Int. J. Heat Mass Transfer. Vol. 36, No. 3, pp. 673–683, 1993
Printed in Great Britain

0017–9310/93 $6.00 + 0.00

An efficient numerical solution for the multidimensional solidification (or melting) problem using a microcomputer

Y. RABIN†
Mechanical Engineering Department, Ben-Gurion University of the Negev, Beer Sheva, Israel

and

E. KORIN
Chemical Engineering Department, Ben-Gurion University of the Negev, Beer Sheva, Israel

(*Received* 21 *August* 1991 *and in final form* 26 *February* 1992)

Abstract—The aim of this paper is to present a simple and efficient numerical technique for solving transient multidimensional heat transfer problems with melting/solidification processes. The proposed technique comprises an enthalpy-based method for solving the problems by a finite difference scheme, lump system behavior being assumed for each node. The computation technique is able to consider all kinds of boundary conditions, i.e. conduction, convection and radiation alone or in combination. The numerical method neglects convection effects in the liquid phase. The importance of this method lies in the fact that solutions are obtained with a personal microcomputer, thus providing a convenient and reliable tool for wide use in solving many problems of practical interest. The proposed method was verified against the two exact solutions available from the literature for a one-dimensional semi-infinite domain, one with constant temperature boundary condition and the second with constant heat flux. The technique was demonstrated by solving four different cases of two-dimensional problems. A comparison of the results obtained with a microcomputer using the technique presented in this paper with numerical results from the literature obtained using conventional methods, i.e. finite differences and finite elements methods, which generally involve the use of large computers, shows good agreement.

1. INTRODUCTION

HEAT TRANSFER accompanied by phase change is of great importance in many industrial applications, e.g. in food processing, casting and solar energy applications. Solidification and melting problems are nonlinear in the mathematical sense due to the existence of a moving boundary (interface) between the two phases associated with the release of latent heat. Neither position nor the velocity of the interface can be predicted in advance. Mathematical analysis becomes yet more complicated when the physical properties of the phase change material (PCM) are temperature dependent and when the boundary conditions are of the type that apply to convection and radiation.

Phase change problems have a limited number of analytical solutions. Most of those available from the literature apply to simplified and idealized systems and are one dimensional. These analytical solutions have been reviewed by Lunardini [1], and some of the basic solutions may be found in the book of Carslaw and Jaeger [2].

Numerical methods appear to offer a more practical approach for solving phase change problems assuming the moving interface can be traced. Reviews of this type of numerical solution have been presented by Shamsundar [3] and by Viskanta [4].

The fast running microcomputers that have appeared in the past few years are very useful for solving engineering problems. However, they have a limited dynamic memory. Thus numerical methods using small matrices and small numbers of operations become extremely important. Unfortunately, most numerical schemes, and especially those methods that are based on finite elements, use large numbers of variables and matrices to trace the moving interface and the changes in the temperature field. Other finite difference schemes use iterative algorithms as part of the numerical solution for tracing the interface.

The objective of this paper is to present an efficient numerical method suitable for a microcomputer for solving multidimensional phase change problems with complicated geometries, for systems with diverse boundary conditions with respect to fixed temperature and to convective and radiative heat transfer. The proposed numerical method can also deal with phase changes caused by internal heat sources, such as an electric current flowing through the PCM.

The proposed method was tested against two exact

† Present address: Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel.

NOMENCLATURE

A	area
$\mathbf{B}$	coefficients matrix
Bio	Biot number
$\mathbf{C}$	heat capacity matrix
C_p	specific heat
C_i	heat capacity of an element
d	coordinate
D	diagonal length
E'''	energy per unit volume
F	radiation shape factor
h_c	convective heat transfer coefficient
h_r	radiative heat transfer coefficient
$\hbar$	interface location
k	thermal conductivity
$\mathbf{K}$	conductivity matrix
L	latent heat
n	number of grid points
$\bar{n}$	normal
$\dot{q}$	heat source, heat flux per unit of volume
q'	line heat source
R	characteristic length
Rs	nondimensional radius
r	coordinate
s	coordinate
St	Stefan number
t	time
T	temperature
v	volume
$\mathbf{V}$	constants vector
x, y, z	Cartesian coordinates.

Greek symbols

α	thermal diffusivity
α_0	reflectivity
β	thermal diffusivity ratio, $\alpha_l\alpha_s$
Γ	boundary
Δ	interval
ε	emissivity
θ	nondimensional temperature
κ	thermal conductivity ratio, k_l/k_s
τ	nondimensional time, $\alpha_s \cdot t/R^2$.

Other symbol

∇	divergence of.

Subscripts

a	air, surrounding
av	average
b	index of equation (12)
c	convection
i	calculated element
j	neighbor element to the i element
l	liquid
m	melting
r	radiation
s	solid
w	wall
0	initial
1	of time t_1.

solutions of phase change problems and against two other numerical methods based on finite differences and finite elements, in four different cases.

2. MATHEMATICAL FORMULATION

The mathematical modeling presented below assumes that heat transfer by convection in the liquid phase of the PCM can be neglected. The heat balance equations that describe the transfer of heat during phase change can be broken up into three equations describing each of the three regions, the liquid phase, the solid phase, and the solid/liquid interface.

In the liquid phase:

$$\operatorname{div}(k_l \operatorname{grad} T_l) + \dot{q} = \rho_l \cdot C_{p_l} \cdot \frac{\partial T_l}{\partial t} \quad T > T_m. \quad (1)$$

In the solid phase:

$$\operatorname{div}(k_s \operatorname{grad} T_s) + \dot{q} = \rho_s \cdot C_{p_s} \cdot \frac{\partial T_s}{\partial t} \quad T < T_m \quad (2)$$

where k is the thermal conductivity, T the temperature, $\dot{q}$ an internal heat source, ρ the density, C_p the specific heat and t the time. The subscripts s and l denote solid and liquid phases, respectively, and m denotes the melting front.

