Decision Diagrams for Optimization

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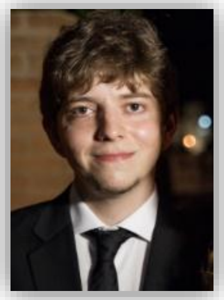

Thiago Serra
Carnegie Mellon University

Our Main Research Goal

Investigate the use of Decision Diagrams for solving discrete optimization problems

Contributions so far

- **New relaxation/bounding technique**
  - Bounds can be superior to state-of-the-art methods in certain problems
- **Generic primal heuristic**
  - Scales to large-scale problems
- **Inference techniques**
  - New types of cuts for MIPs and other optimization technologies
- **Novel complete solution technique**
  - Solved open instances from classical benchmarks
  - **Parallel method** that scales almost linearly with number of processors

Decision Diagrams

- Graphical representation of **Boolean functions**

$$f(x) = \left(x_1 \Leftrightarrow x_2\right) \wedge \left(x_3 \Leftrightarrow x_4\right)$$

x_1	x_2	x_3	x_4	**f(x)**
0	0	0	0	**1**
0	0	0	1	**0**
0	1	1	0	**0**
0	0	1	1	**1**
…	…	…	…	**…**

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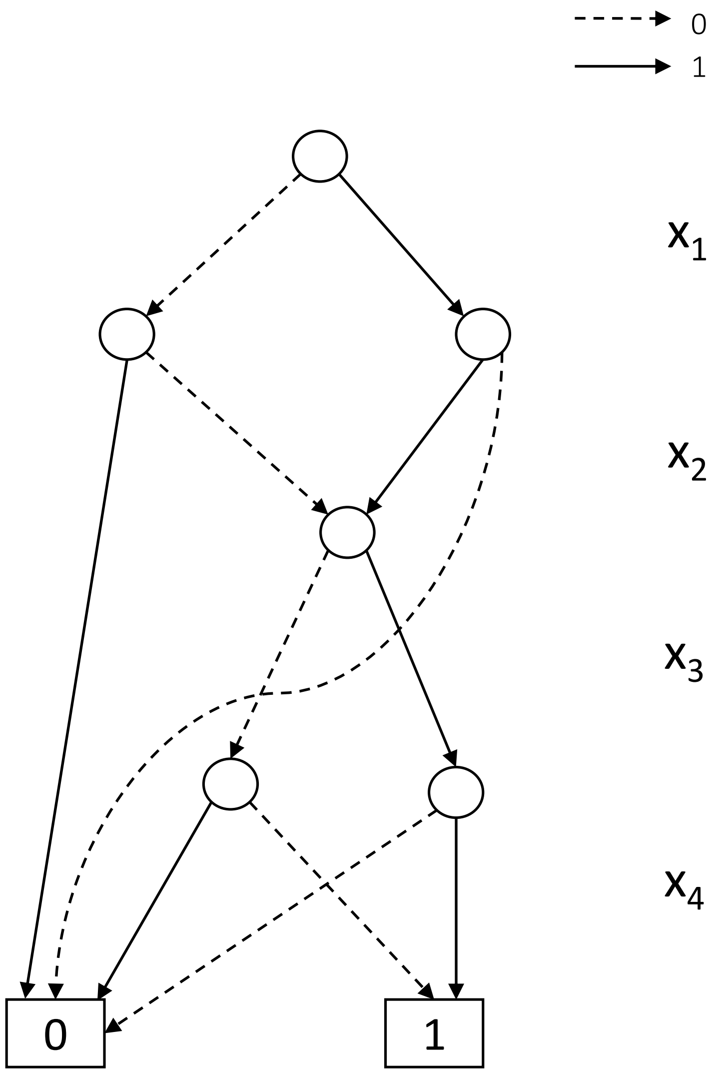

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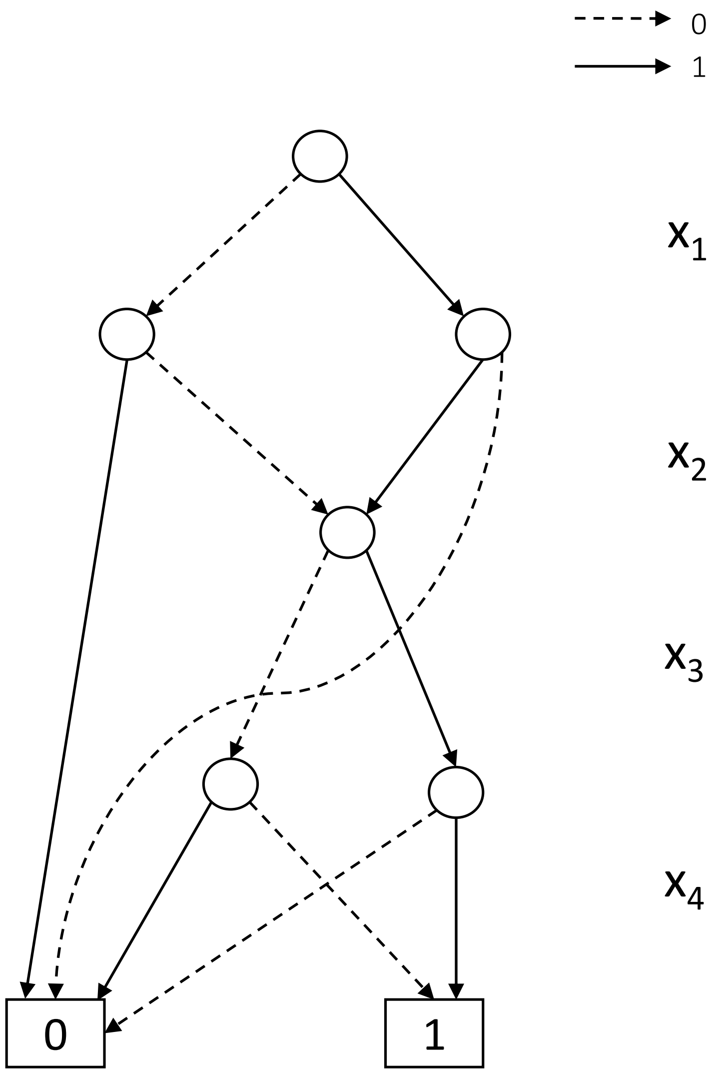

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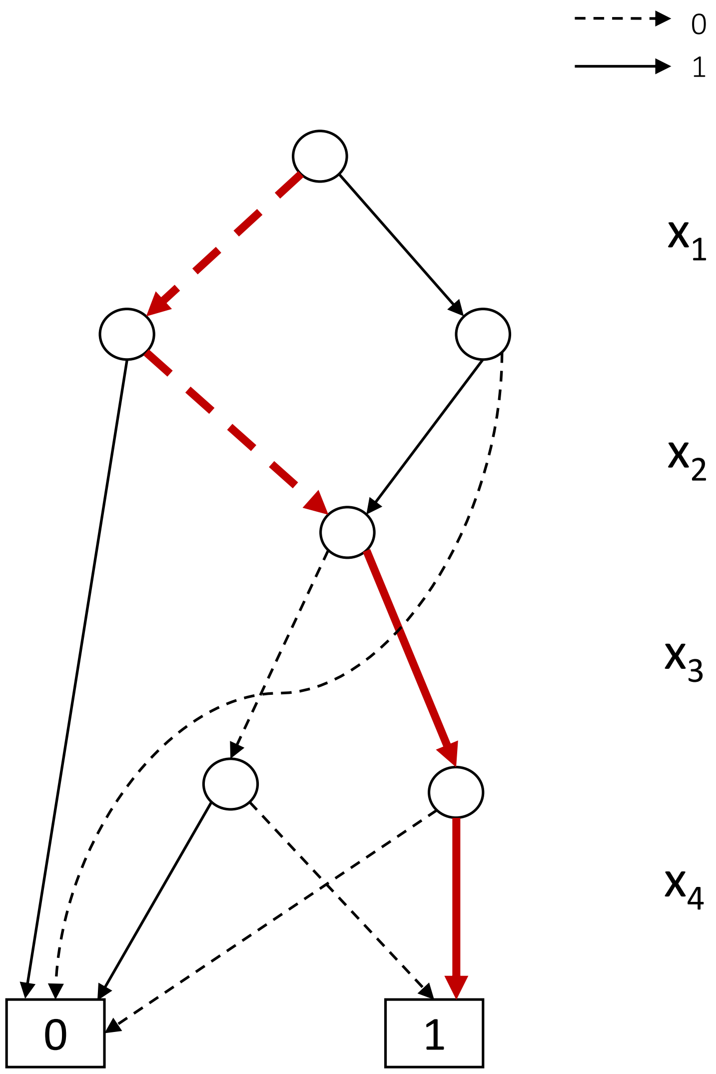

Decision Diagrams

- Graphical representation of **Boolean functions**

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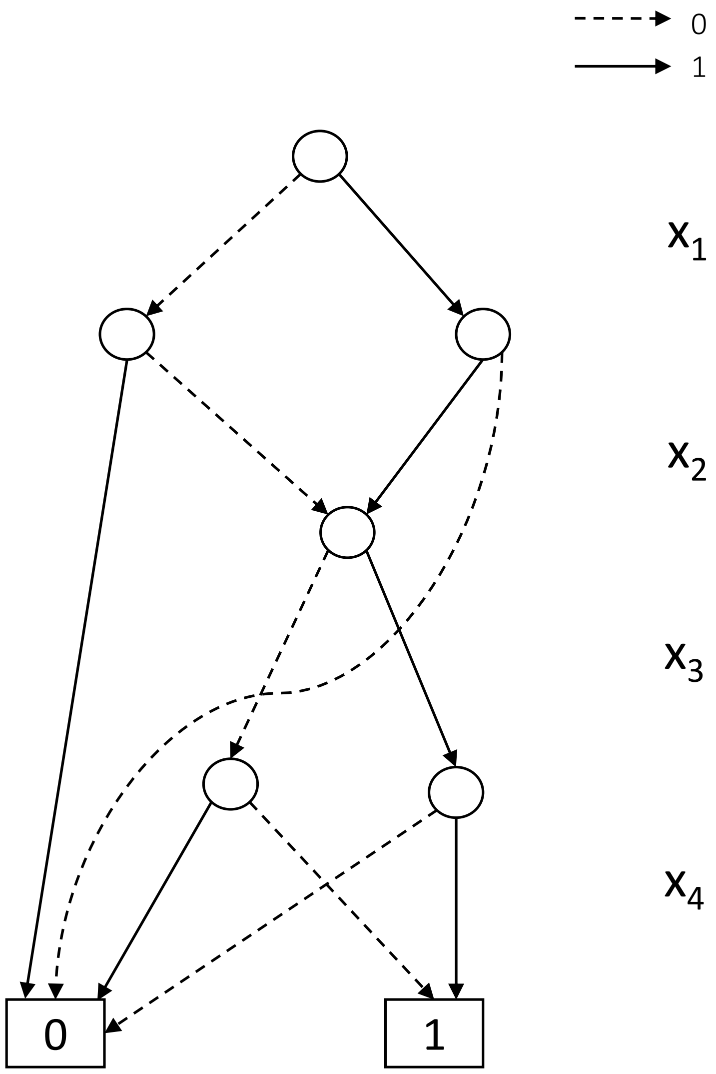
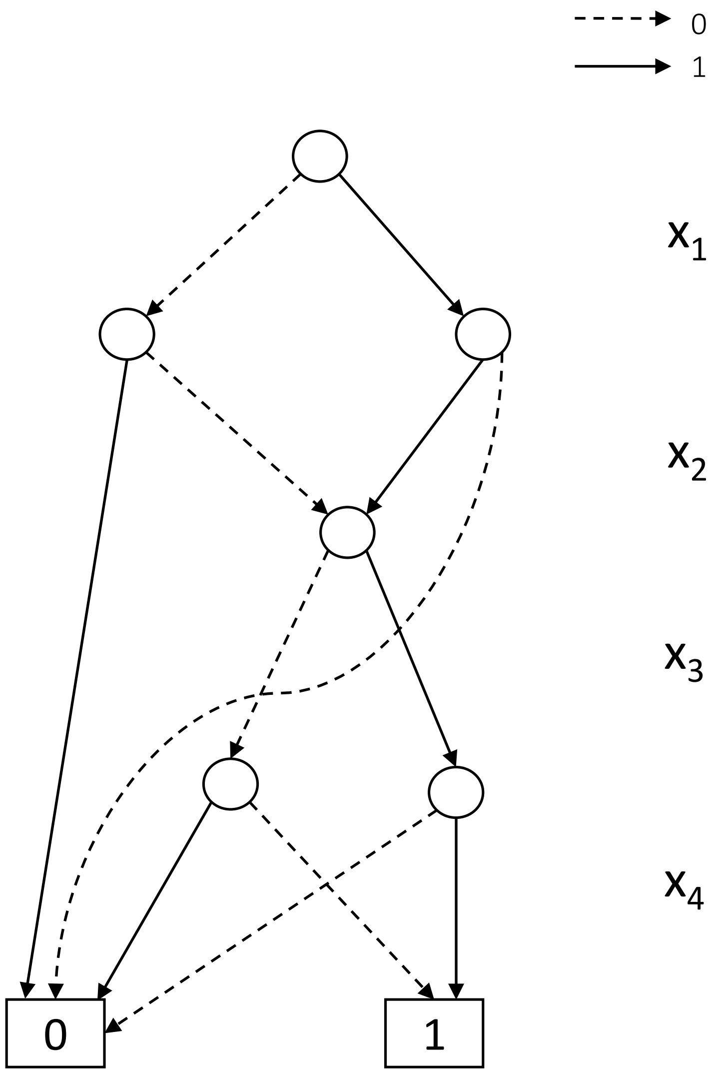
- **Dual role**
  - Computational model
  - Graphical encoding
- [Lee'59, Akers'78, Bryant'86]



Decision Diagrams

- **Application** in several areas
  - Circuit design
  - Formal verification
  - Symbolic model checking
  - …
- Our focus: **Optimization**
  - Literals → variables
  - Arcs → value assignments
  - Paths encode **solutions**

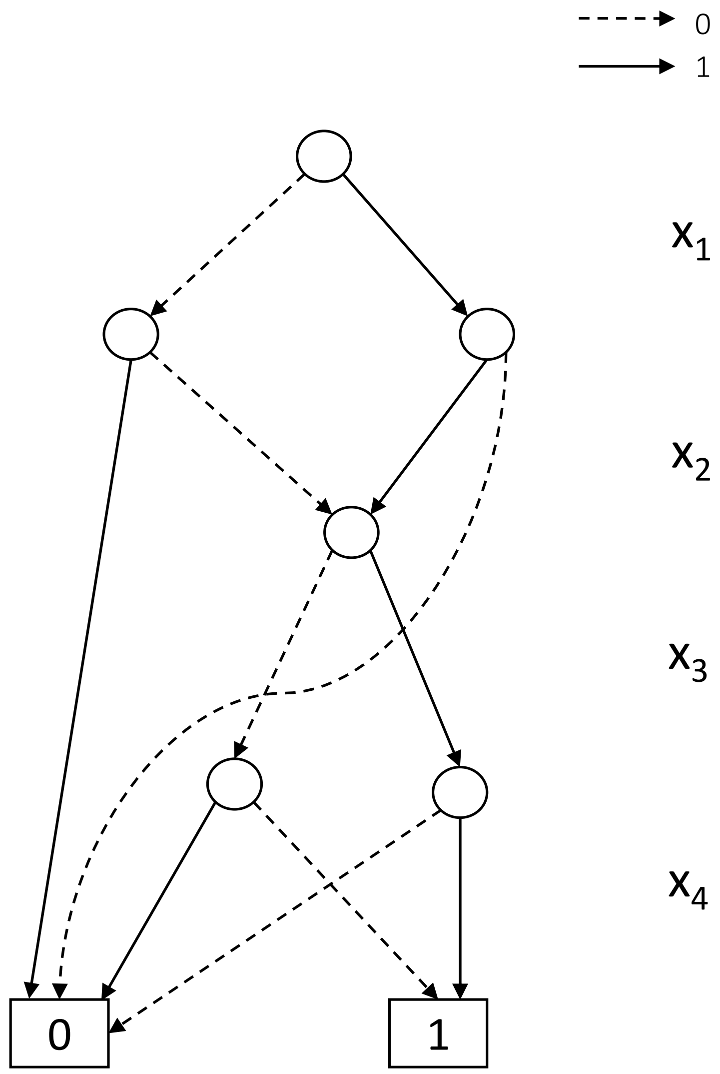


Decision Diagrams

max $2x_1 + x_2 - 4x_3 + x_4$

subject to

$$x_1 - x_2 = 0$$
$$x_3 - x_4 = 0$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$



