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APPROXIMATE SOLUTION OF THE INVERSE-STEPHAN PROBLEM IN CARTESIAN, CYLINDRICAL AND SPHERICAL GEOMETRIES

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INTRODUCTION

One of the most important factors for single cells destruction or survival is the cooling rate at freezing temperatures (McGrath, 1993). It was found experimentally that at very low, or alternatively, at very high cooling rates, maximal cell destruction is achieved. These cooling rates are of the order of a few, or hundreds of Centigrade per minute, respectively, at both ends of the spectrum (Orpwood, 1981). Analogous to the dependency of single cells destruction in the freezing rates, and in order to increase cryosurgical destruction, it was suggested that a special protocol be followed to cool the tissue in order to force a pre-specified cooling rate at the freezing front (Rubinsky and Shitzer, 1976). For the application of this approach, the time-dependent temperature function at the outer surface (termed also the "forcing function") needs to be solved, which is the solution of an inverse Stephan problem in biological tissue. Solutions of inverse Stephan problems are available for one-dimensional cases in Cartesian geometry only (Rubinsky and Shitzer, 1976; Rabin and Shitzer, 1994; Budman et al., 1995; Rabin and Shitzer, 1995).

An approximate solution of the inverse Stephan problem, subject to a pre-specified cooling rate at the freezing front in Cartesian, cylindrical and spherical coordinate systems, is presented. The approximate solution combines a numerical solution in the unfrozen region, including the subregion undergoing phase-transition and an analytical solution in the frozen region. The numerical solution in the unfrozen region is based on the enthalpy approach. A newly defined Fourier number is applied for the analysis of this problem, which resulted from the thermal diffusivity of the frozen region and the latent heat effect of freezing. The approximate solution assumes large Fourier numbers when the time-dependent term in the heat balance equation can be neglected.

ANALYSIS

The problem is assumed to be one-dimensional in semi-infinite domain, where a radial heat flow is assumed in the cylindrical and spherical cases. Two regions are defined for the present analysis: the unfrozen region, which includes the sub-region undergoing phase transition, u, and the frozen region, f. The solution is derived under the following conditions: the thermal conductivity is a step-like function across the freezing front, possessing two constants but different values; the specific heat is a temperature dependent effective property which includes the thermal effect of latent heat at the phase-transition temperature range; both metabolic heat generation and blood perfusion are temperature dependent and decay by freezing; blood temperature entering the cooled region is constant; and, the initial temperature is a known arbitrary function.

The governing equation in the unfrozen region is assumed to be the bio-heat equation:

$$C\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial \eta^2} + \frac{n}{\eta}\frac{\partial T}{\partial \eta}\right) + \hat{w}_b C_b (T_b - T) + q_{met} \qquad \eta > \eta_0 \quad (1)$$

where n equals 0, 1 and 2 for Cartesian, cylindrical and spherical coordinate systems, respectively. The only direction of heat flow is η, which stands for x in Cartesian coordinate system, and for r in cylindrical or spherical coordinate systems. The desired inverse Stephan condition at the freezing front during the freezing process is:

$$\frac{\partial T}{\partial t}\,(\eta = s,\ t) = H_u \qquad (2)$$

Applying a freezing front tracing technique:

$$\xi = \eta - s \qquad (3)$$

the bio-heat equation is transformed into:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial \xi^2} + \left(\frac{\alpha\, n}{\xi + s} + \dot{s} \right) \frac{\partial T}{\partial \xi} - \frac{\hat{w}_b C_b}{C} T + \frac{q_{met} + \hat{w}_b C_b T_b}{C} \qquad \xi > 0 \tag{4}$$

the boundary condition at ξ's origin is a known arbitrary forcing function prior to first ice formation, and a constant temperature thereafter (the lower boundary of phase-transition temperature range). The interface location in Eq. (4) equals 0 prior to first ice formation and is calculated by the transformed inverse Stephan condition thereafter:

$$- \dot{s} \frac{\partial T}{\partial \xi} (\xi = 0,\, t) = H_u \tag{5}$$

Equation (4) and its boundary and initial conditions present a well-defined mathematical problem in the unfrozen region, u, that can be solved independently of the solution in the frozen region, f. Numerical solution of Eq. (4) may be obtained by a modified Crank-Nicholson technique in order to include metabolic heat generation and blood perfusion terms (Rabin and Shitzer, 1994).

