mu) = (A\ M, \eta_M, \mu_M)$
This is a monoid in **Sets** since:

$\eta_M : 1M \to A\ M$ which means $\eta_M : \{\star\} \to A\ M$.

$\mu_M : (A \times A)M \to AM$, which means $\mu_M : A\ M \times A\ M \to A\ M$ as needed.

Furthermore since components of natural transformations distribute over composition and Cartesian product, the associative and unit laws hold for $(A\ M, \eta_M, \mu_M)$ as needed.
Thus M^* preserves monoid objects.

Claim: Every moniod in **Sets** is a functorial image of $\mathcal{U}$.
We will show that $M \cong M^*(\mathcal{U})$.
From above we have that $M^*(\mathcal{U}) = (U\ M, u_M, m_M)$
Clearly the underlying set of M is UM.
From the defintion of u, u_M is the unit of the monoid M.
From the defintion of m, m_M is the monoid multiplication from the monoid M.
Thus $M \cong M^*(\mathcal{U})$

Problem 4:

Claim: $\mathcal{U}$ is in the image of the yoneda embedding $y : \mathbf{Mon}^{\mathrm{op}} \to \mathbf{Sets}^{\mathbf{Mon}}$
We will show that $\mathcal{U} \cong y(F\{a\})$.
Note that for each $M : \mathbf{Mon}$, we can place a monoid structure on $yC\ M$ point–wise.
So for $f, g \in yC\ M$, let $f \cdot_M g = (c \mapsto m_M(f\ c, g\ c))$ and

$e_m : 1M \to yC\ M$ by $e_M(\star) = (c \mapsto u_m\star)$.

As with $\mathcal{U}$ these can be taken as components of natural transformations $\cdot$ and e.
So we must verify the following squares commute for $h : M \to N$:

$$\begin{array}{ccc} 1M & \xrightarrow{e_M} & yCM \\ {\scriptstyle 1h}\downarrow & & \downarrow{\scriptstyle yCh} \\ 1N & \xrightarrow[e_N]{} & yCN \end{array}$$

$$\begin{aligned} (yC)h \circ e_M =& h \circ e_M && \text{[Defn of } yC\text{]} \\ =& e_N && [h \text{ is an arrow in } \mathbf{Mon}] \\ =& e_N \circ 1h && [1h \text{ is an identity arrow}] \end{aligned}$$

So e is a natural transformation.

$$\begin{array}{ccc} yC \times yCM & \xrightarrow{\cdot_M} & yCM \\ {\scriptstyle yC \times yCh}\downarrow & & \downarrow{\scriptstyle yCh} \\ yC \times yCN & \xrightarrow[\cdot_N]{} & yCN \end{array}$$

$$\begin{aligned}
(yC)h\ (f \cdot_M g)\ c =& h \circ (f \cdot_M g)\ c && [\text{Defn of } yC] \\
=& h((f \cdot_M g)c) && [\text{Simplify } \circ] \\
=& h(m_M \langle f\ c, g\ c\rangle) && [\text{Defn of } \cdot_M] \\
=& m_N \langle h(f\ c), h(g\ c)\rangle && [h \text{ respects products}] \\
=& m_N \langle (h \circ f)\ c, (h \circ g)\ c\rangle && [\text{Expand } \circ] \\
=& m_N \langle ((yC)h\ f)\ c, ((yC)h\ g)\ c\rangle && [\text{Defn of } yC] \\
=& m_N ((yC \times yC)\ h\ \langle f\ c, g\ c\rangle) && [\text{Defn of } yC \times yC] \\
=& (((yC)h)\ f \cdot_N ((yC)h)\ g)\ c && [\text{Defn of } \cdot_N]
\end{aligned}$$

So $\cdot$ is a natural transformation as well.
Thus $(yC, e, \cdot)$ is a monoid in $\mathbf{Sets}^{\mathbf{Mon}}$.
We showed in problem 1, that $U \cong y(F\{a\})$ by $\phi_M\ x : F\{a\} \to M$ is defined by taking a to x.
Then $((\phi_M\ x) \cdot_M (\phi_M\ y))\ a = m_M\langle \phi_M\ x\ a, \phi_M\ y\ a\rangle = m_M\langle x, y\rangle$.
Thus $(\phi_M\ x) \cdot_M (\phi_M\ y) = (c \mapsto m_M\langle x, y\rangle)$ (it suffices to check the generators).
Also $m_M(\psi\ f, \psi\ g) = m_M(f\ a, g\ a) = (f \cdot_M g)\ a = \psi(f \cdot_M g)$.
Thus $\cdot_M \cong m_M$.
Similarly $\phi_M\ (u_M\star)\ a = (u_M\star) = e_M\ a$. Thus $\phi_M\ (u_M\star) = e_M$.
Also $\psi\ e_M = e_M(a) = (u_M\star)$ as needed.
Thus $e_M \cong u_M$.
So $\mathcal{U} \cong y(F\{a\})$ as claimed.