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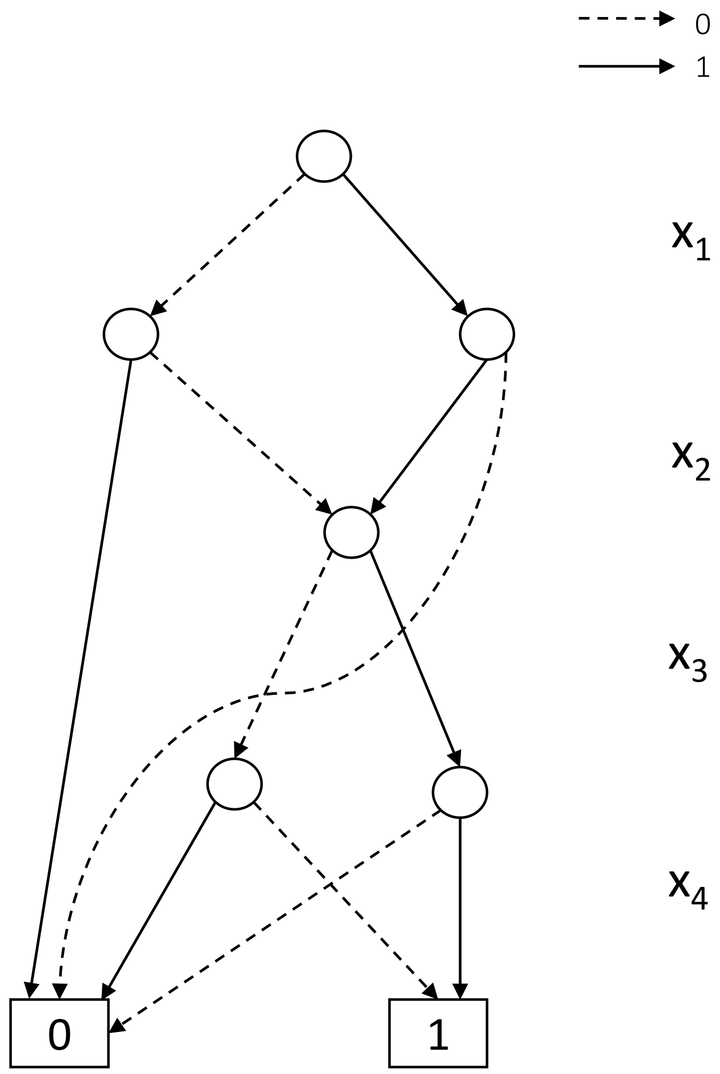

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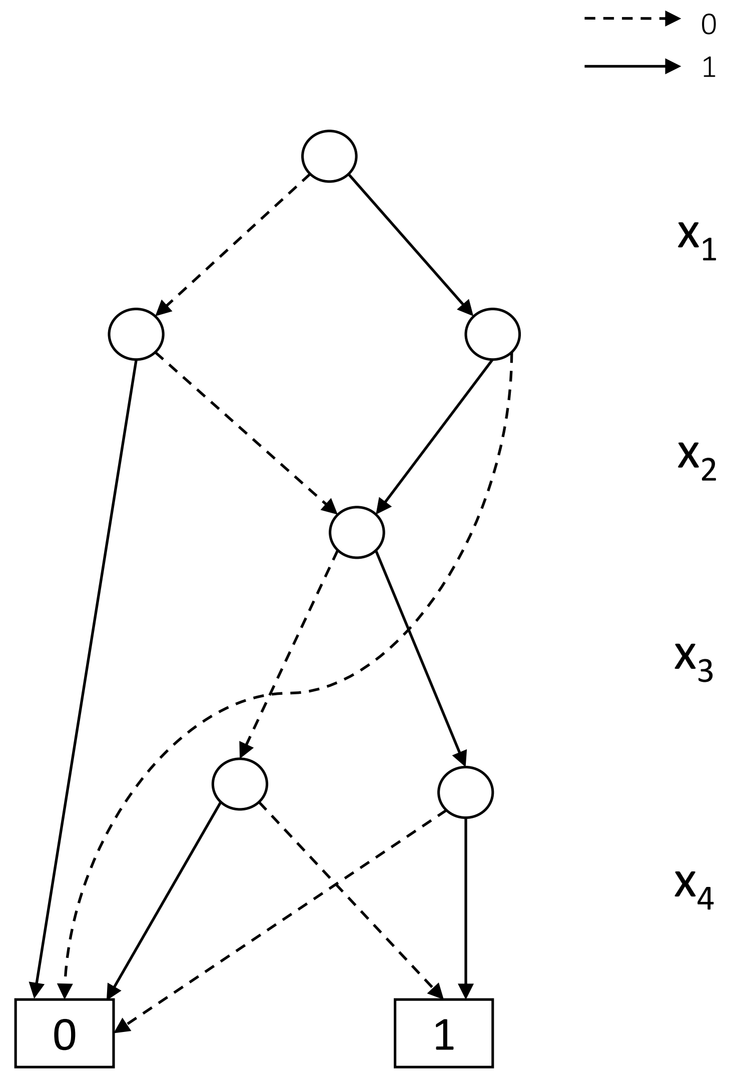

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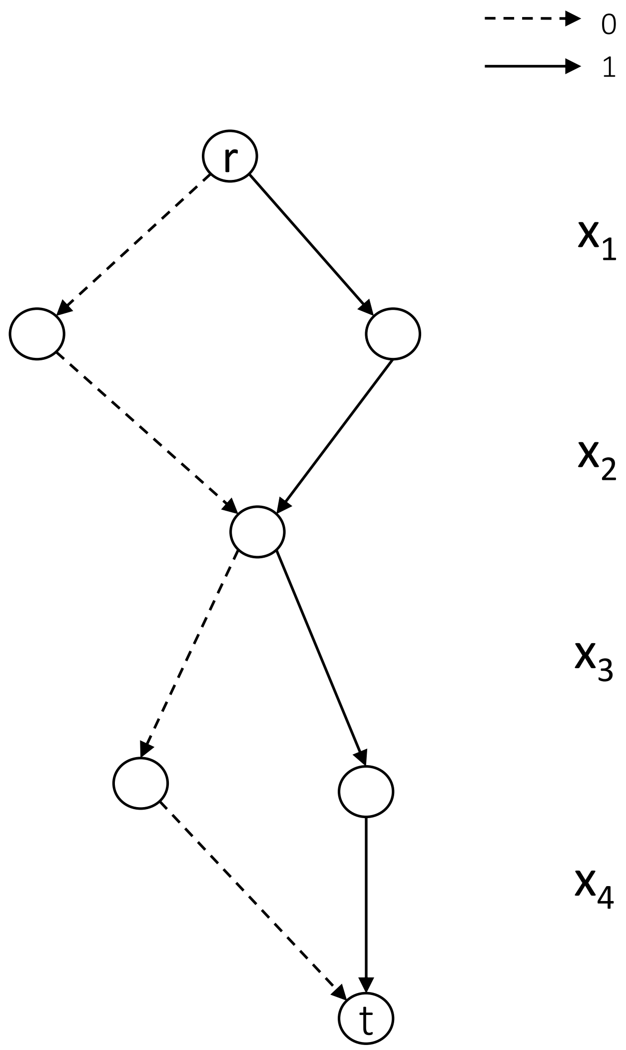

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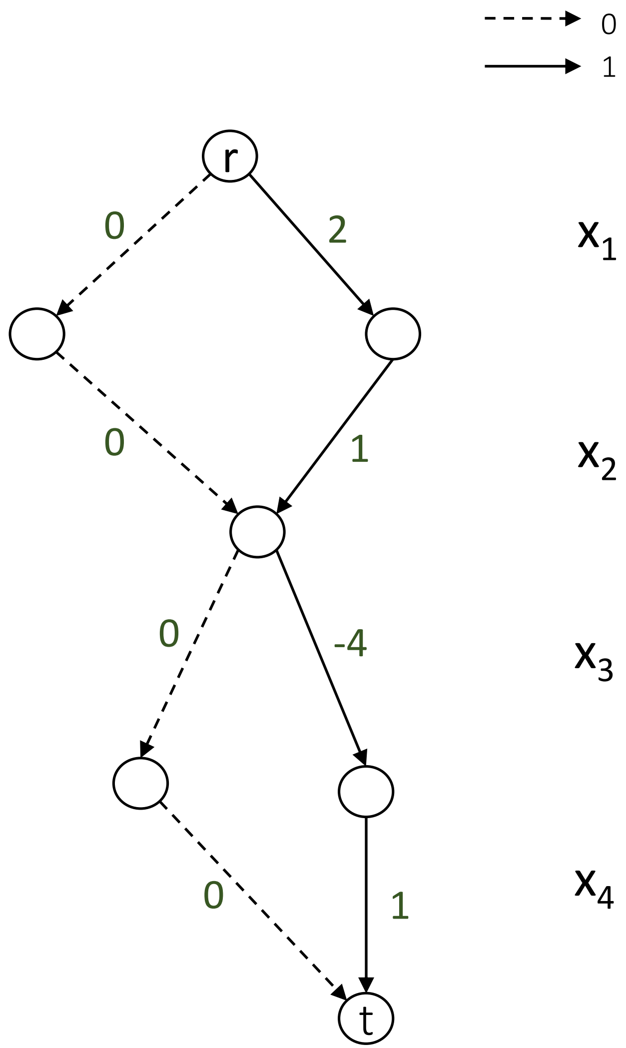
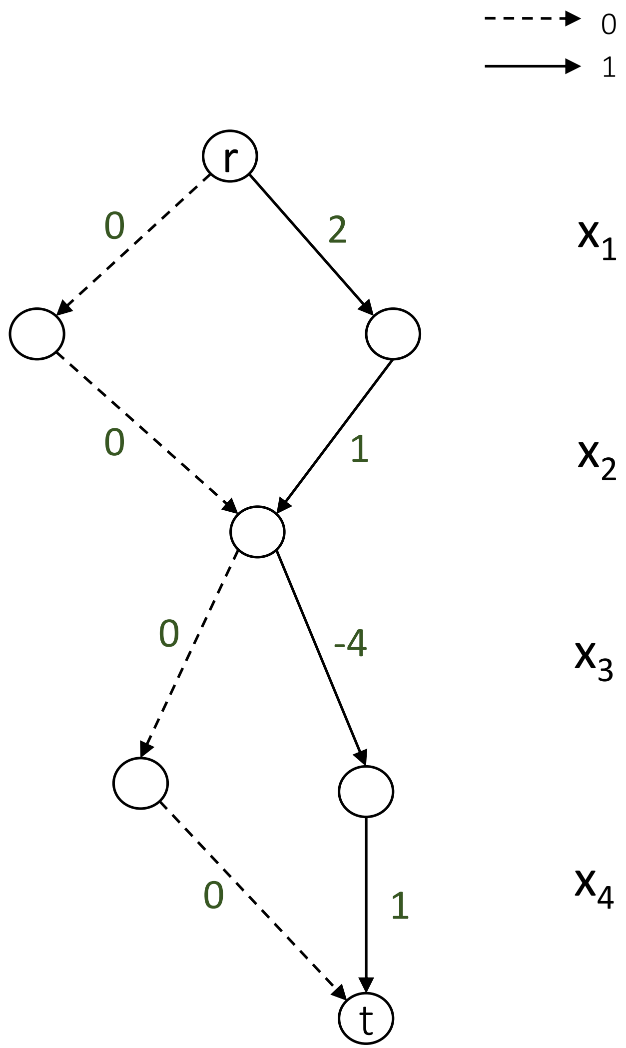


- Maximizing a linear (or separable) function:
  - Arc lengths: contribution to the objective
  - Longest path: optimal solution

Decision Diagrams

- Uses of this framework:
  - Solution counting (Lobbing'96)
  - Large-scale network flows **(Hachtel et al'97)**
  - Postoptimality analysis (Hadzic & Hooker'08)
  - Few others, typically **domain-specific**.

- Our goal: exploit the use of decision diagrams in **generic optimization methods**



Relaxation Methods

E.g., Linear Programming Relaxation

Modeling Framework

E.g., Linear Inequalities

Primal Heuristics

E.g., Feasibility Pump

Generic Optimization Techniques

E.g., Mixed-integer Programming

Inference

E.g., valid cuts

Search

E.g., Branch and bound

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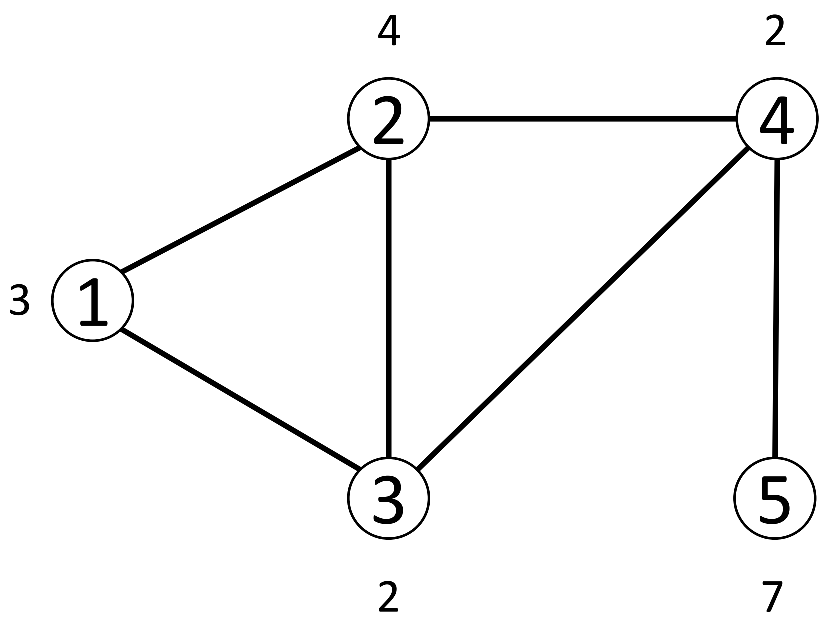
Modeling Framework

Ex.: Maximum independent set problem



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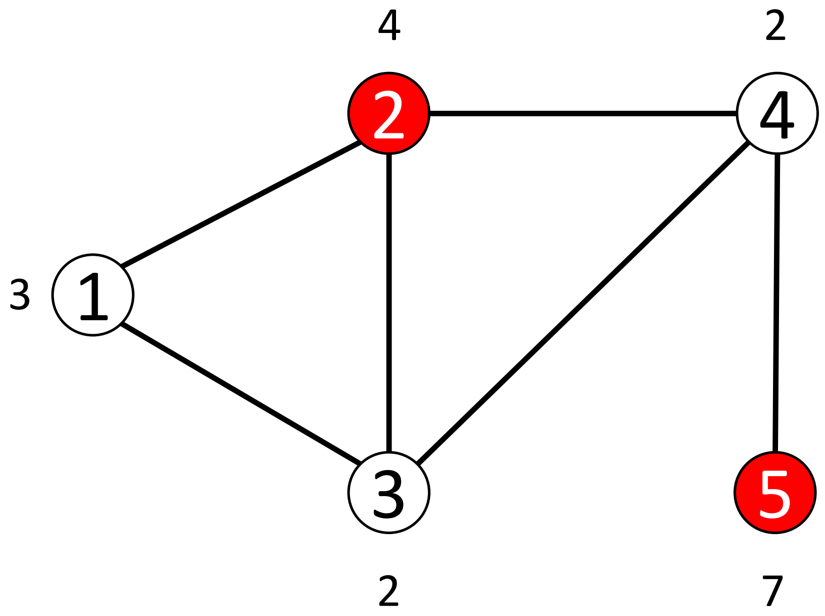

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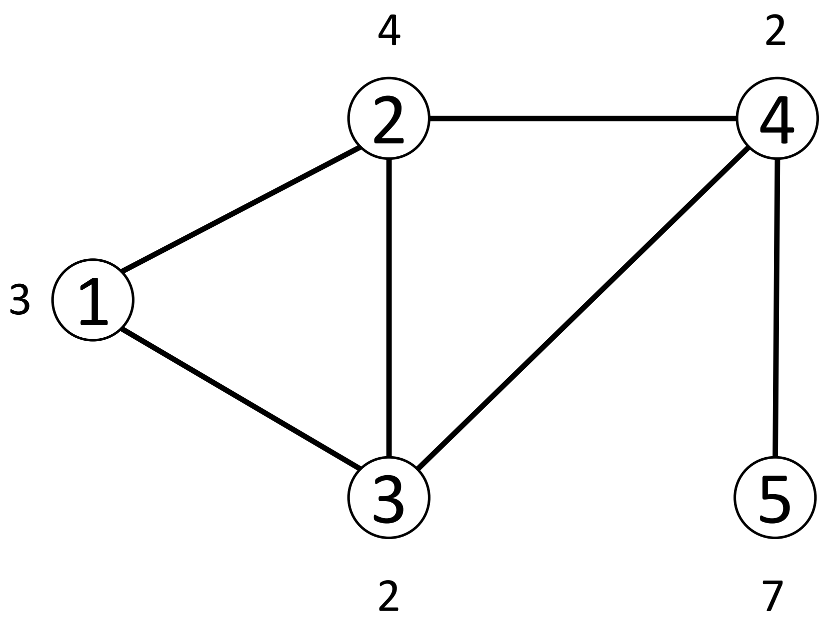