The mathematical analysis of the frozen region is addressed next. It is assumed that metabolic activities and blood perfusion decay by freezing, and therefore the bio-heat equation decays to an ordinary heat balance equation by freezing. For further analysis Fourier number is defined as:

$$Fo = \frac{\alpha\, \tau}{s^2} \tag{6}$$

where τ is the time required for a frozen region of a thickness s to be formed. The definition of Fourier number here, Fo, differs from the ordinary definition of dimensionless time parameter, as both the nominator and the denominator are time dependent. It is noted that Fo number tends to infinity at the initial moment, when s equals 0.

The dimensionless form of the heat balance equation in the frozen region is:

$$\nabla^{*2} T^* = \frac{1}{Fo} \frac{\partial T^*}{\partial t^*} \qquad 1 \geq \eta^* \geq \eta_0^* \tag{7}$$

For typical thermophysical properties of soft biological tissues, Fo is expected to be a very large number and therefore the transient response of the frozen region can be neglected. The solution of the heat balance equation in the frozen region for large Fo numbers, and under boundary conditions of continuity in temperature and in heat flux across the freezing font is:

$$T = \begin{cases} T_{mf} + \dfrac{H_f}{\dot{s}} (s - x) & n=0 \\[2ex] T_{mf} + \dfrac{H_f\, s}{\dot{s}} \ln\left(\dfrac{s}{r}\right) & n=1 \\[2ex] T_{mf} + \dfrac{H_f\, s}{\dot{s}} \left(\dfrac{s}{r} - 1\right) & n=2 \end{cases} \tag{8}$$

Finally, the sought after cryoprobe forcing function is obtained by replacing x and r with 0 and the cryoprobe radius, respectively.

DISCUSSION

A validation test of the approximate solution was performed by comparison with an exact solution of the inverse Stephan problem in Cartesian coordinates (Rabin and Shitzer, 1995). Typical thermophysical properties similar to those of water (thermal diffusivity of $1.4 \cdot 10^{-7}$ and $2 \cdot 10^{-6}$ in the unfrozen and frozen regions, respectively), freezing temperature range as of soft tissue (-1 ÷ -8°C), and extremely high blood perfusion (11.1 s^{-1}) and metabolic heat generation (20 kW/m^3), were assumed for the purpose of that test (Chato, 1985). The forcing function at the initiation stage of the approximate solution was extracted from the exact solution. Results for relatively low and relatively high freezing rates as viewed from the frozen region are presented by Fig. 1.

The agreement between the exact and the approximate solutions in the temperature distribution of the unfrozen region was found to be in the order of 10^{-3}°C for 0.025 mm space and 0.1 s time intervals. The agreement in the unfrozen region becomes better as time and space intervals decrease. A very good agreement was also found in the frozen region at the beginning of the freezing process, which becomes worse with the time as Fo number decreases. As the forcing function reaches -100°C, Fo number is reduced to 5 and 4.5 for the freezing rates of 10°C/min and 100°C/min, respectively. At that point the temperature differences between the approximate and the exact solution in the frozen region were found to be within 10% and 9%, respectively.

By neglecting the time-dependent term in the heat balance equation (7), the transient response of the heat released from the frozen region is neglected. It follows that the approximate solution assumes that less energy is needed to be absorbed from the tissue, and therefore the actual freezing rates at the interface will be lower then the pre-specified value H. This difference starts from zero and increases with time.

A demonstration of the approximate solution in cylindrical and spherical cases is addressed next. For demonstration purposes, typical thermophysical properties of soft tissues (thermal diffusivity of $1.4 \cdot 10^{-7}$ and $1.1 \cdot 10^{-6}$ in the unfrozen and frozen regions, respectively), (Rabin and Shitzer, 1995), and linear temperature-dependent blood perfusion and metabolic heat generation were assumed, all starting from the extreme values presented above. Figure 2 presents the forcing functions for various cryoprobe diameters in Cartesian, cylindrical and spherical geometries. As can be expected from energy considerations, the cooling rate at the cryoprobe surface in the spherical case is much higher than in the cylindrical case, and that in turn is much higher than in the Cartesian case. In general, Fo number decreases as the diameter of the cryoprobe increases, and is much smaller in cylindrical cases than in Cartesian cases, and even smaller in spherical cases. Therefore, the approximation of the temperature distribution in the frozen region, and as a result of the forcing function, becomes better in cylindrical cases, and much better in spherical cases. Furthermore, the approximation becomes better as the cryoprobe diameter decreases.