At the solid/liquid interface:

$$k_l \cdot \frac{\partial T_l}{\partial \bar{n}} - k_s \cdot \frac{\partial T_s}{\partial \bar{n}} = \pm \rho_m \cdot L \cdot \frac{\partial \hbar}{\partial t} \quad T = T_m \quad (3)$$

where $\bar{n}$ is a direction normal to the interface, L the latent heat and $\hbar$ the interface location. The (+) in equation (3) is applicable to solidification and the (−) to melting. The boundary conditions of equations (1) and (2) can include conduction, convection and radiation, singly or in combination. The heat source $\dot{q}$ can describe radiation at the boundary or into the PCM, or any other heat source/sink, such as an electric current flowing through the PCM.

For a constant thermal conductivity coefficient, equations (1) and (2) can be written as

$$k_l \cdot \nabla^2 T + \dot{q} = \rho_l \cdot C_{p_l} \cdot \frac{\partial T}{\partial t} \quad T > T_m + \Delta T_m \quad (4)$$

$$k_s \cdot \nabla^2 T + \dot{q} = \rho_s \cdot C_{p_s} \cdot \frac{\partial T}{\partial t} \quad T < T_m - \Delta T_m. \quad (5)$$

In order to simplify the mathematical model, the

physical fact that in nonpure materials the phase change takes place in a temperature interval $T_m \pm \Delta T_m$ can be used. The PCM then acquires a new set of physical properties in the region of the melting temperature interval (subscripts m), and equations (3)–(5) can be written as

$$k_m \cdot \nabla^2 T + \dot{q} = \rho_m \cdot C_{p_m} \cdot \frac{\partial T}{\partial t} \quad T \in T_m \pm \Delta T_m. \quad (6)$$

The new set of PCM properties (subscripts m) have to be chosen in such a way as to ensure that the heat balance integral on the interface of the PCM will not change. In order to find the new set of properties, the heating process for a small unit of volume of PCM was analyzed (Fig. 1). Let us assume that the density and the specific heat are

$$\rho_m \cdot C_{p_m} = \tfrac{1}{2}(\rho_l \cdot C_{p_l} + \rho_s \cdot C_{p_s}) + \rho_m \cdot \frac{L}{2 \cdot \Delta T_m} \quad (7)$$

where $C_s \cdot \rho_s$ and $C_l \cdot \rho_l$ are constant values in the temperature interval $T_m \pm \Delta T_m$. With the new properties, the stored energy in a unit volume of PCM may be expressed as

$$E''' = \int_{T_0}^{T_m - \Delta T_m} \rho_s \cdot C_{p_s} \cdot dT + \int_{T_m - \Delta T_m}^{T_m + \Delta T_m} \rho_m \cdot C_{p_m} \cdot dT + \int_{T_m + \Delta T_m}^{T_l} \rho_l \cdot C_{p_l} \cdot dT \quad (8)$$

thus

$$E''' = \int_{T_0}^{T_m - \Delta T_m} \rho_s \cdot C_{p_s} \cdot dT + \int_{T_m - \Delta T_m}^{T_m} \rho_s \cdot C_{p_s} \cdot dT + \int_{T_m - \Delta T_m}^{T_m + \Delta T_m} \tfrac{1}{2} \rho_m \cdot \frac{L}{\Delta T_m} \cdot dT + \int_{T_m}^{T_m + \Delta T_m} \rho_l \cdot C_{p_l} \cdot dT + \int_{T_m + \Delta T_m}^{T_l} \rho_l \cdot C_{p_l} \cdot dT. \quad (9)$$

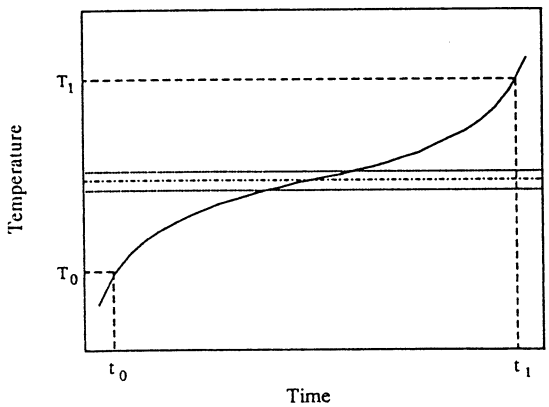

FIG. 1. Time-dependent temperature of a unit of volume of PCM during heating from T_0 (below the melting temperature) to T_l (above the melting temperature). The dot-and-dash line represents the melting temperature and the dotted lines represent the boundaries of the phase change region.

Equation (9) may be simplified to

$$E''' = \int_{T_0}^{T_m} \rho_s \cdot C_{p_s} \cdot dT + \rho_m \cdot L + \int_{T_m}^{T_l} \rho_l \cdot C_{p_l} \cdot dT. \quad (10)$$

Equation (10) describes the amount of energy stored in a unit volume as if there were no need to use the new set of PCM properties. The density ρ_m on the right-hand side of equation (7) should correspond to a solid in the case of melting and to a liquid in the case of solidification.

The thermal conductivity coefficient can be estimated as an arithmetic average

$$k_m = \tfrac{1}{2} \cdot (k_l + k_s) \quad T \in T_m \pm \Delta T_m. \quad (11)$$

The new set of PCM properties makes it possible to express the phase change process in terms of a single equation

$$k_b \cdot \nabla^2 T + \dot{q} = \rho_b \cdot C_{p_b} \cdot \frac{\partial T}{\partial t} \quad (12)$$

$$T < T_m - \Delta T_m \quad b = s$$
$$T_m - \Delta T_m < T < T_m + \Delta T_m \quad b = m$$
$$T_m + \Delta T_m < T \quad b = l.$$

Comparing equations (8) and (10), one can see that, even for pure materials, which melt at a fixed temperature and not over a temperature interval, use of the new PCM properties and a dummy temperature interval can facilitate a simple solution.

