Explicit near-fully X-Ramanujan graphs

Xinyu Wu
Carnegie Mellon University

Joint work with Ryan O’Donnell
6-regular infinite tree $\mathbb{T}_6$
Goal: approximate spectrum and structure of infinite graphs with finite graphs
Approximating infinite graphs with finite graphs

\( d \)-regular Ramanujan graphs approximate the structure and spectrum of infinite \( d \)-regular tree

- Ramanujan graphs: best possible expanders
- Expanders have many applications in TCS
Beyond d-regular graphs

Approximations of more complicated infinite graphs = expanders with local constraints

Example: typical instances of random constraint satisfaction problems
First step: algebraic recipe to describe infinite graphs
6-regular infinite tree $\mathbb{T}_6$
\( T_6 \) as a color-regular graph
All green arrows = permutation of tree

Represent $T_6$ as sum of 3 infinite permutations

$$g_1 + g_2 + g_3 +$$
$$g_1^{-1} + g_2^{-1} + g_3^{-1}$$

Formally: generators of $\mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$ or $\mathbb{F}_3$
Adjacency matrix

\[ A = P_{g_1} + P_{g_2} + P_{g_3} + P_{g_1}^* + P_{g_2}^* + P_{g_3}^* \]

\((P_{g_1}^* = p_{g_1}^{-1})\)

Express as a polynomial

\[ p(X_1, X_2, X_3) = X_1 + X_2 + X_3 + X_1^* + X_2^* + X_3^* \]

Then \( A = p(P_{g_1}, P_{g_2}, P_{g_3}) \)

Also written \( p(g_1, g_2, g_3) \)

Add inverse perms \( \rightarrow \) undirected graph
Consider the polynomial
\[ q(X_1, X_2) = X_1 + X_2 + X_1X_2 + X_1^* + X_2^* + X_2^*X_1^* \]
What is \( q(g_1, g_2) \)?
Vertices = vertices infinite 4-regular tree
\[ q(X_1, X_2) = X_1 + X_2 + X_1X_2 + X_1^* + X_2^* + X_2^*X_1^* \]
\[ q(X_1, X_2) = X_1 + X_2 + X_1X_2 + X_1^* + X_2^* + X_2^*X_1^* \]
Approximate spectrum and structure of graphs described by polynomials
Approach: finite permutations

• Replace infinite permutations with finite permutations on $[n]$: $P_{\sigma_1}, P_{\sigma_2}, P_{\sigma_3}$

• $p(X_1, X_2, X_3) = X_1 + X_2 + X_3 + X_1^* + X_2^* + X_3^*$

• $p(\sigma_1, \sigma_2, \sigma_3)$ is adj matrix of 6-regular graph on $[n]$
\[ q(X_1, X_2) = X_1 + X_2 + X_1X_2 + X_1^* + X_2^* + X_2^*X_1^* \]

Applied to random permutations:

Compare to infinite graph
What does it mean to approximate?

• We say that $G$ covers $H$ if there’s a surjection from $G$ to $H$ which is a local bijection
  • $q(g_1, g_2)$ covers $q(\sigma_1, \sigma_2)$

• For any base graph $H$ there is a unique (usually infinite) tree (the *universal covering tree*) that covers $H$
Approximating using random permutations

• Random permutations actually form a good approximation

• Friedman’s theorem: random $d$-regular graph structurally + spectrally approximates the infinite $d$-regular tree (is almost Ramanujan)
Spectral approximation
Closeness in spectrum

Sequence of permutations such that
\[ \text{spec} \left( p(\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n}) \right) \rightarrow \text{spec} \left( p(g_{1}, g_{2}, g_{3}) \right) \]

Spectrum: \( \{ \lambda: (\lambda I - A) \text{ is not invertible} \} \)
Trivial eigenvalues

Eigenvalues of all 1s vector
All 1s is not an eigenvector of infinite graph since it’s not bounded
How to find these? They are the eigenvalues of \( p(1,1,1) \)
  • \( p \) applied to identity permutation on [1]
  • Or, degree of the graph
Spectral approximation

Infinite 6-reg tree $T_6$

$-2\sqrt{5}$ to $2\sqrt{5}$

Nontrivial spectrum of finite graph $G$

Notion of convergence:
For $n$ large enough, every point in $\text{spec}(G)$ is within $\varepsilon$ of a point of $\text{spec}(T_6)$ and vice versa

“Convergence in Hausdorff distance”
Ramanujan graphs

$A$ — adj matrix of $n$-vertex, $d$-regular graph

Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

Trivial eigenvalue $\lambda_1 = d$

Alon–Boppana theorem:

$$\max(\lambda_2, |\lambda_n|) \geq 2\sqrt{d - 1} - o_n(1)$$

Spectral radius of infinite $d$-regular tree
Ramanujan graphs

Alon–Boppana theorem:
\[ \max(\lambda_2, |\lambda_n|) \geq 2\sqrt{d - 1} - o_n(1) \]

Ramanujan graphs:
\[ \max(\lambda_2, |\lambda_n|) \leq 2\sqrt{d - 1} \]

Explicit constructions:
- [Margulis ’88], [Lubotsky–Phillips–Sarnak ’89], [Morgenstern ’94] (\(d - 1\) prime power)
- [Marcus–Spielman–Srivastava ’15] (bipartite)
Friedman’s theorem

