Parametric Equations and Vectors Calculus of a Single Variable X. Du

- **Parametric Equations**
  - In $\mathbb{R}^2$, a curve in the xy-plane can be expressed as a two functions of a parameter *t*, i.e. $x(t)$ and $y(t)$.
  - Number of parameters is the number of free variables.
    - Curves may only have one free variable (i.e. one parameter)
    - Surfaces have two free variables (i.e. two parameters)
    - Solids have three free variables (i.e. three parameters)
  - This allows complex curves to be more easily written as a parametric equation.
  - There are an infinite number of ways to parameterize a function!
  - Trigonometric identities may help parameterize circles and ellipses.
  - Master this concept! It will become critical for multivariable calculus.
  - **Smooth** - A parameterization is smooth on an interval if $x(t)$ and $y(t)$ have continuous first derivatives on the interval except possibly at the endpoints of the interval.
  - **Piecewise smooth** - A parameterization is piecewise smooth on an interval if the parameterization is smooth along subintervals of the interval.
  - Derivatives - Use chain rule. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
  - Arc length: $s = \int_{t_1}^{t_2} |\vec{r}(t)| dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
  - Area of a surface of revolution: $S = 2\pi \int_a^b r(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
- **Vectors**
  - Have magnitude and direction.
  - Contrast to scalar quantities
  - Dot product:
    - Let $\vec{a} = \langle a_1, a_2, a_3, \ldots a_n \rangle$ and $\vec{b} = \langle b_1, b_2, b_3, \ldots b_n \rangle$.
    - $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + a_n b_n$
    - Scalar!
  - Cross product:
    - Only valid in three-space: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.
    - $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$
    - Vector! Direction - use right hand rule.
  - In the xy-plane in two-space, a parameterized curve can be expressed with the vector-valued position function $\vec{r}(t) = \langle x(t), y(t) \rangle$ $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$
  - Magnitude: $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$