3. NUMERICAL SOLUTION

There are several ways of solving equation (12) and calculating the temperature field during the phase change process [5]. In order to solve equation (12) in a multidimensional form, the following numerical scheme based on finite differences was chosen

$$T_i^{p+1} = T_i^p + \frac{\Delta t}{C_i}\left[\sum_j \frac{T_j^p - T_i^p}{R_{ij}} + \dot{q}_i\right] \quad (13)$$

where p is the time index, i the node index, and j the index for all the nodes in the neighborhood of node i.

One can visualize the numerical scheme as dividing space into small elements. Since each element is very small, it functions as a small lumped capacity system. The heat capacity of each element is a function of the state

$$C_i = \rho_i \cdot C_{p_i} \cdot V_i. \quad (14)$$

The thermal resistance R_{ij} to heat transfer by conduction from the i element to its neighbors j, in the x direction in a Cartesian system, is

$$R_{ij} = \frac{0.5 \cdot \Delta x_i}{\Delta y_i \cdot \Delta z_i \cdot k_i} + \frac{0.5 \cdot \Delta x_j}{\Delta y_j \cdot \Delta z_j \cdot k_j}. \quad (15)$$

The thermal resistance to heat transfer by convection

from the boundary node i to the surroundings at the temperature T_j can be given as

$$R_{ij} = \frac{1}{A_i \cdot h_c}. \tag{16}$$

The thermal resistance to heat transfer by radiation between two bodies as a result of a small temperature difference can be described using the radiation coefficient h_r instead of h_c in equation (16).

For large temperature differences between the surface and the surroundings, the heat source $\dot{q}$ can be used to describe the effect of thermal radiation. In cases of thermal radiation, as a result of the difference between the element temperature T_i and the temperature of the surroundings T_a, the heat source $\dot{q}$ of element i will be given by $\varepsilon\sigma A(T_a^4 - T_i^4)$, where ε is the thermal emissivity, σ is the Stefan–Boltzmann constant and A is the area exposed to the radiation.

The heat source $\dot{q}$ can also be used to calculate phase changes resulting from an internal heat source, such as an electric current, the amount of energy transferred to each element is calculated, then used to represent the heat source in equation (13).

As presented by Carnahan *et al.* [5], the numerical scheme (13) has the following stability criterion

$$\Delta t \leqslant \left[\frac{C_i}{\sum_j 1/R_{ij}}\right]_{\min}. \tag{17}$$

In calculating the time interval (17), one has to pay special attention to the fact that both the heat capacity C_i and the thermal conductivity coefficient change during phase changes. In cases of radiation the stability criterion becomes

$$\Delta t < \left[\frac{C_i}{\sum_j 1/R_{ij} + \sigma \cdot (\varepsilon \cdot T_i^2 + \alpha_0 \cdot T_a^2) \cdot (\varepsilon \cdot T_i + \alpha_0 \cdot T_a)}\right]_{\min}. \tag{18}$$

In addition, one must also devote special attention to the changes taking place in the temperature field in the PCM. To ensure energy conservation in the numerical scheme, it is very important to eliminate the possibility of the temperature changing over the entire melting temperature interval in a single time step.

4. RESULTS AND DISCUSSION

There are only a limited number of analytical solutions to phase change problems. Most are one dimensional and do not take the difference between liquid and solid densities into account. As the first verification step, the present numerical solution was compared with two exact solutions of the Stefan problem from the book of Carslaw and Jaeger [2].

Case 1. A one-dimensional melting problem in a semi-infinite body with constant wall temperature. The initial temperature is uniform and below the melting temperature $\theta_w = 0$ when $\theta = (T - T_0)/(T_m - T_0)$. The wall temperature is suddenly changed to $\theta = 1.25$ for every $\tau > 0$. The material $CaCl_2 \cdot 6H_2O$ was taken as the PCM. It has the following physical properties: $St = 0.175$ when $St = C_{p_s}/L \cdot (T_m - T_0)$, $\beta = 2.1$ and $\kappa = 0.725$. Figure 2 presents the numerical and the exact solutions. An infinite slab was taken as the semi-infinite body for the numerical solution (length of 0.2). It can be seen that there is good agreement between the analytical and the numerical solutions as long as the slab acts as a semi-infinite body from the thermal point of view ($\tau = 0.054$).

Case 2. A one-dimensional solidification problem in an infinite body with constant heat flux. This type of problem is found in solidification around pipes in very large media due to coolant flow. The initial temperature is uniform and above the melting temperature $\theta_w = 1$ when $\theta = (T - T_m)/(T_0 - T_m)$. A uniform line heat source (with a negative sign) $q' = -100$ W m^{-1} starts acting and thereafter stays constant for $\tau > 0$. The material $Mg(NO_3)_2 \cdot 6H_2O$ (61.5 wt%), NH_4NO_3 (38.5 wt%) was taken as the PCM. It has the following physical properties: $St = 0.5125$ where $St = C_{p_s}/L \cdot (T_0 - T_m)$, $\beta = 0.895$ and $\kappa = 0.78$. Figure 3 presents the numerical and the analytical temperature solutions as a function of the dimensionless radius Rs, which is the position of the thermal influence ($\theta = 0.01$) at $\tau = 0.3$. The solution of a melting problem with electric current flow would be similar to that for the positive line heat source.

In cases 1 and 2, the one-dimensional phase change problems with constant temperature and constant heat flux boundary conditions were tested in Cartesian and cylindrical coordinate systems. It can be seen that there is a good agreement between the analytical and the numerical solutions of these one-dimensional problems.

Two-dimensional analytical problems are necessarily limited to simplified cases. The numerical solutions of the two-dimensional problems for various boundary conditions were tested against the numerical solution of Lazaridis [6] based on finite differences and that of Hsiao and Chung [7] based on finite elements. The two-dimensional phase change problem of a long prism is presented in Fig. 4. Because of the symmetry of the geometry, only the temperature field in one eighth of the prism had to be solved. Cases 3–6 present phase change problems of that prism for different boundary conditions and different initial temperatures.