*Random* graphs are almost Ramanujan
For any $\varepsilon > 0$, a random $d$-reg graph whp has
$$\max(\lambda_2, |\lambda_n|) \leq 2\sqrt{d} - 1 + o_n(1)$$

Explicit construction in recent work of Mohanty–O’Donnell–Paredes ’20
Beyond Friedman’s theorem

• Generalizing Friedman’s theorem to other algebraically-described graphs
• Concept: random lifts and universal covering
Lifts

Start with a base graph $G$

Replace each vertex with $n$ vertices
Replace each edge with a matching \( n \)-lift
Lifts

- \( n \)-lift \( G_n \) covers the base graph \( G \)
- Limiting object (\( \infty \)-lift) is the universal covering tree
- Spectrum: Trivial eigenvalues are \( \text{spec}(G) \)
  - Consider newly added eigenvalues
- *Random* \( n \)-lift: iid uniform random random matchings for edges
X-Ramanujan graphs

A sequence of graphs \( \{G_n\} \) such that
1. Nontrivial spectrum of \( G_n \) is \( \varepsilon \)-close to \( \text{spec}(X) \) in Hausdorff distance
2. \( X \) covers \( G_n \)
Generalized Friedman’s conj.

A random $n$-lift $G_n$ of a base graph $H$ has nontrivial spectrum $\text{spec}(G_n) \setminus \text{spec}(H)$ $\varepsilon$-close to $\text{spec}(G_\infty)$ with high probability

Proved by Bordenave–Collins ’19

More general result: stated in terms of polynomials
Our result

- Explicit form of Bordenave–Collins’ theorem
- Consequently: Given a base graph $H$ and $\varepsilon > 0$, we have a poly$(n)$-time algorithm which constructs an $n$-lift with nontrivial spectrum $\varepsilon$-close to that of the universal covering tree of $H$.
- Includes more infinite graphs too
Up next

• What graphs can we describe with polynomials?
• Make some natural algebraic generalizations of polynomials
Weighted graphs

\[ p(X_1, X_2, X_3) = 0.15X_1 + 0.25X_2 + 0.1X_3 + \cdots \]

Interpretation: weighted random walk on tree
Matrix-weighted graphs

Matrix weighted graph with same adjacency matrix
Matrix-weighted graphs
Matrix polynomials

\[ p(X_1, \ldots, X_6) = a_1X_1 + \cdots + a_6X_6 + a_1^*X_1^* + \cdots \]

Evaluate as

\[ p(\sigma_1, \ldots, \sigma_6) = \sum_{i=1..6} P_{\sigma_i} \otimes a_i + P_{\sigma_i}^* \otimes a_i^* \]

is \( p(1, \ldots, 1) \)
Matrix polynomials

\[ p(\sigma_1, \ldots, \sigma_6) = \sum_{i=1 \ldots 6} P_{\sigma_i} \otimes a_i + P_{\sigma_i}^* \otimes a_i^* \]

\( \sigma_1, \ldots, \sigma_6 \) are permutations

is a lift of

\[ p(g_1, \ldots, g_6) \text{ is the } \textit{universal covering tree} \]

\( g_1, \ldots, g_6 \) are infinite cyclic permutations/generators of \( \mathbb{Z} \)
For any matrix polynomial $p(X_1, \ldots, X_d)$, iid unif random permutations $\{\sigma_{1,n}, \ldots, \sigma_{d,n}\}$ satisfy
\[
\text{spec} \left( p(\sigma_{1,n}, \ldots, \sigma_{d,n}) \right) \rightarrow \text{spec}(p(g_1, \ldots, g_3))
\]
(minus trivial eigenvalues)
whp in Hausdorff distance
Graphs from matrix polys

We show that they include:

• Free products of finite vertex transitive graphs/rooted graphs

• “Additive products” [Mohanty–O’Donnell ’20]

• “Amalgamated free products” [Vargas–Kulkarni ’20]

• And others
Our results

Given $X, \varepsilon > 0$, we give a poly($n$) time algo which produces a graph $G$ on $n' \sim n$ vertices

- $G$ is covered by $X$

- $G'$s nontrivial spectrum is $\varepsilon$-close in Hausdorff distance to $X'$s spectrum
Not covered

Some non-examples:

• Grids — matrix polynomial graphs have finite treewidth and are hyperbolic
Graphs from matrix polys

Some non-examples:

• “Grandparent graph” — matrix polynomial graphs are unimodular

Proof steps

1. Linearization
   • Reduce to proving theorem for only linear polys

2. Matrix-valued Ihara–Bass formula
   • Relate spectra of adjacency operator and non-backtracking operator
   • Norm bounds for NB operator ➔ Hausdorff distance bounds on adj operator

3. Prove norm bounds on NB operator
   • Trace method with matrix weights
Conclusion

• Non-commutative polynomials
  • Recipe for constructing many infinite graphs and finite graphs covered by them

• Explicit constructions of finite graphs spectrally close to infinite graphs

• Open question: is there a similar recipe for other combinatorial objects, e.g. high dimensional expanders?
Proof – norm bounds

• Further reductions following [MOP ’20] (Ideas from [Bilu–Linial ’06])

• Main technical difficulty: construct 2-lifts which all new eigenvalues of nonbacktracking matrix are bounded

• Key step: trace method with matrix weights
  • Matching walks on the finite lift with walks on the infinite graph