Furthermore $y(C + C) \cong yC \times yC$ since y is contravariant and preserves (co)limits.
($\phi([F, G]) = F \times G$ for $F, G : C \to \mathbf{Sets}$ is a clear iso)
Also $1 : \mathbf{Mon} \to \mathbf{Sets}$ is isomorphic to yI where I is the trivial monoid on one element $\star$.
(I is a terminal monoid so $(yI)\ M$ has only one element as does $1M$.)
So $\cdot : y(C + C) \to yC$ and $e : yI \to yC$.
Consider $C = F\{a\}$.
Since y is full and faithful this means that there are arrows in $\mathbf{Mon}^{\mathrm{op}}$, $m^* : C + C \to C$ and $u^* : I \to C$ such that $\cdot = y\ m$ and $e = y\ u$.
Since $+$ is a coproduct in $\mathbf{Mon}$ it is a product in $\mathbf{Mon}^{\mathrm{op}}$. So (C, u^*, m^*) is a monoid in $\mathbf{Mon}^{\mathrm{op}}$ and thus (C, u, m) is comonoid in $\mathbf{Mon}$ as we needed.
We let $u\ c = \star$ for all $c \in F\{a\}$.
Since $yI \cong 1$ we identify the unique f in $(yI)M$ with $\star$ in $1M$.
Then $(y\ u)_M : (yI)M \to (yC)M$ by $(y\ u)_M f = f \circ u$.
So for this unique $f : I \to M$: $((y\ u)_M\ f)\ c = (f \circ u)c = f(u\ c) = f(\star) = u_M = (e_M\star)c$.
Thus $(y\ u)_M = (e_M)$ and $y\ u = e$

We also know that the free monoid functor preserves coproducts.

So $F\{a\}+F\{a\} \cong F(\{a\}+\{a\}) \cong F\{\ell,r\}$.

So to define $m : F\{a\} \to F\{a\}+F\{a\}$ we will define $\hat{m} : F\{a\} \to F\{\ell,r\}$ and compose with the unique map from $\xi : F\{a,b\} \to F\{a\}+F\{a\}$.

($\xi(\ell) = i_1(a)$ and $\xi(r) = i_2(a)$ suffices to define ξ).

To define $\hat{m}$ it suffices to specify where $\hat{m}$ takes the generator a.

Let $\hat{m}(a) = \ell r$. So then $m = \xi \circ \hat{m}$.

Since $yC \times yC \cong y(C+C)$ we identify $[f,g]$ with $\langle f,g\rangle$.

We want $\cdot = y\ m$ so we need $\cdot_M = (y\ m)_M$.

Given $\langle f,g\rangle (= [f,g])$, $(f \cdot_M g)a = m_M\langle f\ a, g\ a\rangle$.

On the other hand :

$(y\ m)_M[f,g]\ a = ([f,g]\circ m)\ a = [f,g](m\ a) = [f,g](\xi(\ell r)) = [f,g]((i_1\ a)(i_2\ a)) = m_M(f\ a, g\ a)$

So $(y\ m)_M[f,g]$ and $\cdot_M\langle f,g\rangle$ agree on the generator and thus agreee everywhere.

Thus $y\ m = \cdot$ as we claimed.

Problem 5:

Claim: $(C,m,u) \models s = t$ in $\mathbf{Mon}^{\mathrm{op}}$ if and only if for all monoids $M : \mathbf{Mon}$, $M \models s = t$ in **Sets**.

Proof: Assume $(C,m,u) \models s = t$ in $\mathbf{Mon}^{\mathrm{op}}$.

This gives us a comuting diagram in $\mathbf{Mon}^{\mathrm{op}}$.

Applying yoneda to this diagram gives a commuting diagram in $\mathbf{Sets}^{\mathbf{Mon}}$.

Since $yC \cong \mathcal{U}$ this shows that $\mathcal{U} \models s = t$ in $\mathbf{Sets}^{\mathbf{Mon}}$.

Then for any monoid M in sets, M^* is a monoid preserves functor.

So applying M^* to the diagram in $\mathbf{Sets}^{\mathbf{Mon}}$ gives a commuting diagram in **Sets**.

Furthermore we showed $M \cong M^*(\mathcal{U})$. Thus this diagram shows us that $M \models s = t$ in **Sets**.

Since this held for an arbitrary monoid it holds for all monoids in **Sets**.

For the reverse, if an equation, $s = t$, holds for all monoids M in **Sets** by defintion it holds for all components of the natural transformation u and m in $\mathcal{U}$.

Thus the diagram commutes for $\mathcal{U}$ and $\mathcal{U} \models s = t$.

Furthermore $\mathcal{U} = yC$ so $yC \models s = t$.

In addition yoneda is full and faithful so a commuting diagram in the image of yoneda comes from a unique comuting diagram in the domain.

So the diagram in $\mathbf{Sets}^{\mathbf{Mon}}$ came from a commuting diagram in $\mathbf{Mon}^{\mathrm{op}}$ using (C,u,m) as the monoid.

Thus $(C,u,m) \models s = t$ as needed.

Recall $T \vdash s = t$ if and only if "for all monoids M in **Sets**, $M \models s = t$".

We just proved "for all monoids M in **Sets**, $M \models s = t$" if and only if "$(C,u,m) \models s = t$ in $\mathbf{Mon}^{\mathrm{op}}$".

So $T \vdash s = t$ if and only if $(C,u,m) \models s = t$