Case 3. A solidification problem with constant wall temperature. The initial state of the PCM is liquid at its melting temperature $\theta = 1$ where $\theta = (T - T_w)/(T_m - T_w)$. The prism is suddenly exposed to a uniform wall temperature $\theta = 0$ for $\tau > 0$. $St = 0.641$ where $St = C_{p_s}/L \cdot (T_m - T_w)$. Figures 5 and 6 show the interface positions as a function

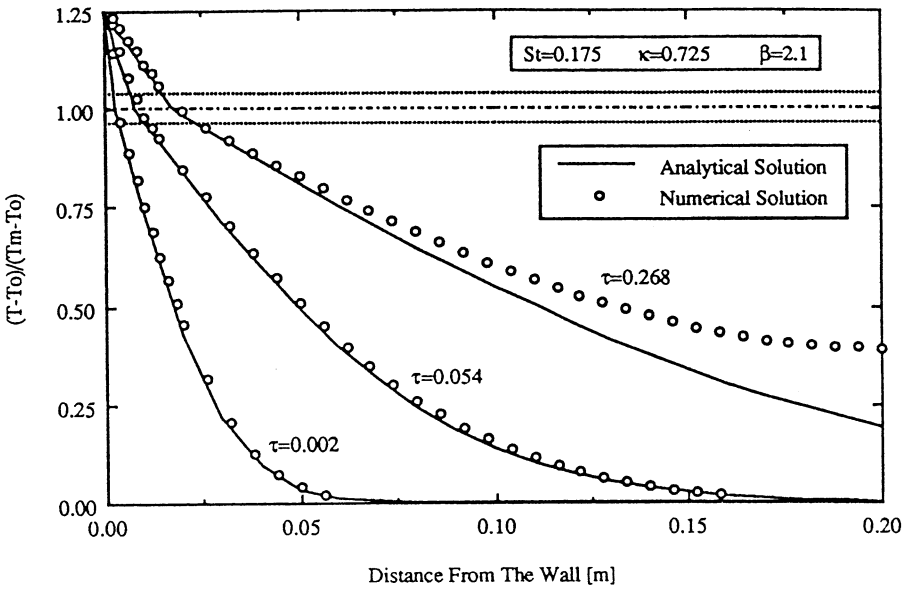

FIG. 2. Melting problem due to a constant wall temperature.

of time along the diagonal and the center lines, respectively. On the assumption that the phase changes occur in the melting temperature interval, the use of the new set of physical properties $\rho_m \cdot C_m$ enables a good estimation of the temperature field in the PCM, but makes estimation of the interface location more difficult. Here interface location is predicted by parabolic interpolation. In order to take into account all the latent heat effects, the initial temperature was chosen as $T_m + \Delta T_m$ instead of T_m. From the agreement between the numerical solutions, one can see that $T_m + \Delta T_m$ is a good choice for the initial temperature.

Case 4. A solidification problem with constant wall temperature. The initial temperature of the PCM is $\theta = 9/7$, where $\theta = (T - T_w)/(T_m - T_w)$. The prism is suddenly exposed to a uniform wall temperature $\theta = 0$ for $\tau > 0$. $St = 2$ where $St = C_{p_s}/L \cdot$

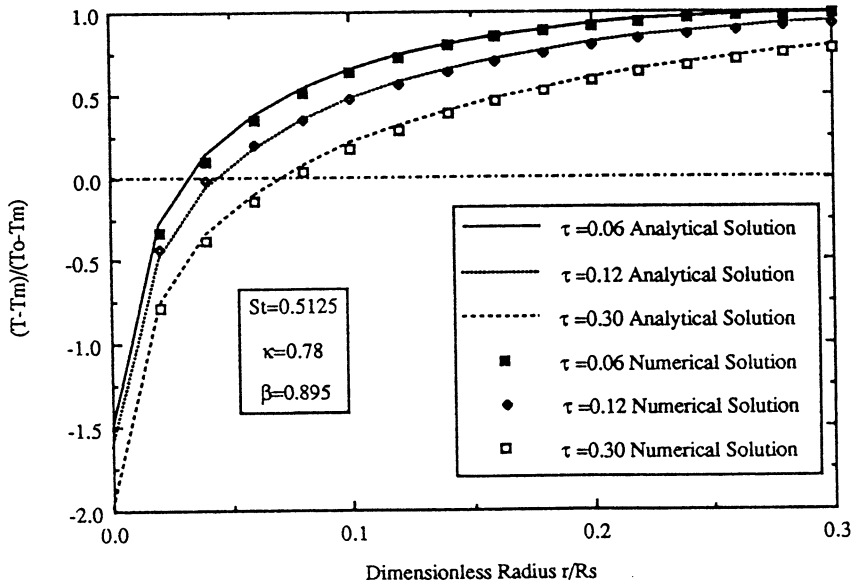

FIG. 3. Solidification problem due to a constant heat sink at the center of an infinite cylinder.

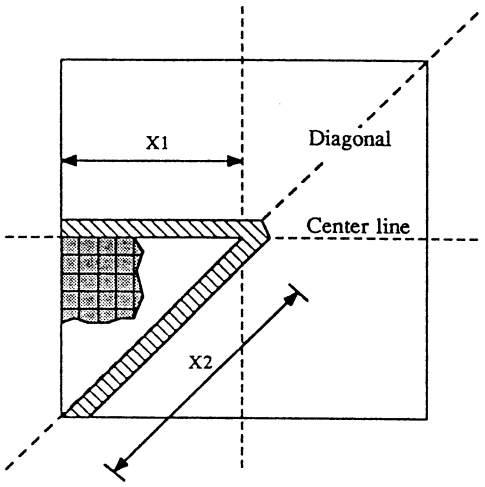

FIG. 4. Pictorial representation of the cross-section of the prism involved in the two-dimensional phase change problems solved in cases 3–6. The solution referred to one eighth of the prism.

$(T_m - T_w)$, $\beta = 0.9$ and $\kappa = 0.9$. Figures 7 and 8 show the interface positions as a function of time along the diagonal and the center lines, respectively.