- Integer Programming Formulation:

max $3x_1 + 4x_2 + 2x_3 + 2x_4 + 7x_5$

subject to

$x_1 + x_2 \leq 1$

$x_1 + x_3 \leq 1$

$x_2 + x_3 \leq 1$

$x_3 + x_4 \leq 1$

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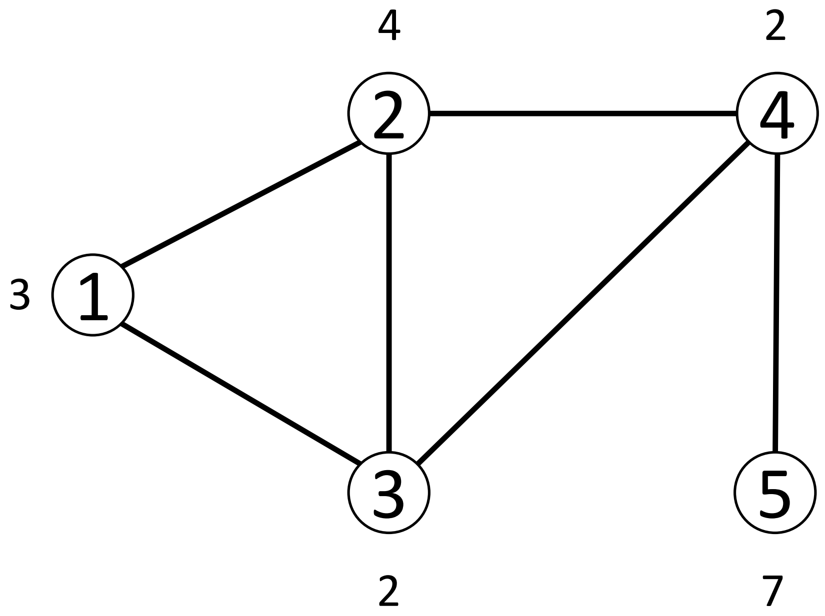


- Our model: **Dynamic Programming**
  - Exploit *recursiveness*
  - Model is formulated through **states**
  - **Decisions** (or *controls):* define state transitions

- Decision diagram: **State-Transition Graph**
  - **Nodes** corresponds to **states**
  - **Arcs** are **state transitions**
  - **Arc weights** are **transition costs**

Modeling Framework

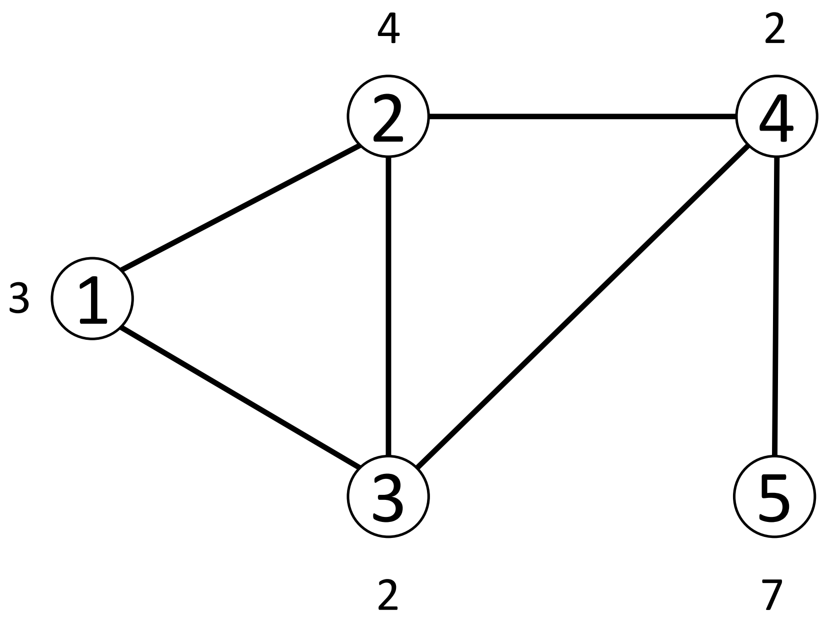
- DP model for the maximum independent set:
  - **State:** vertices that can be added to an independent set (**eligible vertices**)
  - **Decision:** select or not a vertex i from the eligibility set

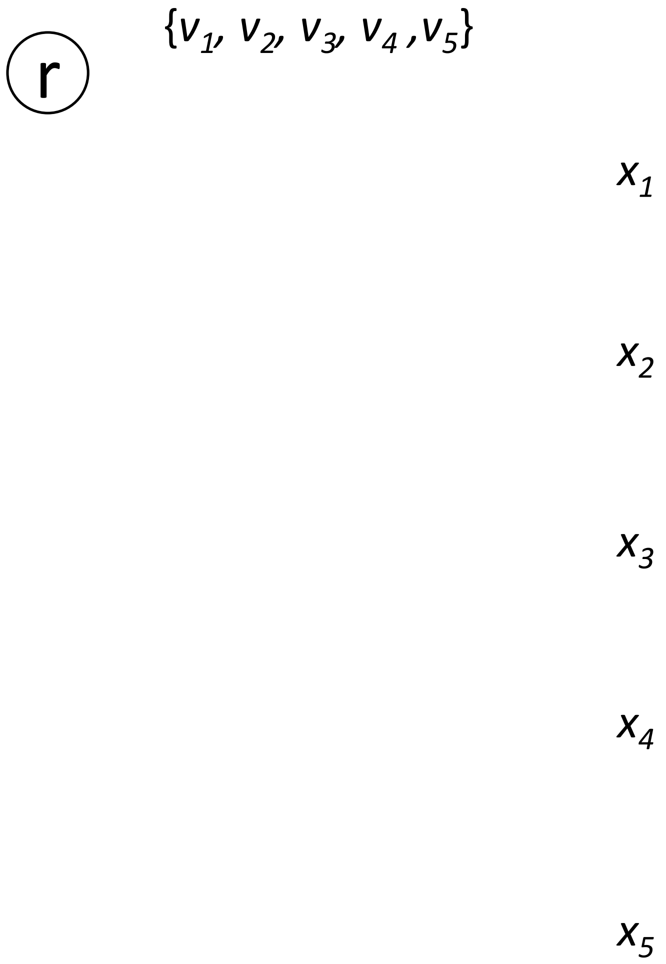
- Formal model*:*

$$V_i(S) = \begin{cases} max\left\{V_{i-1}(S \setminus \{i\}), V_{i-1}\big(S \setminus N(i)\big) + 1\right\}, & i \in S \\ V_{i-1}\big(S \setminus N(i)\big), & o.w. \end{cases}$$

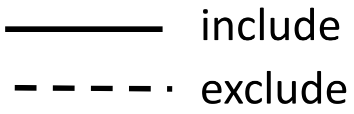
$$V_i(\emptyset) = 0, \qquad i = 1, \ldots, 5$$

Maximum Independent Set Problem





State: set of eligible vertices



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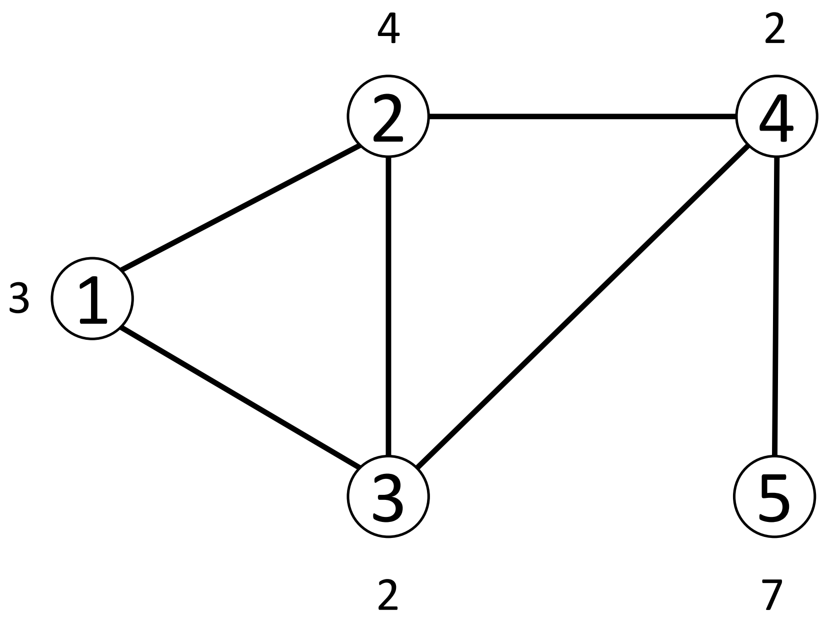

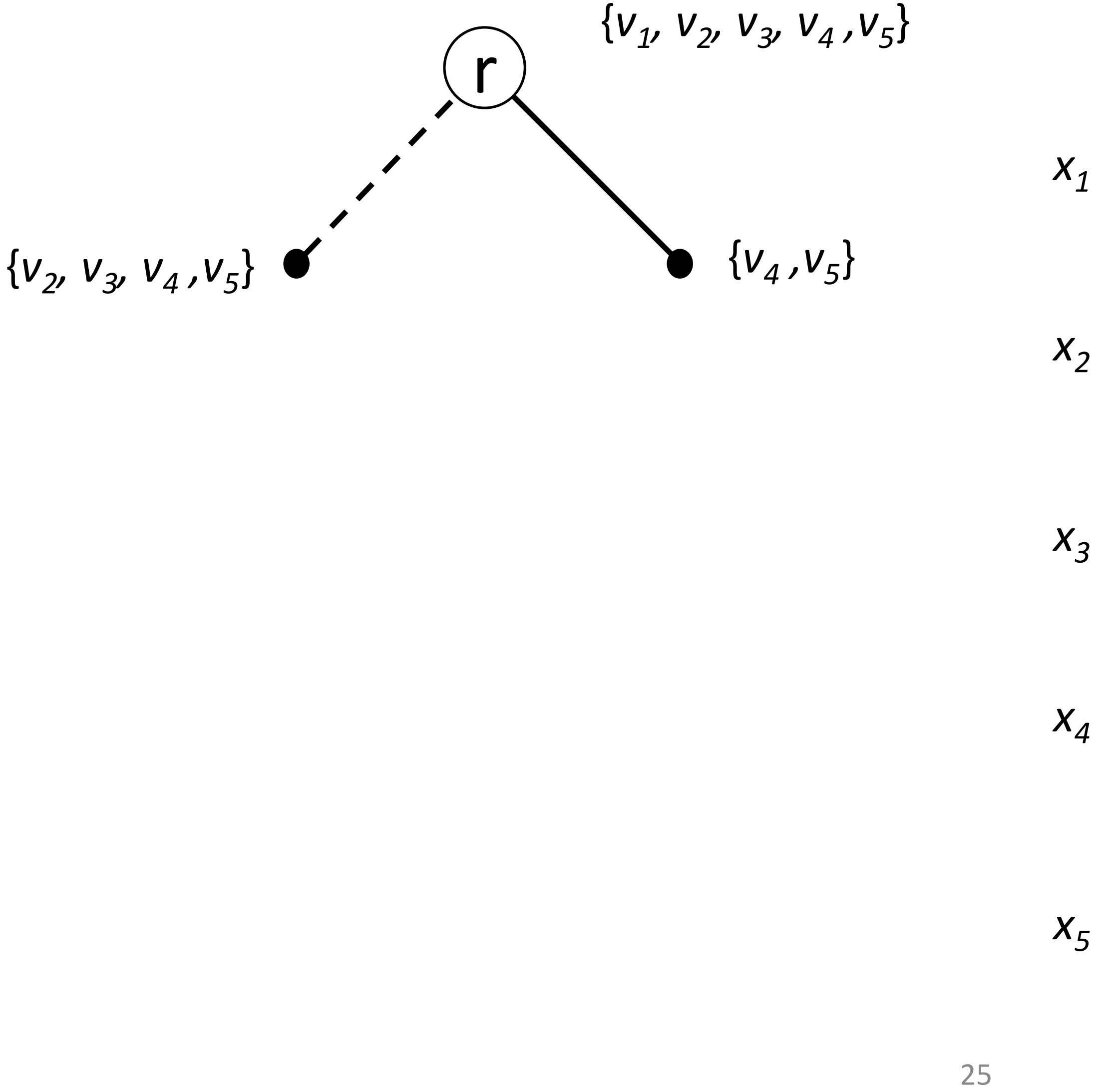

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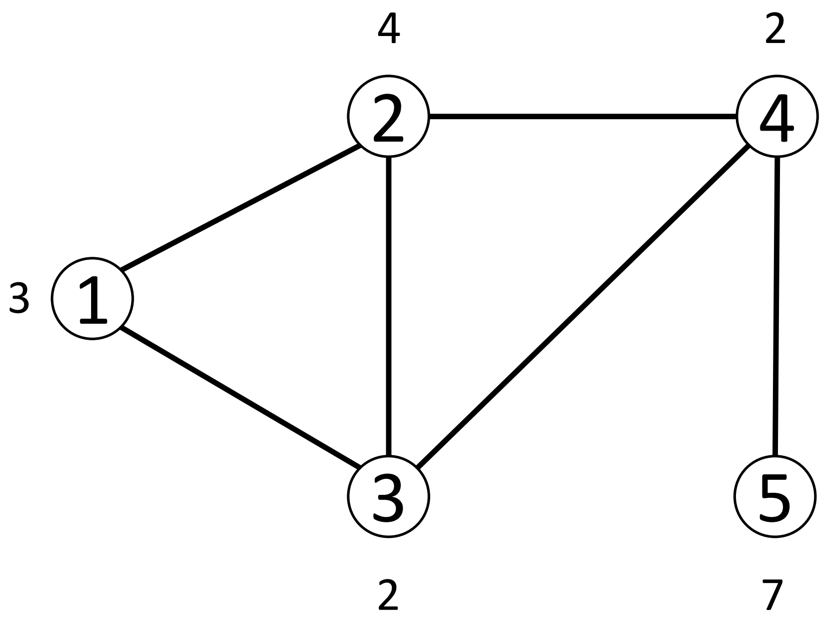

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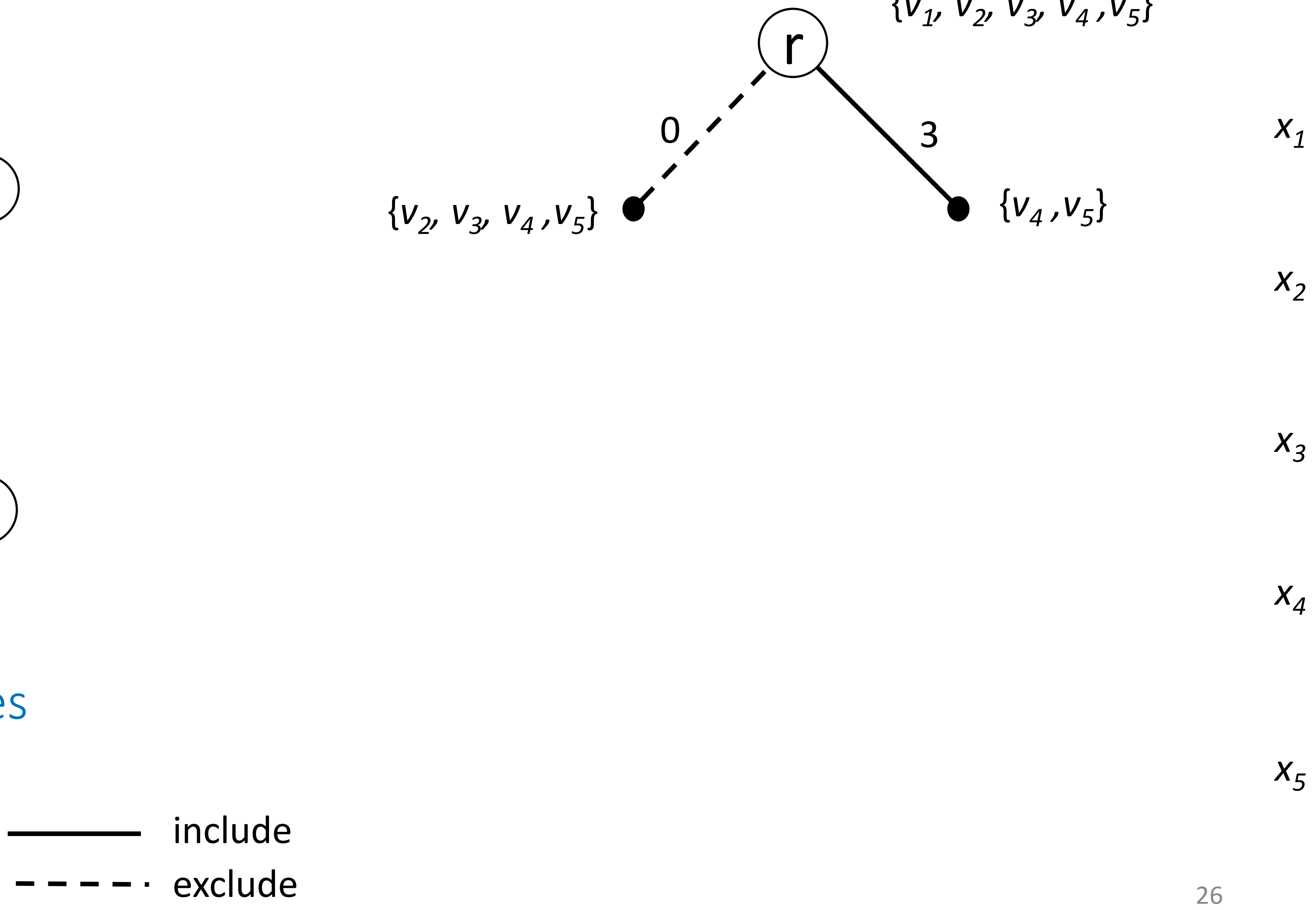