It is customary to assume that the transient response in the frozen region for small Stephan numbers, St, can be neglected, where Stephan number is defined as the ratio of sensitive energy released/absorbed to the latent heat. However, St number does not include the transient response behavior of the medium as reflected through the thermal diffusivity. All solutions in Fig. 2 have the same St numbers but very different Fo numbers. Therefore it is strongly suggested that the newly defined Fo number be used as a quality indicator of the transient response negligence in the frozen region.

SUMMARY

An approximate solution for the inverse Stephan problem in Cartesian, cylindrical and spherical coordinate systems is presented. The temperature distribution in the unfrozen region, including the subregion undergoing phase-transition, the interface location and the interface velocity, are solved numerically. The temperature distribution in the frozen region is approximated by neglecting the time dependent term in the heat balance equation. A newly defined Fo number is introduced as a quality indicator of the approximation.

Very good agreement for cryosurgical applications of low freezing rates was found between an existing exact solution and the approximate solution. This agreement is strongly dependent on Fo number. The approximate solution is expected to be more accurate in the cylindrical geometry than in the Cartesian geometry and even more accurate in spherical geometry. The freezing rates that will actually be caused by application of the approximate solution will be less than the pre-specified freezing rate H, as Fo number decreases.

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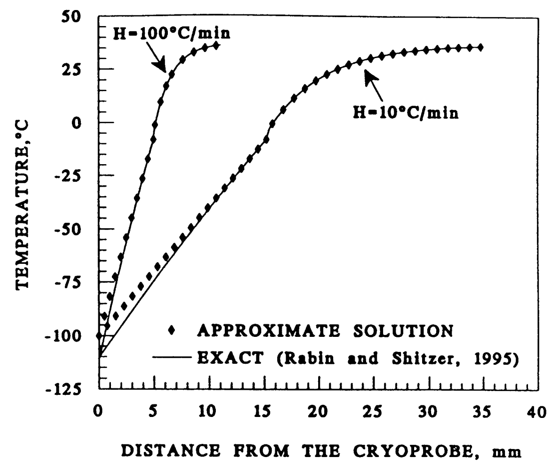

Fig. 1: Comparison of the temperature distribution between the approximate solution and an exact solution (Rabin and Shitzer, 1995), in Cartesian coordinates.

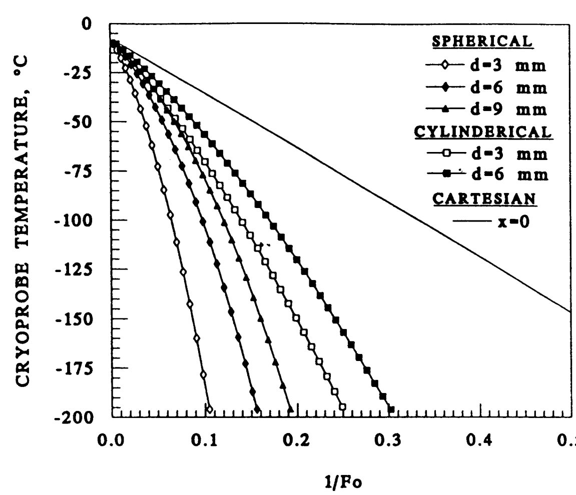

Fig. 2: Forcing function in Cartesian, cylindrical and spherical cases subject to a constant freezing rate of 10°C/min, for typical thermophysical properties of soft tissue, and maximal blood perfusion and metabolic heat generation.

NOMENCLATURE

C	specific heat, J/m^3-°C
d	cryoprobe diameter, m
Fo	Fourier number
H	cooling rate at the freezing front, °C/s
k	thermal conductivity, W/m-°C
n	coordinate system number; Cartesian=0, cylindrical=1, spherical=2
r	radial coordinate, m
s	interface location, m
$\acute{s}$	freezing front velocity, m/s
t	time, s
t^*	dimensionless time, t/τ
T	temperature, °C
T^*	dimensionless temperature, $(T-T_{mf})/\Delta T_{max}$
x	Cartesian coordinate, m
$\hat{w}_b$	blood perfusion, 1/s
q_{met}	metabolic heat generation, W/m^3

Greek

α	thermal diffusivity, m^2/s
η	coordinate in the direction of heat flow, m
η^*	dimensionless coordinate, η/s
ξ	transformed coordinate of the frozen region, m, Eq. (3)
τ	the time required for a frozen region of a thickness s to form, s

Indexes

b	blood
f	frozen
max	maximal
mf	lower boundary of phase-transition temperature range
u	unfrozen