Case 5. A solidification problem with convection from the wall of the prism to the surroundings. The initial temperature of the PCM is $\theta = 9/7$ where $\theta = (T - T_a)/(T_m - T_a)$. The prism is suddenly exposed by convection to a surrounding region at a temperature $\theta = 0$. $St = 2$ where $St = C_{p_s}/L \cdot (T_m - T_a)$, $\beta = 0.9$, $\kappa = 0.9$ and $Bio = h \cdot s/k_s = 2$. Figures 9 and 10 show the interface positions as a function of time along the diagonal and the center lines, respectively.

Cases 3–5 show good agreement between the three different numerical solutions. One can see that the present work agrees better with the work of Hsiao and Chung (finite elements) along the diagonal of the prism and with Lazaridis' work (finite differences) along the center line.

The last case (case 6) presents a similar problem but with thermal radiation at the boundary. Such problems are solved by using the heat source $\dot{q}$ in each of the boundary elements. Assuming a gray body, the heat source $\dot{q}$ can be expressed by: $\dot{q}_i = \sigma \cdot A_i \cdot F \cdot \varepsilon \cdot (T_a^4 - T_i^4)$. The validity of this scheme is shown by the good agreement between the different numerical solutions of the convection boundary problem (case 5). The term $(T_a^4 - T_i^4)$ can be estimated by writing the term $4 \cdot T_{av}^3 \cdot (T_a - T_i)$, where T_{av} is the average temperature between T_a and T_i. Using the new term, one can write the heat radiation coefficient as $h_r = 4 \cdot \sigma \cdot F \cdot \varepsilon \cdot T_{av}^3$. One can then compare the different solutions of the same radiative phase change problem obtained using the heat source $\dot{q}$ and the radiation coefficient h_r.

Case 6. Solidification due to thermal radiation from the wall of the prism to the surroundings. The initial temperature of the PCM is $\theta = 9/7$, where $\theta = (T - T_a)/(T_m - T_a)$. The prism is suddenly exposed by convection to a surrounding region at a temperature $\theta = 0$. $St = 2$ where $St = C_{p_s}/L \cdot (T_m - T_a)$, $\beta = 0.9$, $\kappa = 0.9$. The initial temperature of the prism is 363 K and the prism is exposed to the surroundings at a temperature of 273 K. In that temperature

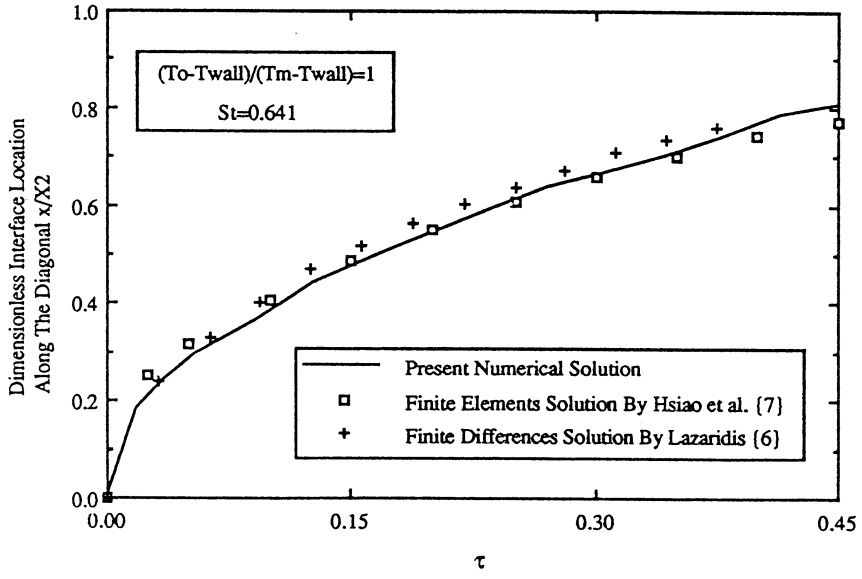

FIG. 5. Interface location along the diagonal of a square region of a solidification problem due to a constant wall temperature, initially at the melting temperature.

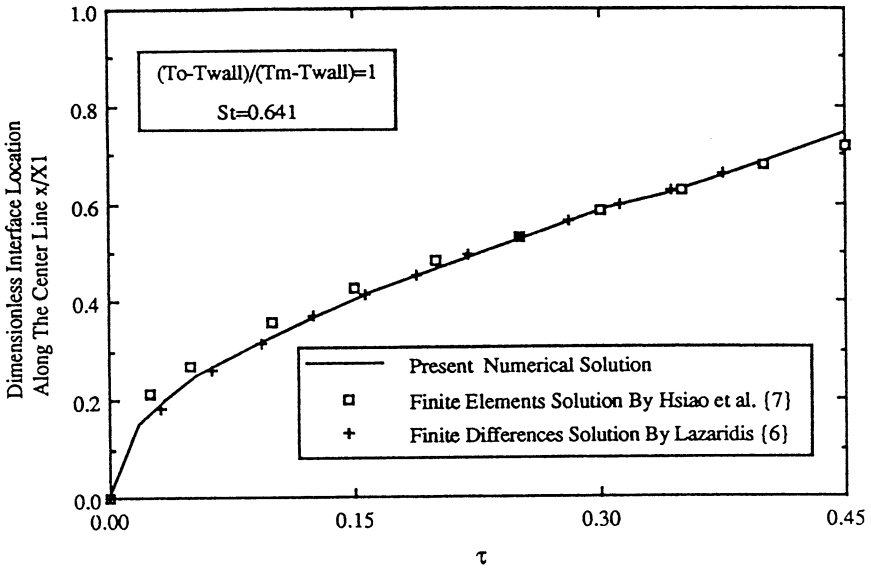

FIG. 6. Interface location along the center line of a square region in a solidification problem due to a constant wall temperature, initially at the melting temperature.

range the use of the term $4 \cdot T_{av}^3 \cdot (T_a - T_i)$ instead of the term $(T_a^4 - T_i^4)$ involves an error range of 2%. The radiation coefficient h_r is calculated for each boundary element in each time interval. In the numerical solution, the Biot number was changed from 9.15 at the beginning to an average of 5.25 at $\tau = 0.225$. Figure 11 shows the interface positions as a function of time along the diagonal and the center lines, respectively.