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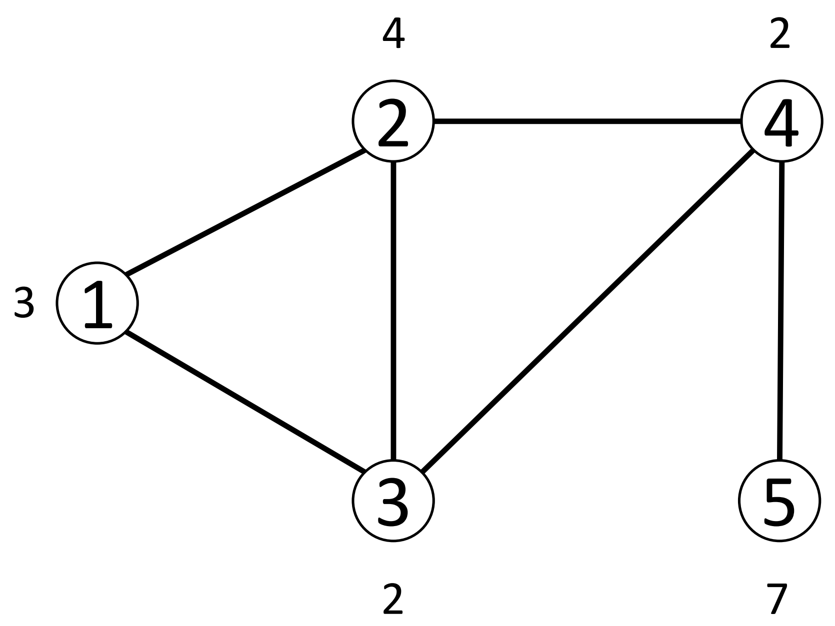

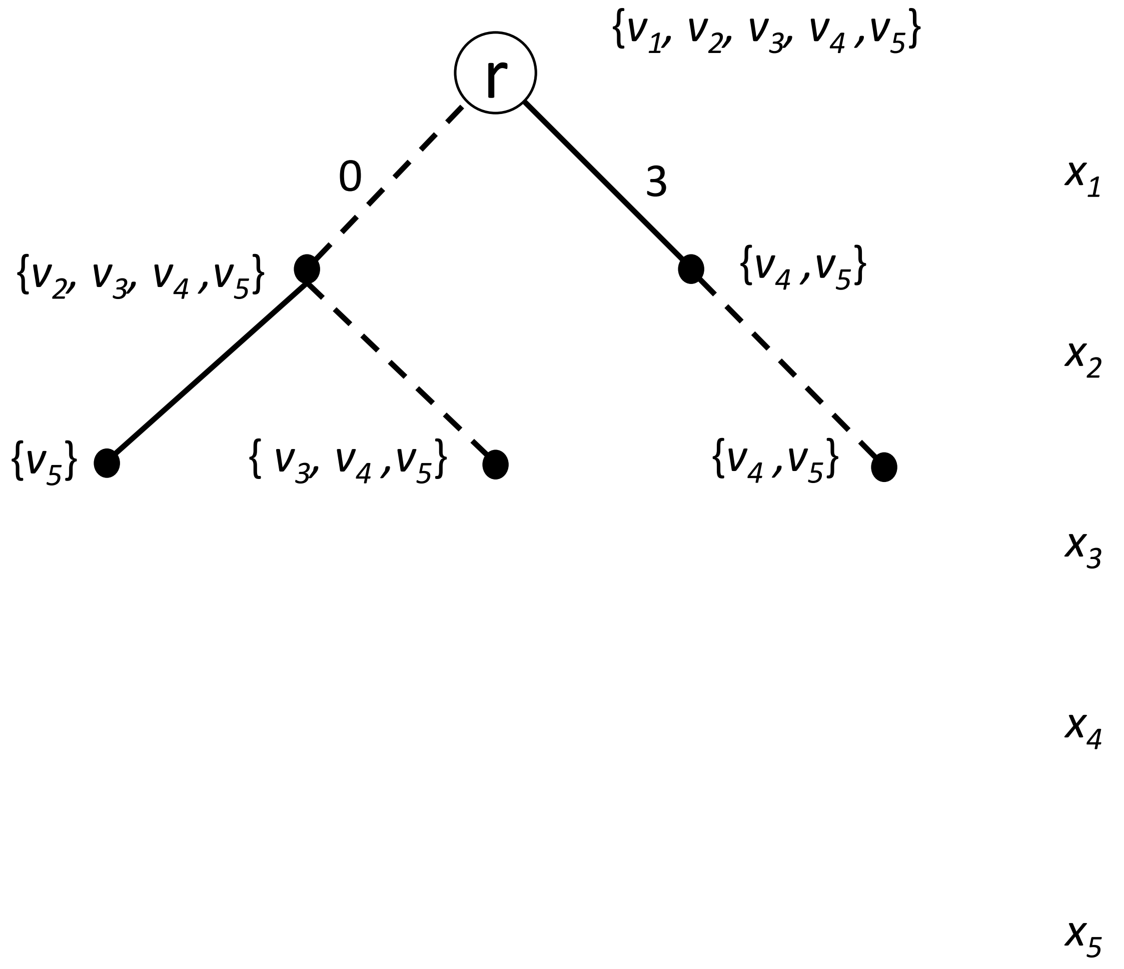

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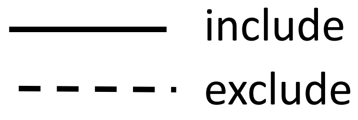

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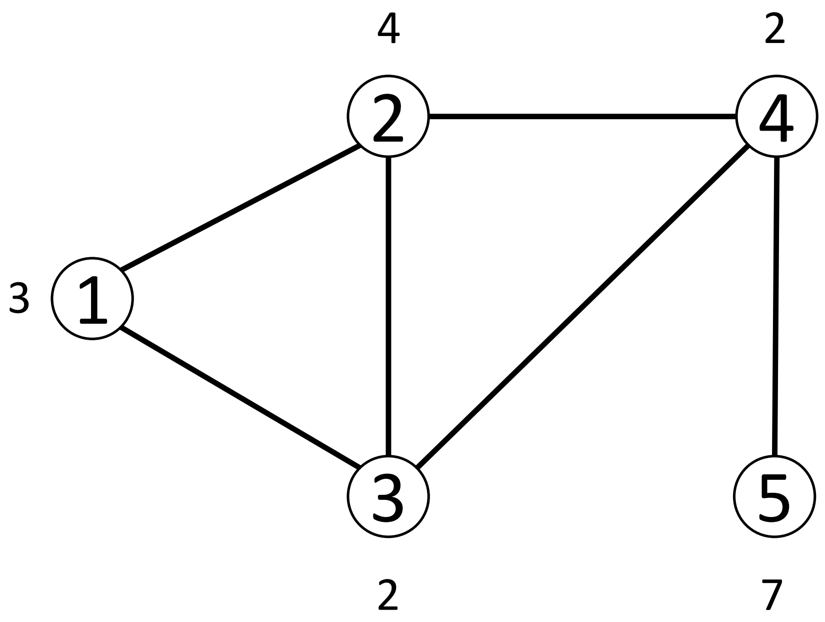

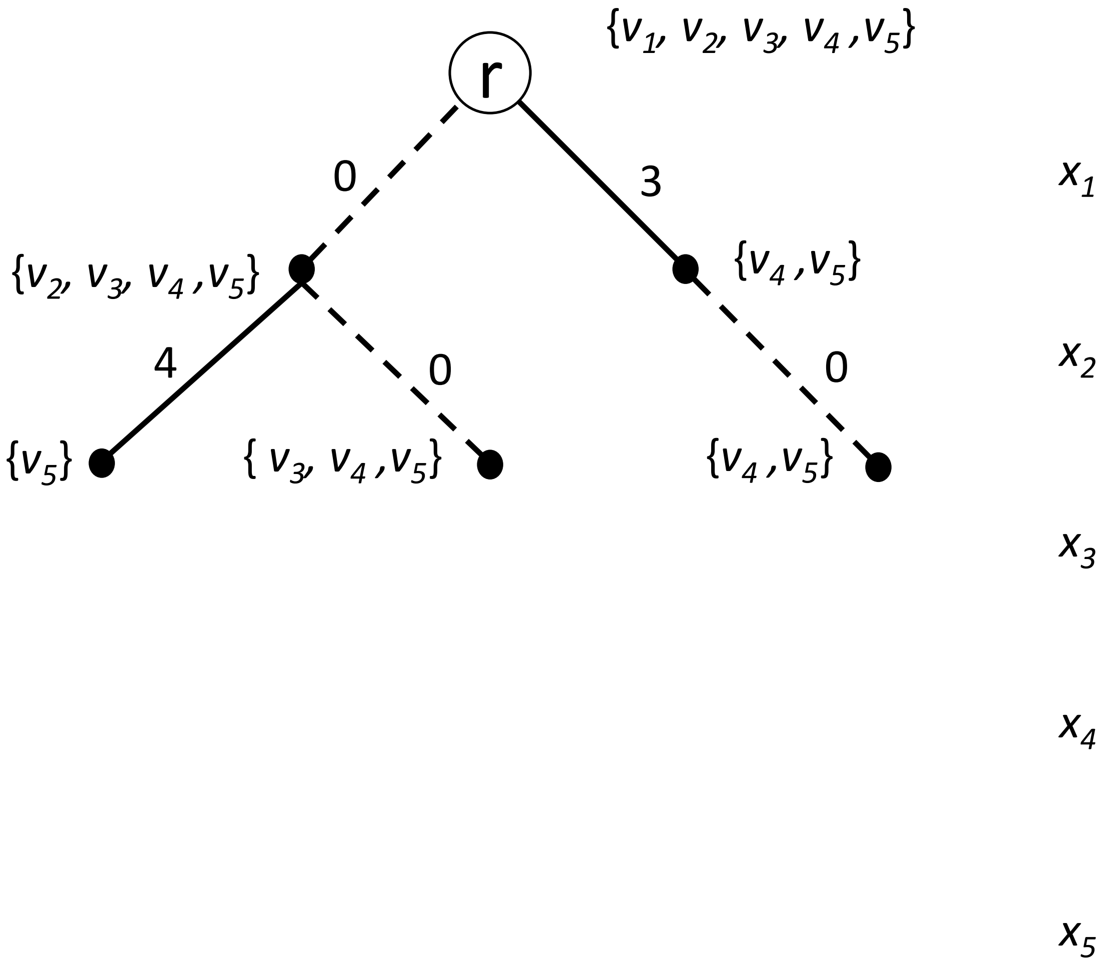

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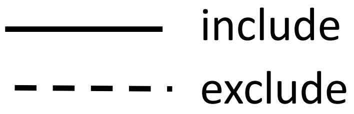

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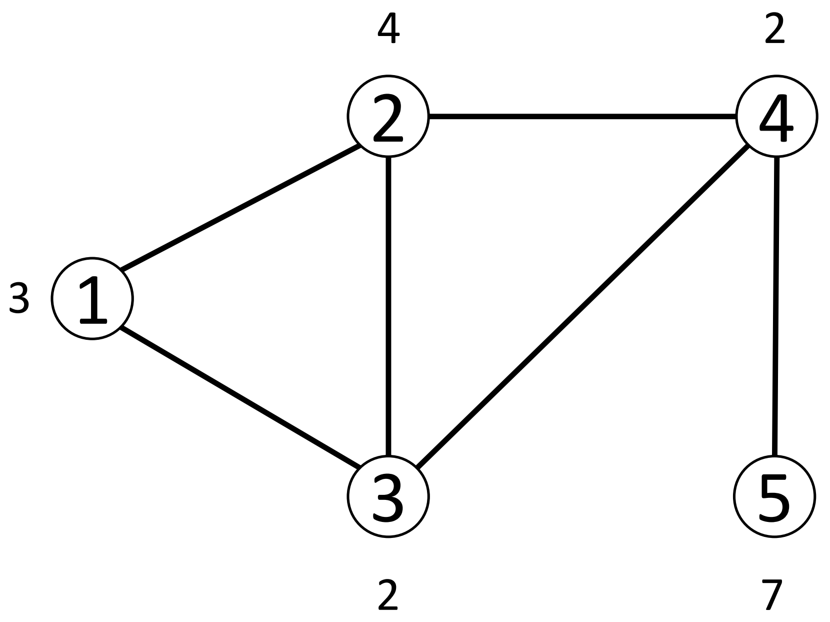

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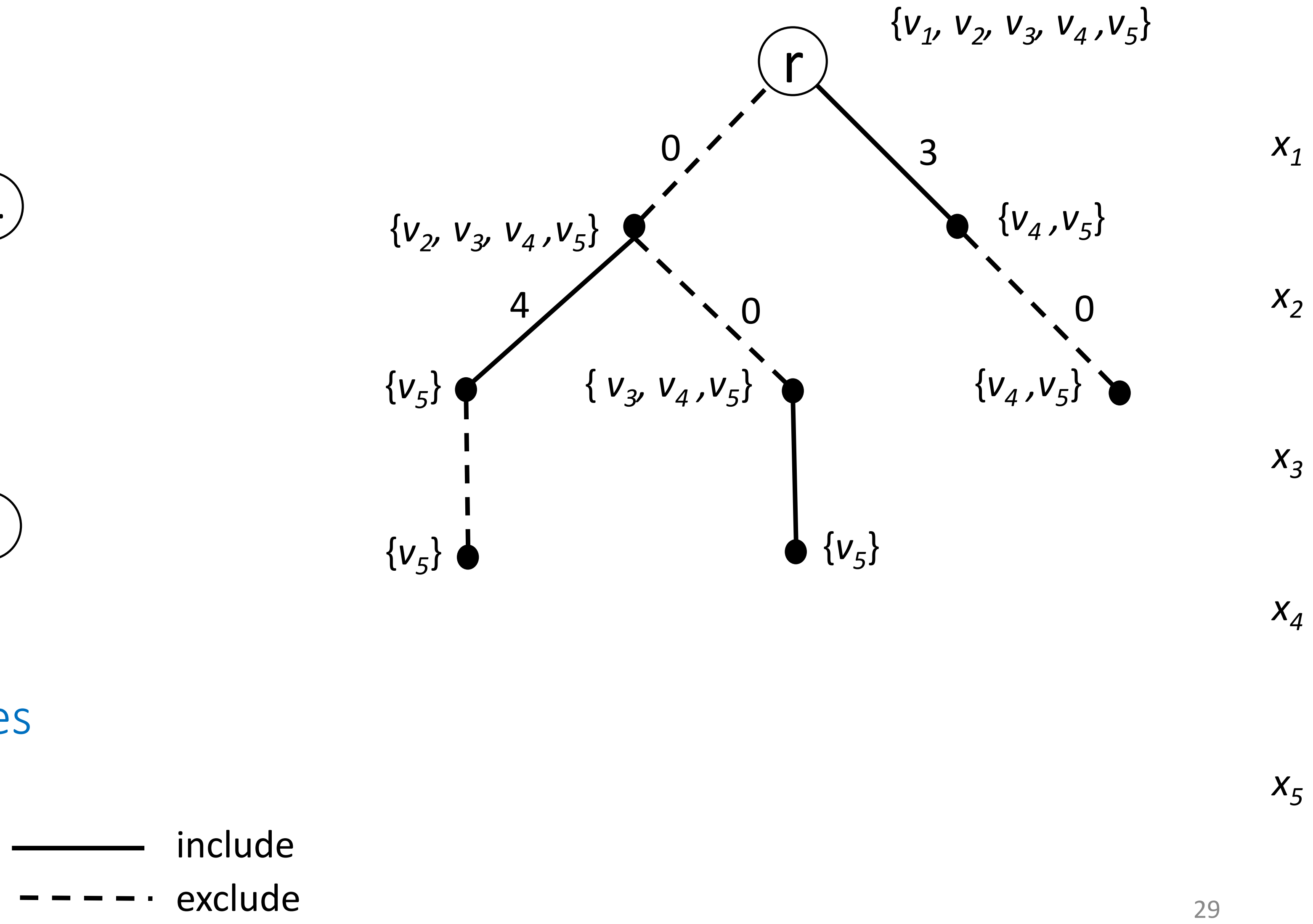

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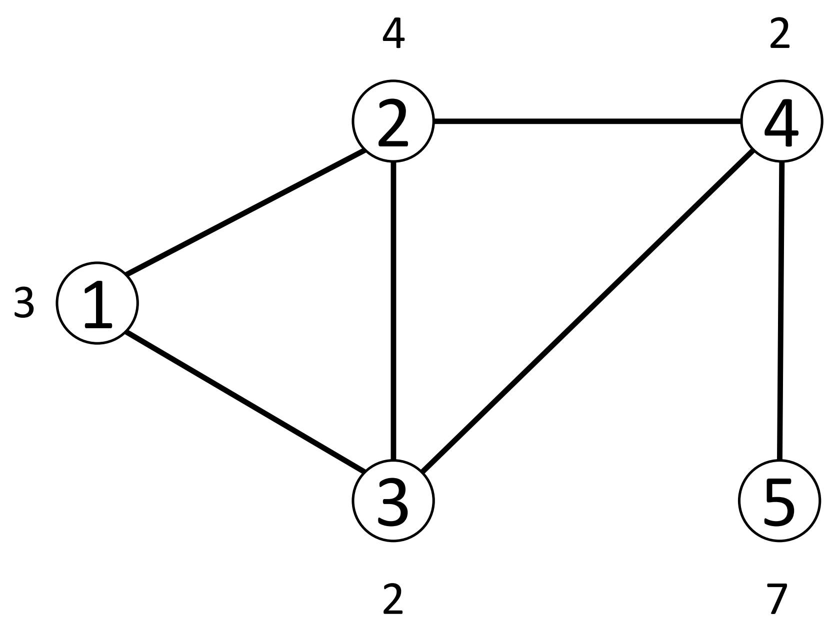

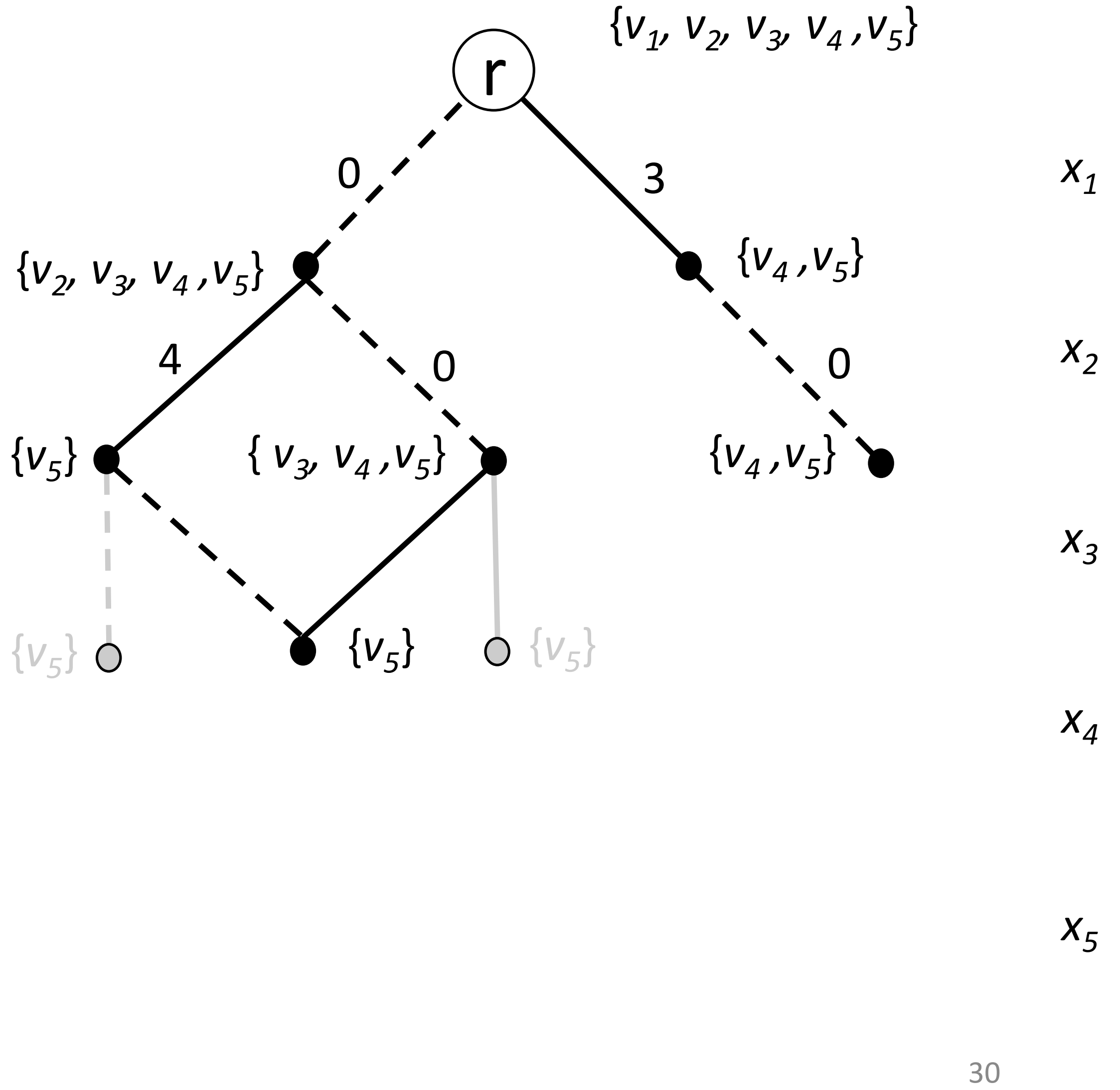

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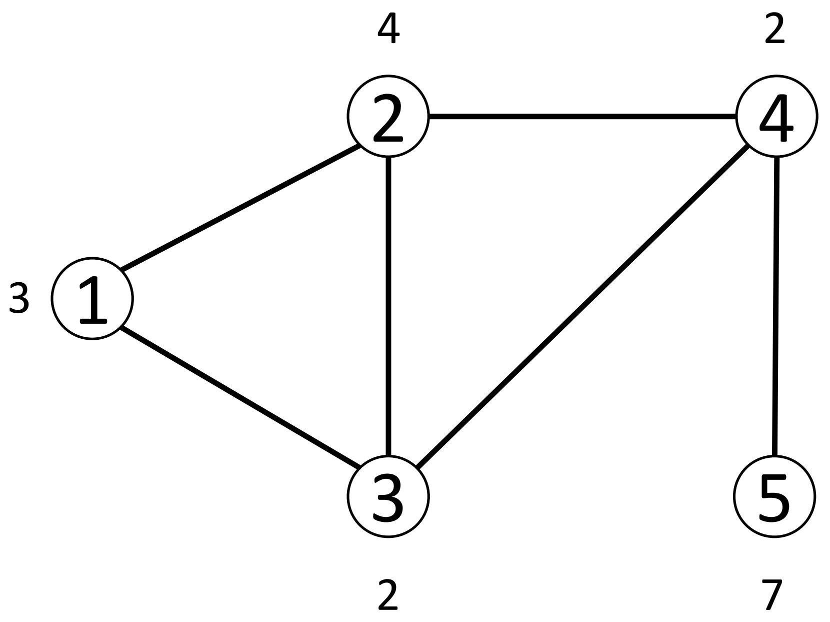

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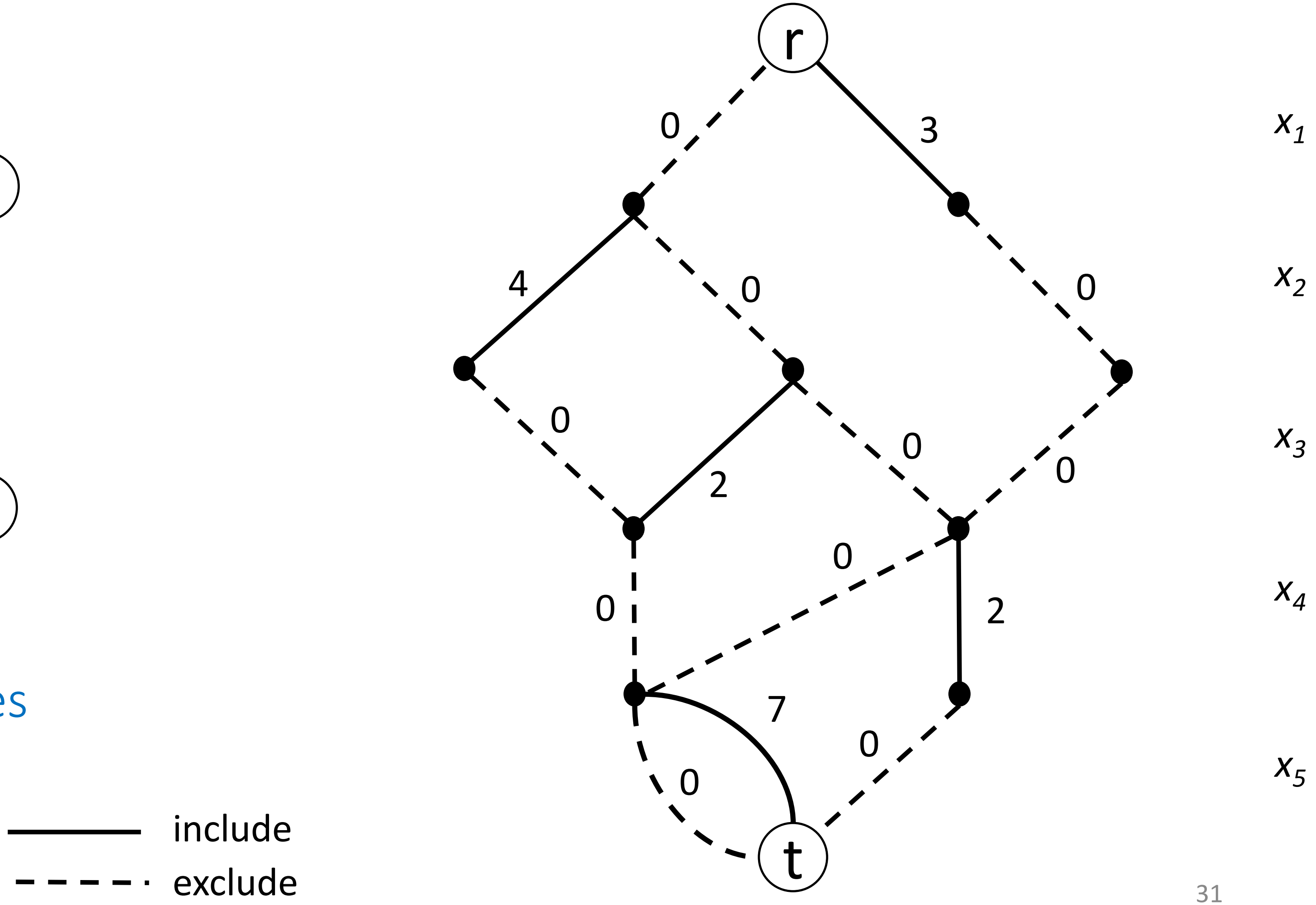
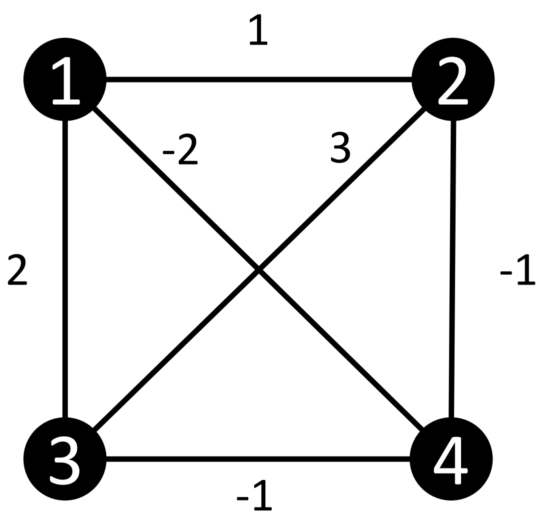
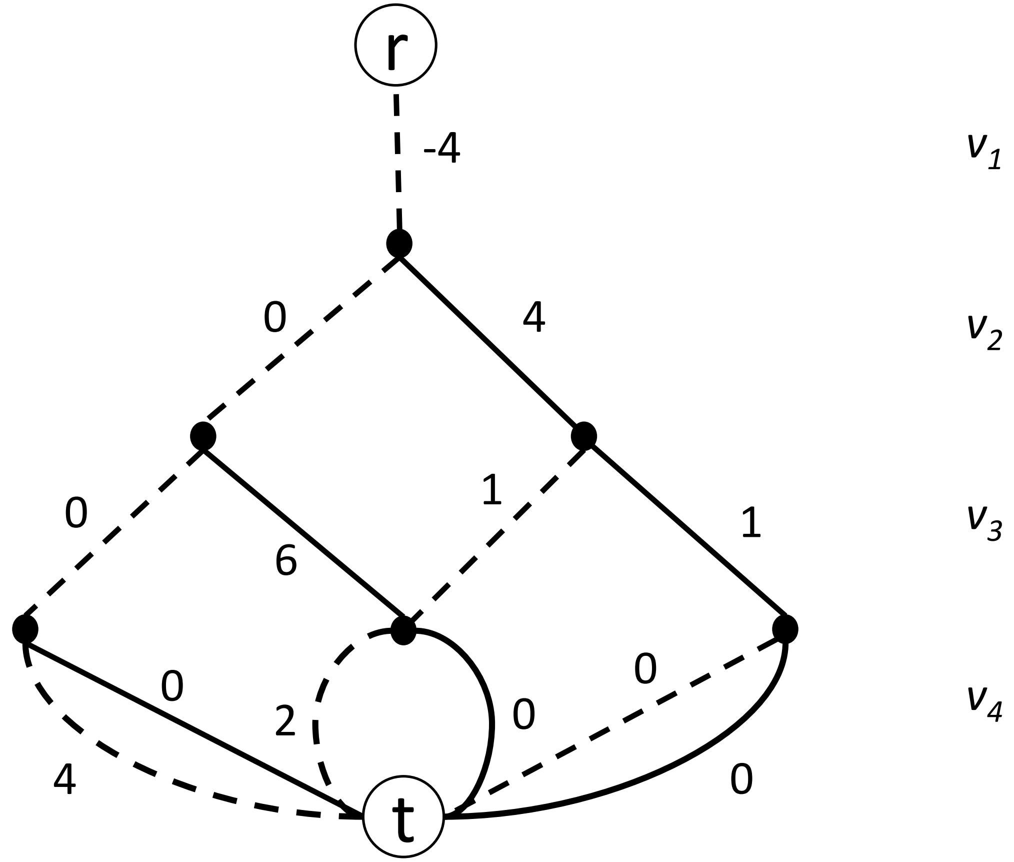
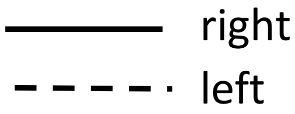