The agreement between the two methods of solution of the same problem, i.e. using h_r or $\dot{q}$, is of the order of 97%. We conclude that the translation of thermal radiation at the boundary to heat sources at the boundary elements is good. Use of the radiation

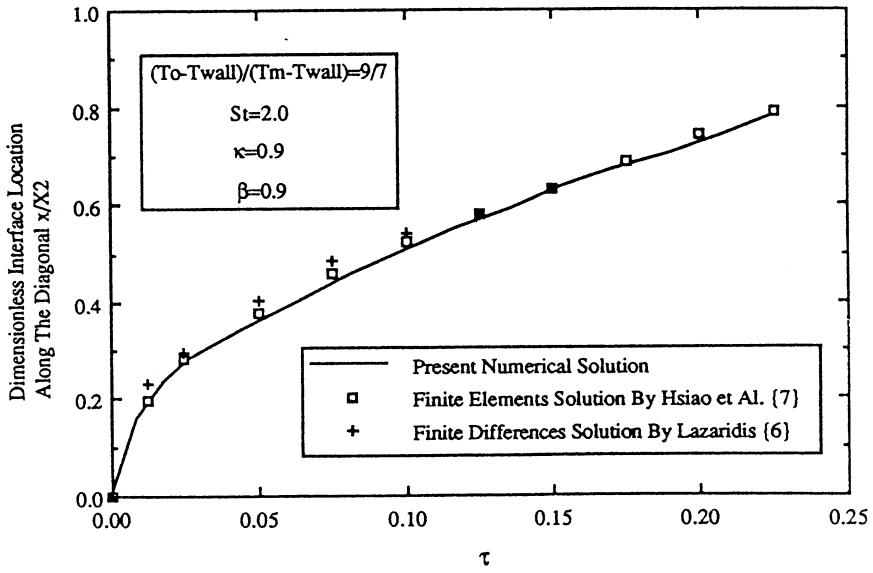

FIG. 7. Interface location along the diagonal of a square region in a solidification problem due to a constant wall temperature, initially at a liquid temperature.

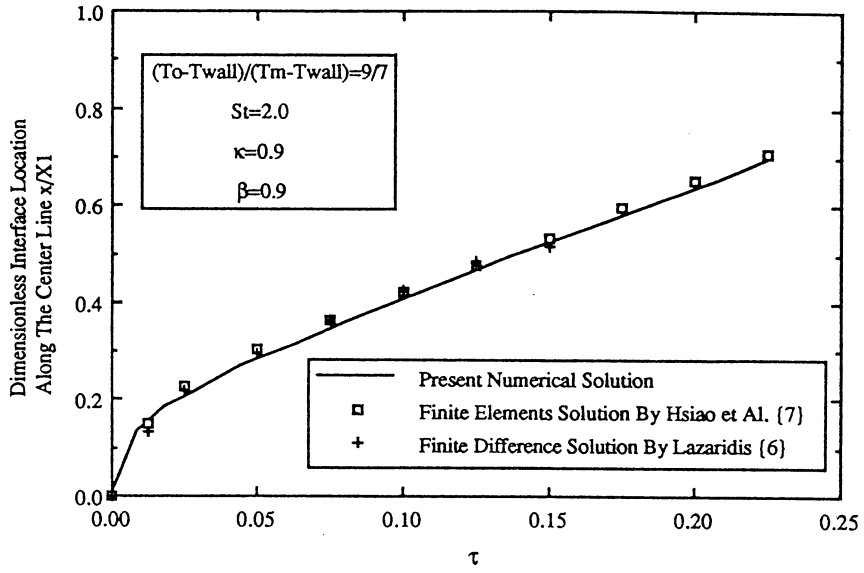

FIG. 8. Interface location along the center line of a square region in a solidification problem due to a constant wall temperature, initially at a liquid temperature.

coefficient becomes less accurate as the temperature difference between the boundary and the surroundings increases.

In order to evaluate the efficiency of the proposed numerical solution from the standpoint of application with a microcomputer with a limited memory capacity, let us compare it with other common numerical solutions based on finite elements. Basically, to solve heat transfer problems by using finite element methods, one has to solve the following equation:

$$\mathbf{C}\cdot\dot{\mathbf{T}}+\mathbf{K}\cdot\mathbf{T}+\mathbf{G}=0 \tag{19}$$

where C and K denote the heat capacity and the conductivity matrices, respectively, G is the vector of heat

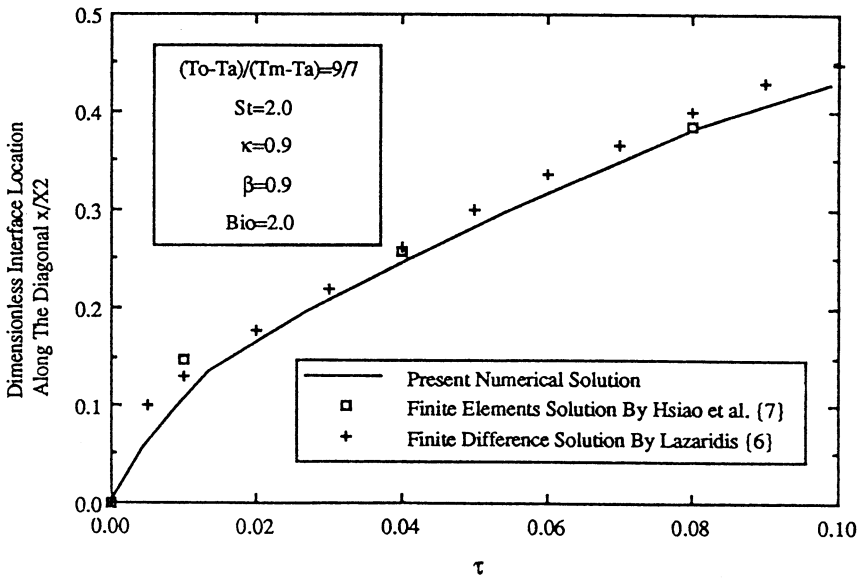

FIG. 9. Interface location along the diagonal of a square region in a solidification problem due to a convection at the boundary, initially at a liquid temperature.