Other Example: Maximum Cut Problem







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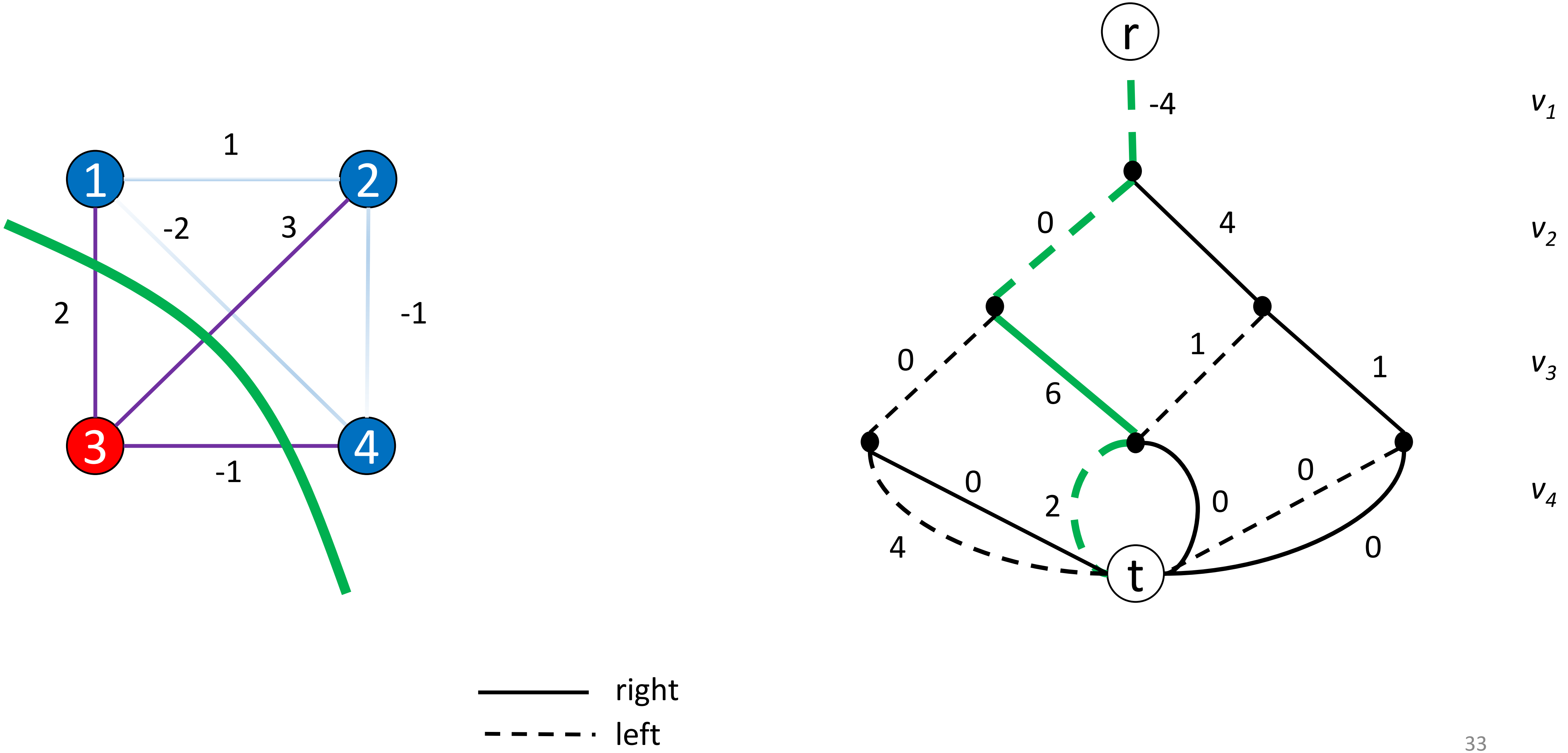


Some quick observations

- Variable ordering plays a big role on **size**
  - Closely connected to **treewidth** and **bandwidth**
  - Independent Set: polynomial for certain classes of graphs
  - TSP: parameterized-size depending on precendence relations

- In general, decision diagrams grow **exponentially** large
  - Proof: Extended Formulations for the Independent Set Problem

Relaxation Methods

E.g., Linear Programming Relaxation

Modeling Framework

E.g., Linear Inequalities

Generic Optimization Techniques

E.g., Mixed-integer Programming

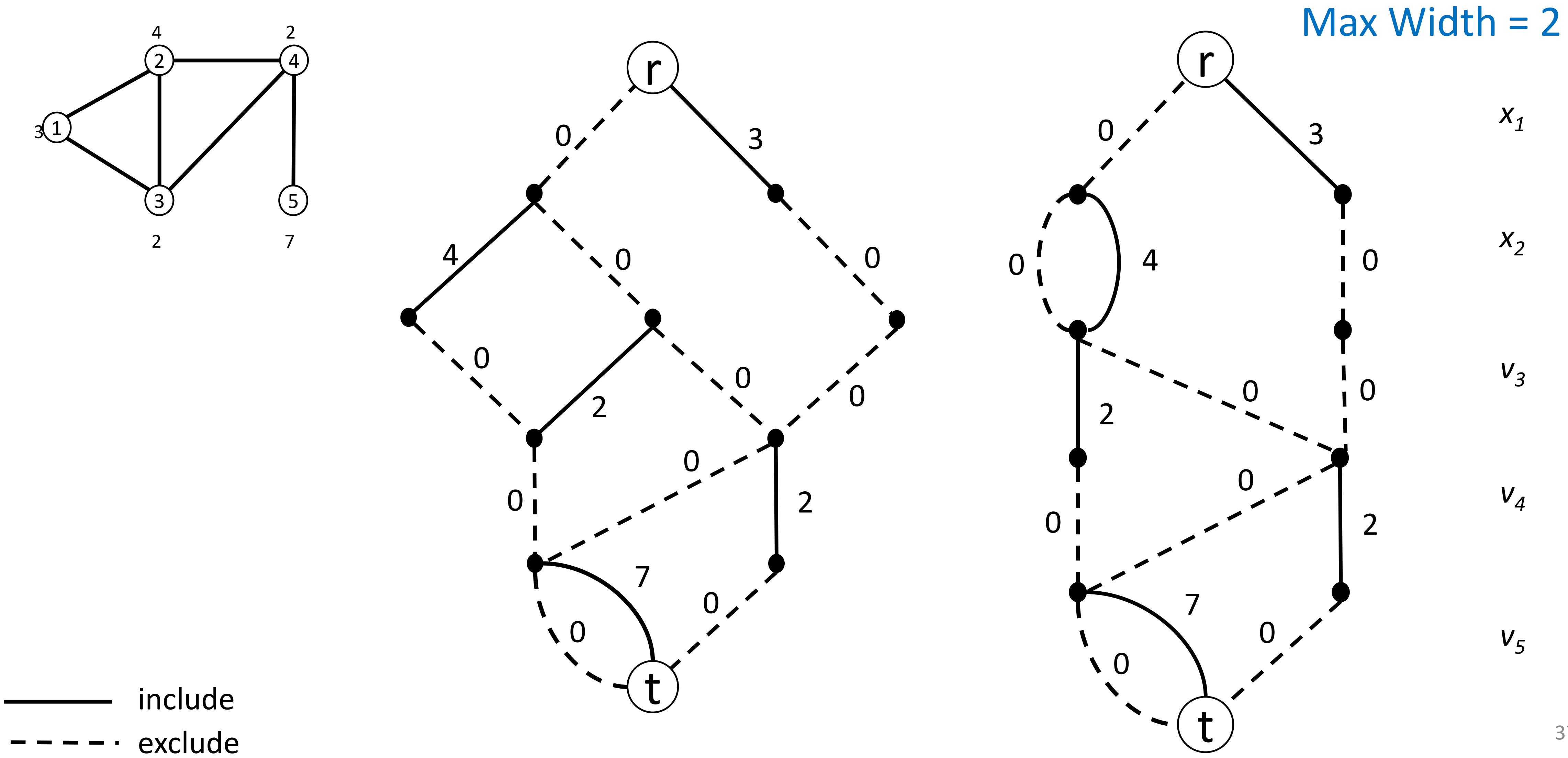
Relaxed Decision Diagrams

- In practice, we cannot work with exact diagrams

- Alternative: limit the size to **approximate** the feasible space
  - Parameter on the **width** of the diagram
  - *Relaxed Decision Diagrams:* Over-approximation

- Introduced by [Andersen et al'07]

Relaxed Decision Diagrams



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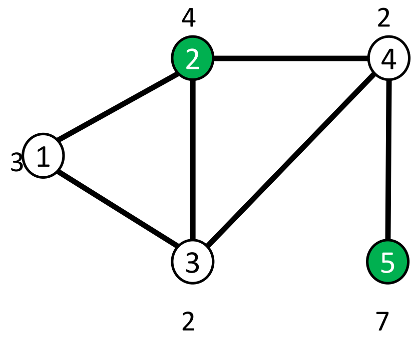
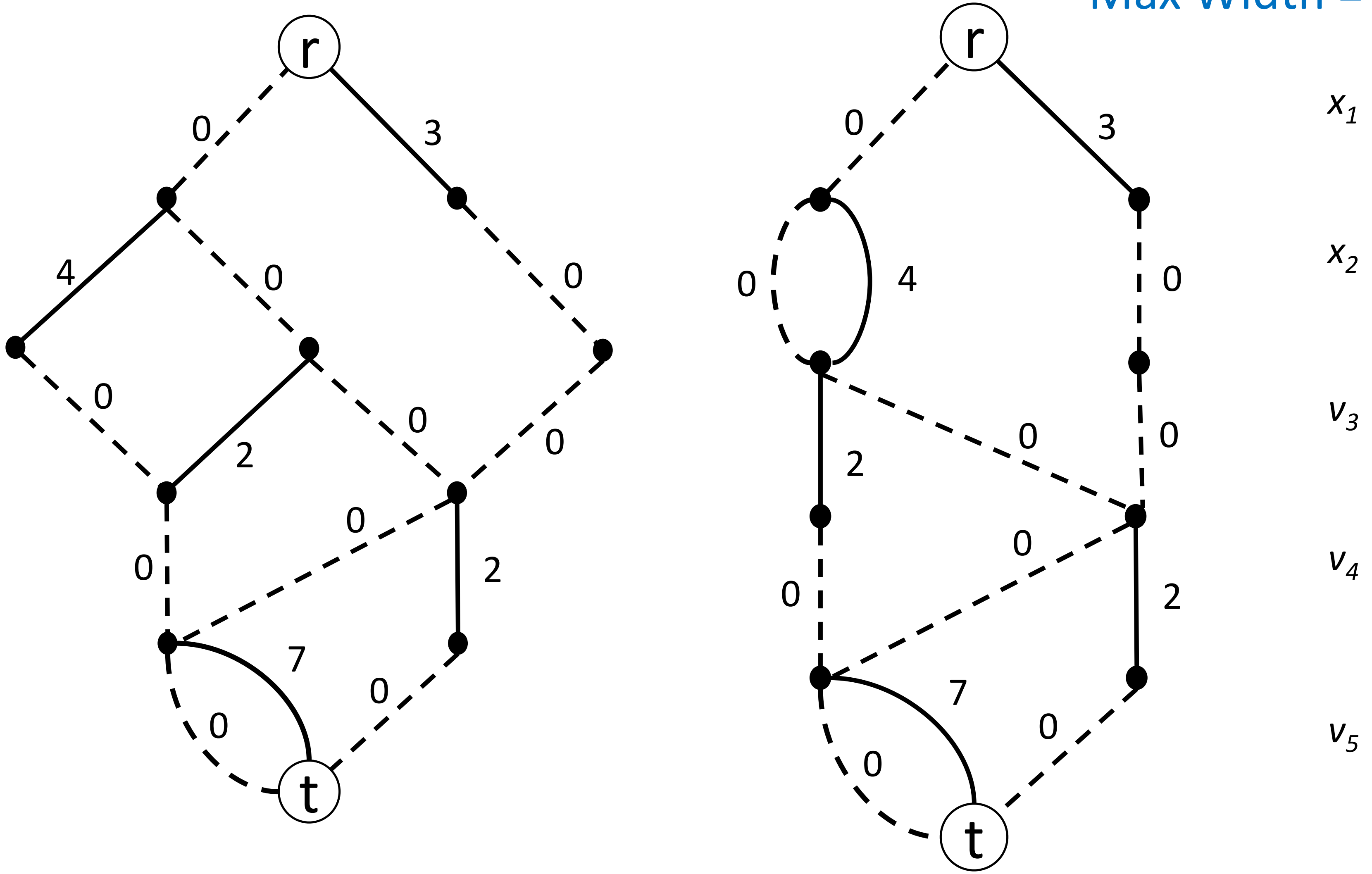


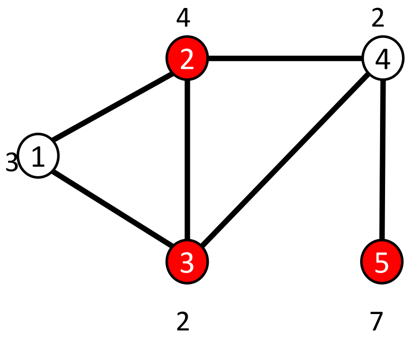
Max Width = 2

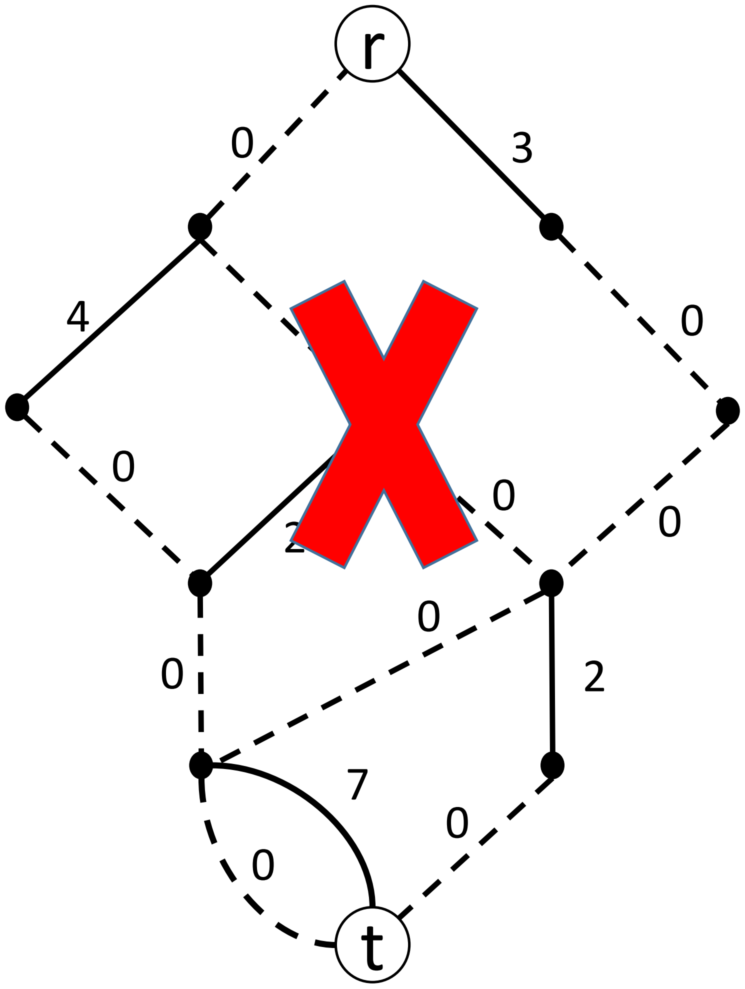


x = (0, 1, 0, 0, 1)
Solution value = 11

include
exclude

Relaxed Decision Diagrams





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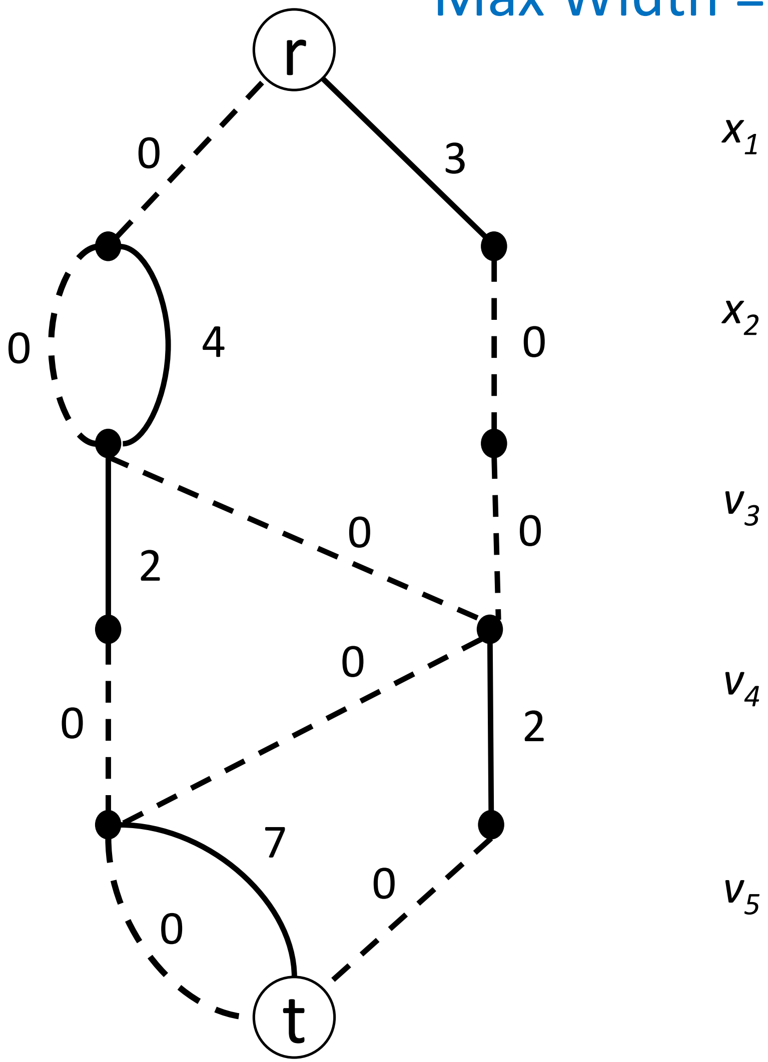


x = (0, 1, 1, 0, 1)
Upper bound = 13

include
exclude

Compiling Relaxed Decision Diagrams

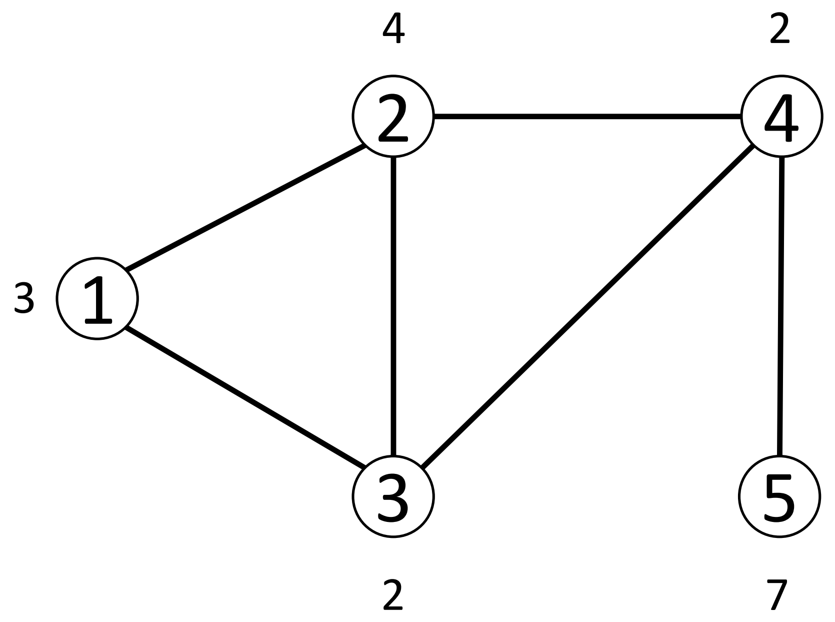
- Model is **augmented** with a **state agregation** operator
  - Recipe on how to merge nodes so that no feasible solution is lost

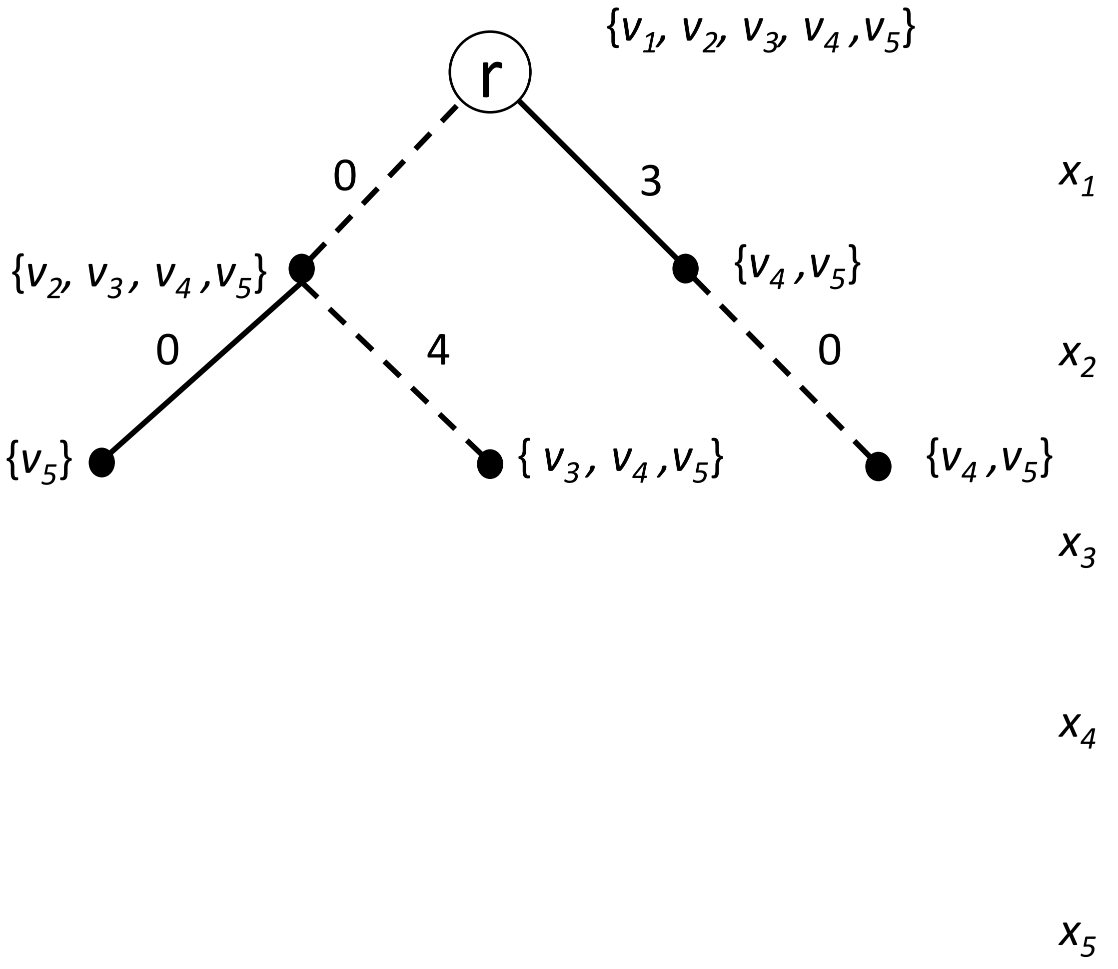
- $$V_i(S) = \begin{cases} max\,\{V_{i-1}(S \setminus \{i\}), V_{i-1}(S \setminus N(i)) + 1\}, & i \in S \\ V_{i-1}(S \setminus N(i)), & o.w. \end{cases}$$

  $$V_i(\emptyset) = 0, \qquad i = 1, \dots, 5$$

- $\Delta(S_1, S_2) = S_1 \cup S_2$

Building Relaxed Decision Diagrams

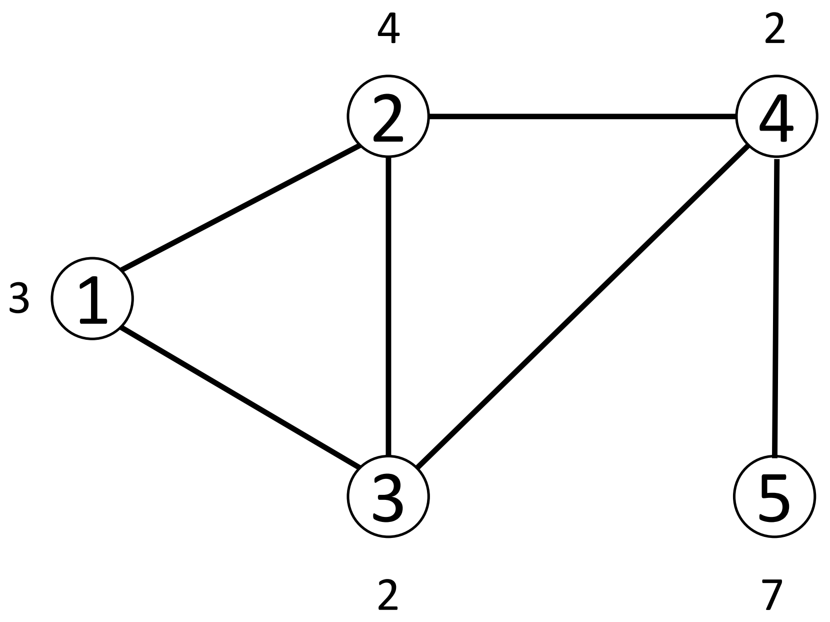




Max Width = 2

include
exclude

Building Relaxed Decision Diagrams



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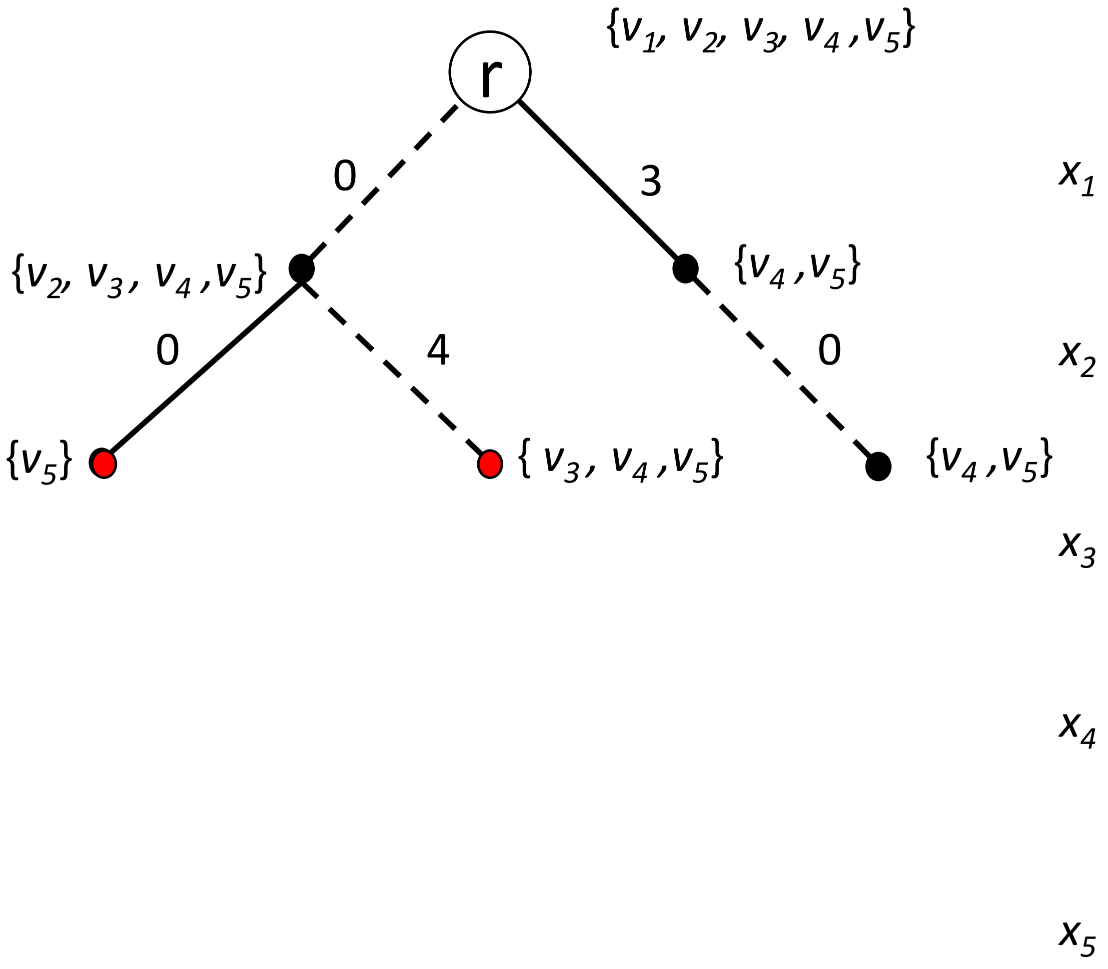
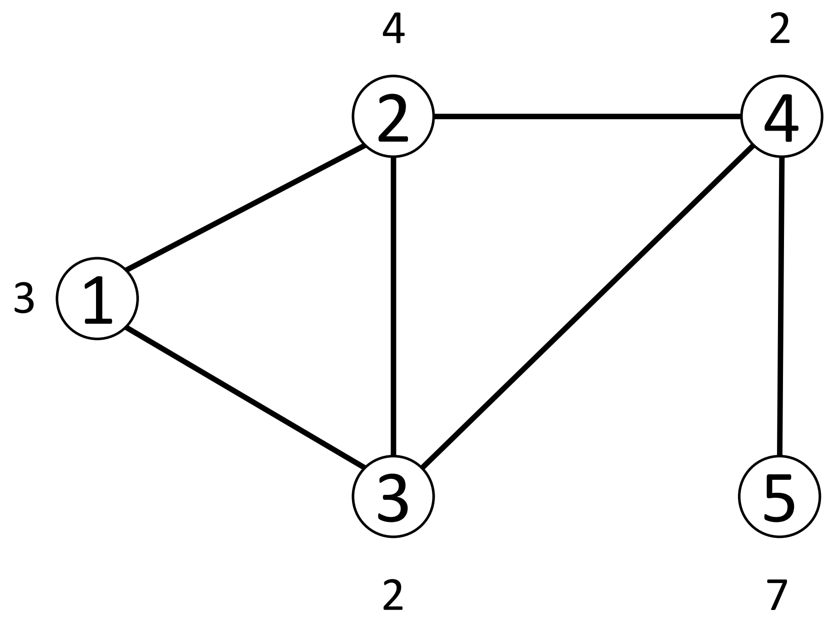