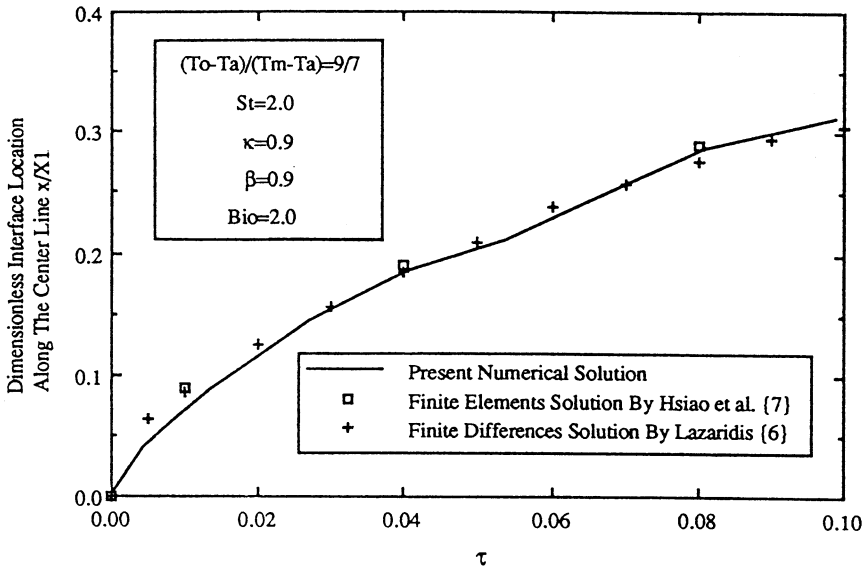

FIG. 10. Interface location along the center line of a square region in a solidification problem due to a convection at the boundary, initially at a liquid temperature.

sources and boundary conditions, and T denotes the vector of the calculated temperatures.

A detailed comparison of different numerical methods based on finite elements for solving heat transfer problems with phase changes is given by Dalhuijsen and Segal [8]. As suggested by Dalhuijsen and Segal, we will focus this discussion on the linear elements. There are several ways to accomplish discretization of equation (19) in time. The algorithm of Lees [9] was chosen for the present discussion, as was suggested by Morgan *et al.* [10]

$$\mathbf{C}^p \cdot (\mathbf{T}^{p+1} + \mathbf{T}^{p-1}) \cdot \frac{1}{2 \cdot \Delta t} + \mathbf{K}^p \cdot (\mathbf{T}^{p+1} + \mathbf{T}^p + \mathbf{T}^{p-1}) \cdot \tfrac{1}{3} + \mathbf{G}^p = 0. \quad (20)$$

This scheme is straightforward and involves three time levels, where **C**, **K** and **G** are evaluated at the time

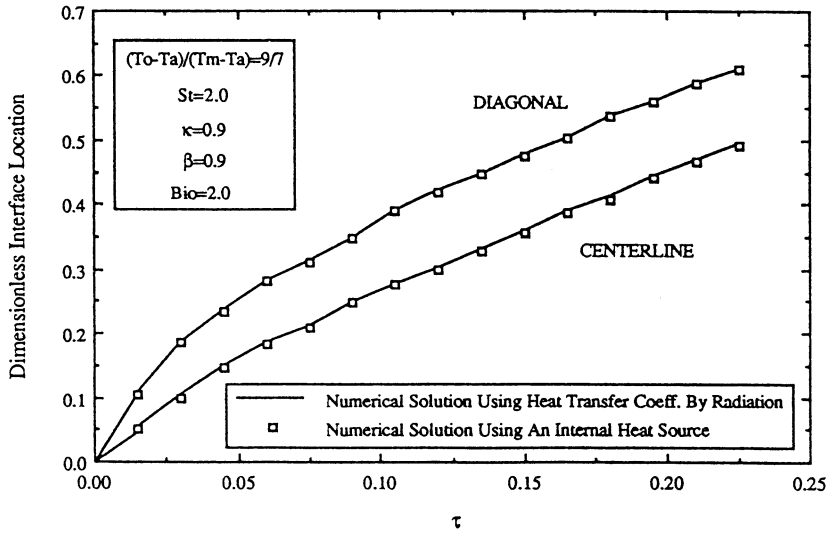

FIG. 11. Interface location in a square region in a solidification problem due to radiation at the boundary, initially at a liquid temperature.

level p. For the numerical solution, the last scheme will be in the form

$$\mathbf{B}\cdot\mathbf{T}^{p+1} = \mathbf{V} \tag{21}$$

where

$$\mathbf{B} = (\mathbf{C}^p + \tfrac{2}{3}\cdot\Delta t\cdot\mathbf{K}^p) \tag{22}$$

$$\mathbf{V} = -\tfrac{2}{3}\cdot\Delta t\cdot\mathbf{K}^p\cdot\mathbf{T}^p + (\mathbf{C}^p - \tfrac{2}{3}\cdot\Delta t\cdot\mathbf{K}^p)\cdot\mathbf{T}^{p-1} + \tfrac{2}{3}\cdot\Delta t\cdot\mathbf{G}^p. \tag{23}$$

The matrices **K** and **C** and therefore **B** are of order $n\cdot n$, where n is the number of grid nodes. During the computer simulation matrix **B** and vector **V** can be calculated directly, saving the unnecessary evaluation of **K**, **C** and **G**. In cases of one-dimensional heat transfer problems (using linear elements) matrix **B** has three non-zero diagonals, and therefore equation (21) can be solved by a simple algorithm. In fact, in one-dimensional problems, as in cases 1 and 2 above, it is preferable to save only $3\cdot n$ of the $n\cdot n$ coefficients of matrix **B** in the computer memory. In two-dimensional problems there are more than three non-zero diagonals. In simple two-dimensional problems, such as cases 3–6 above, there are five non-zero diagonals in matrix **B**, however, solution of equation (21) requires saving at least $(n+1)\cdot n/2$ coefficients of matrix **B** in memory. However, the solution of the general three-dimensional problems requires saving all $n\cdot n$ coefficients of matrix **B**. When estimating the amount of dynamic memory required for numerical solution, one must bear in mind that there are additional vectors of orders n to be placed in the computer memory: $\mathbf{T}^{p+1}$, $\mathbf{T}^p$, $\mathbf{T}^{p-1}$ and **V**. (In some schemes other than Lees' the vector $\mathbf{T}^{p-1}$ is omitted but the remaining vectors and matrices do not change.)

In contrast, in the proposed straightforward numerical scheme, represented by equation (13), only two vectors of order n need be placed in memory: $\mathbf{T}^{p+1}$ and $\mathbf{T}^p$. The size of the dynamic memory required for this scheme is only a function of the number of grid points. The heat capacity C_i and the thermal resistance to heat flow R_{ij} can be calculated for each node at the relevant stage. The computer memory capacities needed for the finite elements and finite difference solutions are summarized in Table 1.

To obtain some idea of the running time required for solving this kind of problem using microcomputers, the solution of case 5 was tested for two different meshes on three versions of IBM-PC compatible computers. The physical properties were $k = 1\ \mathrm{W\,m^{-1}\,K^{-1}}$, $\alpha_l = 10^{-6}\ \mathrm{m^2\,s^{-1}}$ and $R = 1$ m. Case 5 was solved using a mesh of 10×10 and 20×20 grid points (Fig. 4). The stability criteria (17) required nondimensional time intervals of $\Delta\tau < 2.5\times10^{-3}$ for the first mesh and $\Delta\tau < 6.25\times10^{-4}$ for the second. In order to eliminate the possibility of the temperature changing from below to above the melting temperature interval in one time step, i.e. to enforce energy conservation, the time intervals were reduced to $\Delta\tau = 2.0\times10^{-4}$ for the first mesh and to $\Delta\tau = 6.0\times10^{-5}$ for the second. Table 2 gives the results of the running time test using a turbo PASCAL 5.0 compiler.

Table 1. A comparison of the computer memory capacities required for solving the heat transfer problem using finite element and finite difference methods, where n is the number of grid points

Proposed solution	Finite elements	Problem dimensionals
$2\cdot n$	$3\cdot n+4\cdot n$	One-dimensional
$2\cdot n$	$(n+1)\cdot n/2+4\cdot n$	Two-dimensional
$2\cdot n$	$n\cdot n+4\cdot n$	Three-dimensional

Table 2. A comparison of the running times required for solving case 5, using three versions of IBM-PC compatible microcomputers and with different numbers of grid nodes

20×20 grid points	10×10 grid points	Microcomputer version
2266.0 s	173.4 s	XT, 8 MHz, 8086+8087
269.4 s	20.8 s	AT, 25 MHz, 80386+80387
84.1 s	6.5 s	AT, 33 MHz, 80486

5. CONCLUSIONS

The proposed numerical scheme is capable of solving phase change problems with boundary conditions of constant temperature and convection and/or radiation. Radiation and convection boundary conditions can easily be combined in the same problem. The resulting scheme is simple and straightforward and requires neither iteration nor tracing of the interface location. Since the physical properties of the PCM—the heat capacity and the thermal resistance to heat flow of each element—are determined by its temperature, the proposed numerical method can deal with more than one liquid/solid interface.

Since the scheme is based on two temperature vectors only, the properties of each node being determined at the beginning of each time interval according to the previous temperature field, it is suitable for solving phase change problems using a microcomputer with a limited dynamic memory.

Comparison of the proposed numerical solution with exact solutions and with other numerical solutions based on finite differences and finite elements showed good agreement. Thus, the proposed numerical scheme is a simple and reliable tool for obtaining fast solutions to many phase change problems.

All the numerical solutions in the present work were obtained with an IBM PC microcomputer using a turbo PASCAL compiler. The running time for each numerical solution was a few minutes.

REFERENCES

1. V. J. Lunardini, *Heat Transfer in Cold Climates*. Van Nostrand Reinhold, New York (1981).

2. H. S. Carslaw and C. J. Jaeger, *Conduction of Heat in Solids*, Chap. 11. Clarendon Press, Oxford (1959).
3. N. Shamsundar, *Comparison of Numerical Methods for Diffusion Problems with Moving Boundaries in Moving Boundary Problems* (Edited by D. G. Wilson, A. D. Solomon and P. T. Boggs). Academic Press, New York (1978).
4. R. Viskanta, *Solar Heat Transfer Storage: Latent Heat Material* (Edited by G. A. Lane). CRC Press, Florida (1983).
5. B. Carnahan, H. A. Luther and J. O. Wilkes, *Applied Numerical Methods*. Wiley, New York (1969).
6. A. Lazaridis, A numerical solution of the multi-dimensional solidification (or melting) problem, *Int. J. Heat Mass Transfer* **13**, 1459–1477 (1970).
7. J. S. Hsiao and B. T. F. Chung, An efficient algorithm for finite element solution to two-dimensional heat transfer with melting and freezing, ASME 84-HT-2 (1984).
8. A. J. Dalhuijsen and A. Segal, Comparison of finite elements techniques for solidification problems, *Int. J. Numer. Meth. Engng* **23**, 1807–1829 (1986).
9. M. Lees, A linear three-level difference scheme for quasilinear parabolic equations, *Math. Comp.* **20**, 516–622 (1966).
10. K. Morgan, R. W. Lewis and O. C. Zienkiewicz, An improved algorithm for heat conduction problems with phase change, *Int. J. Numer. Meth. Engng* **12**, 1191–1195 (1978).