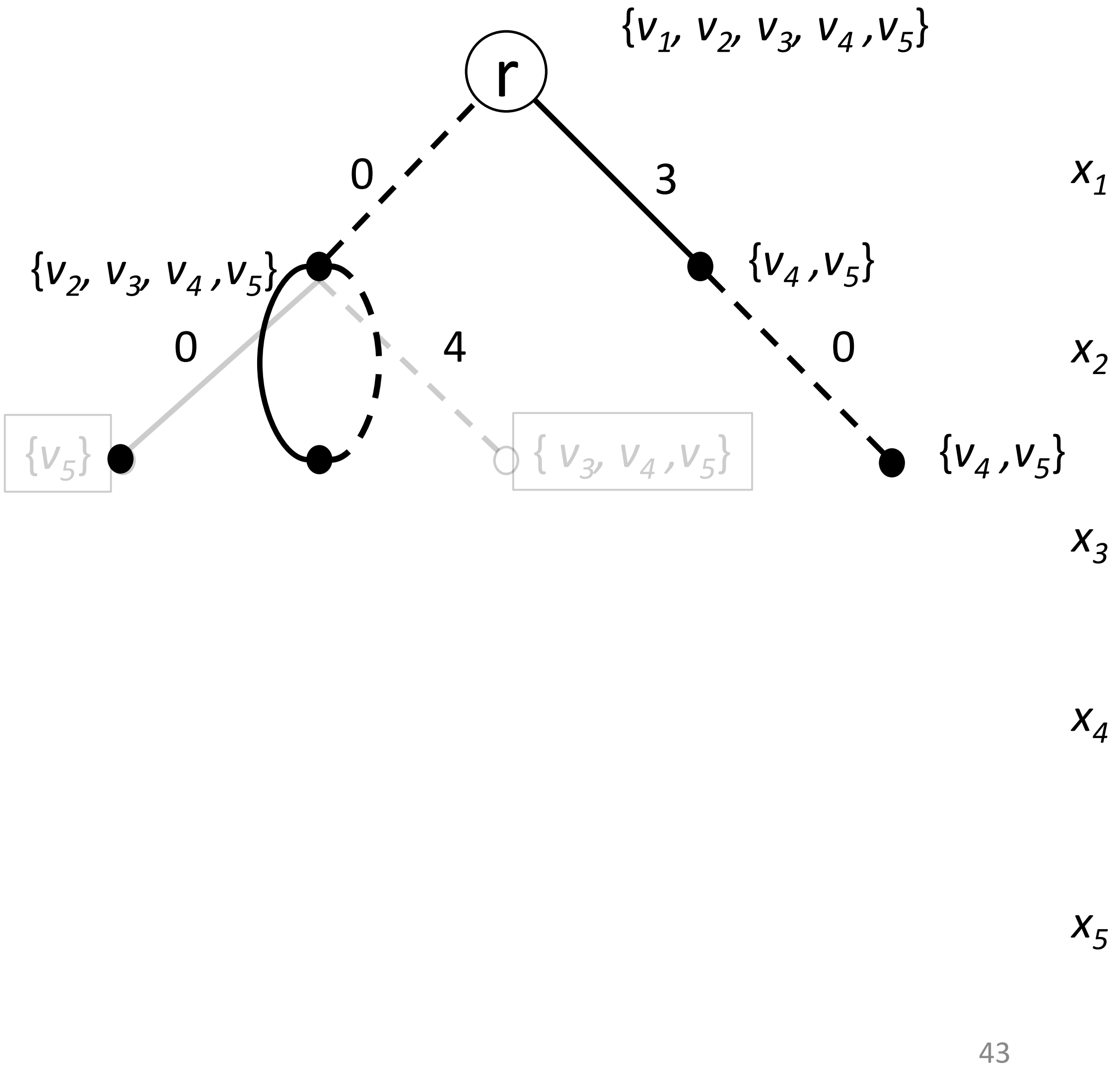
include

exclude

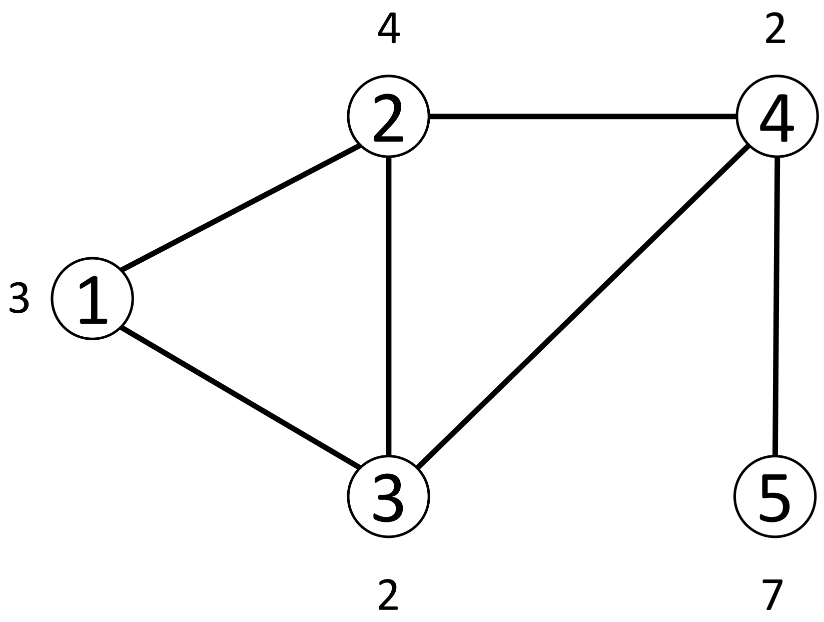
Building Relaxed Decision Diagrams



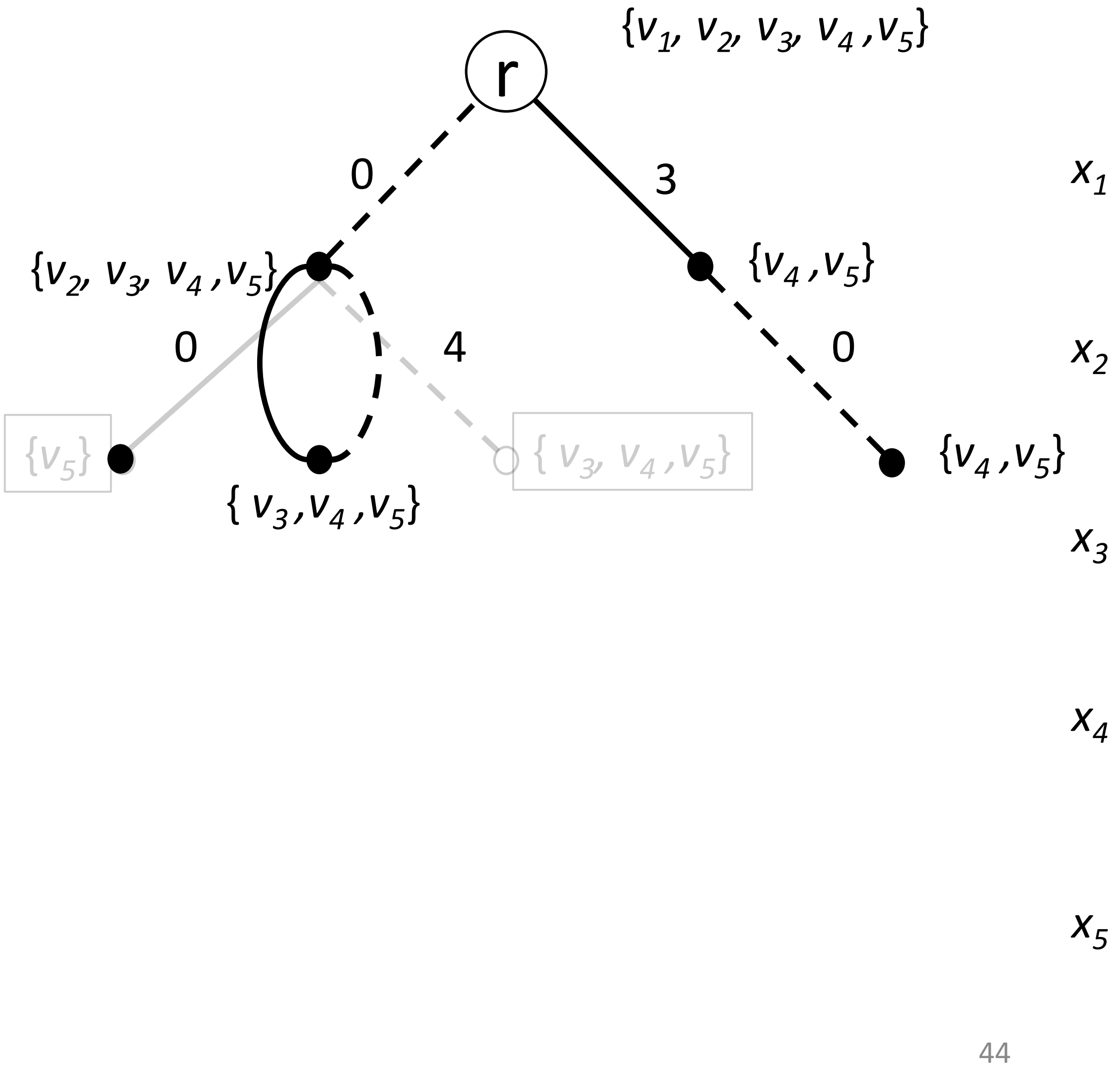
Max Width = 2



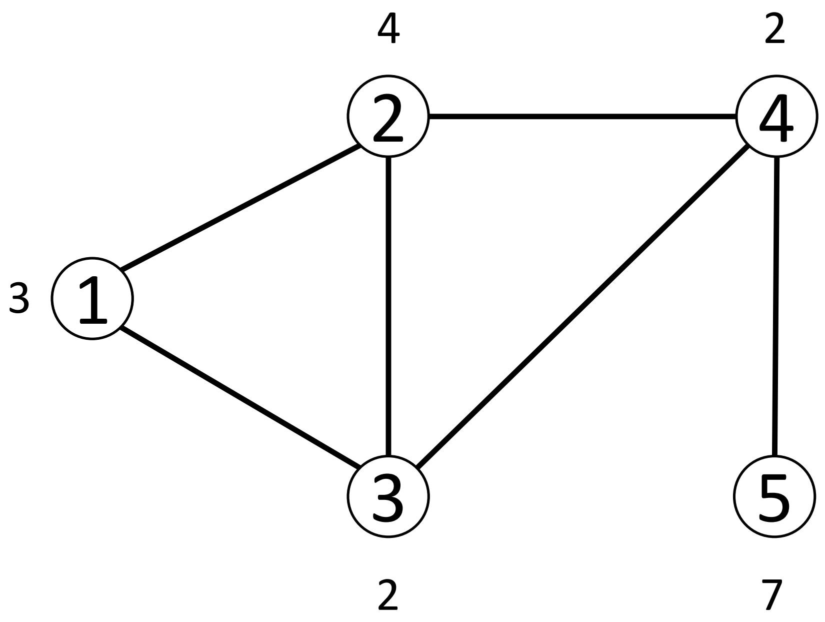
Building Relaxed Decision Diagrams



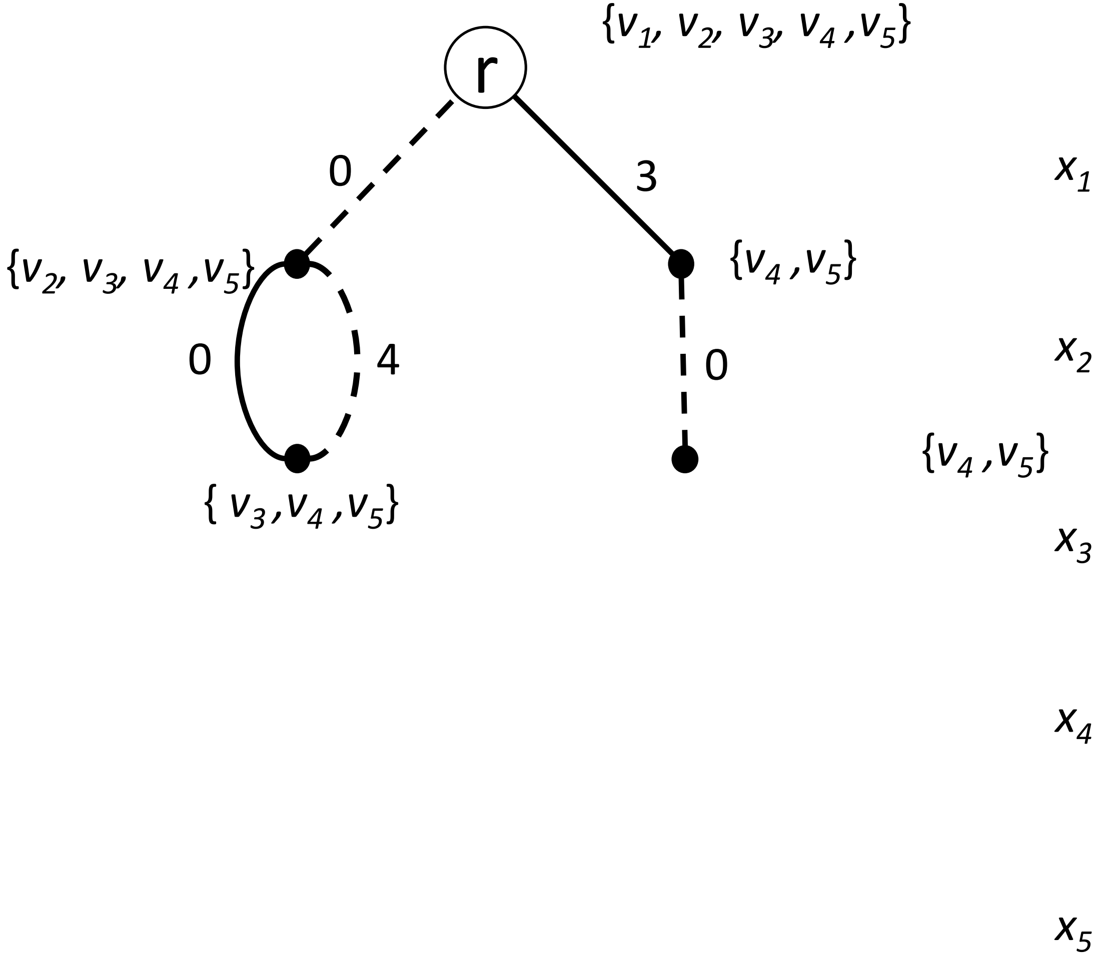
Max Width = 2



Building Relaxed Decision Diagrams

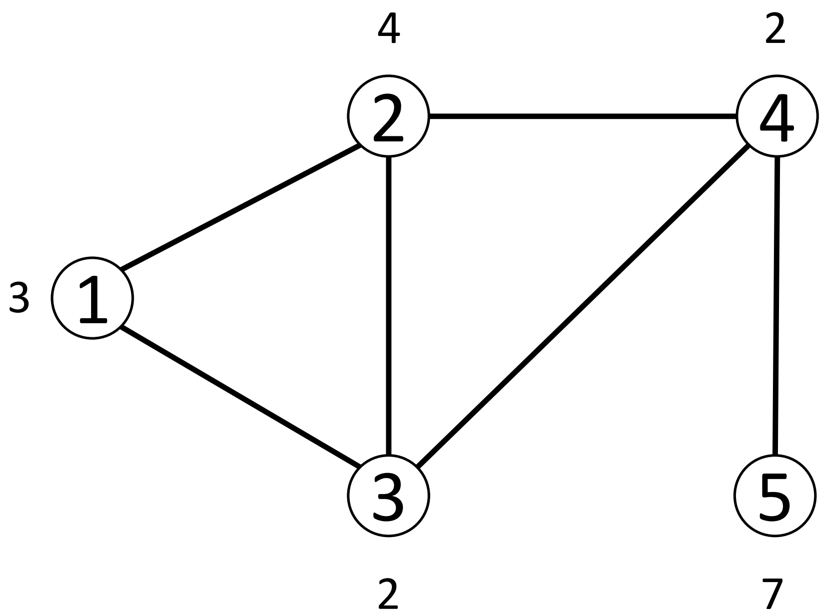


Max Width = 2

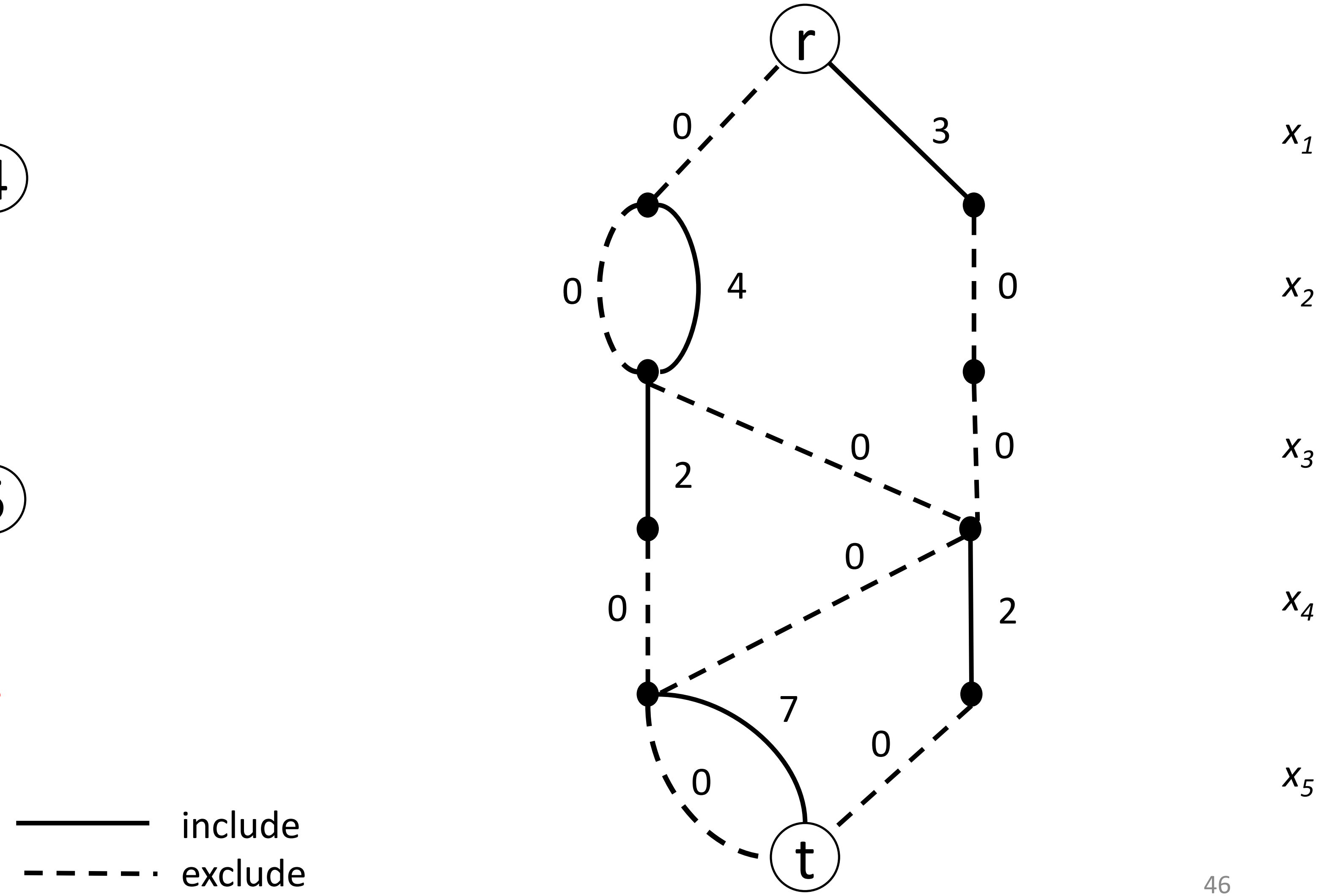


include
exclude

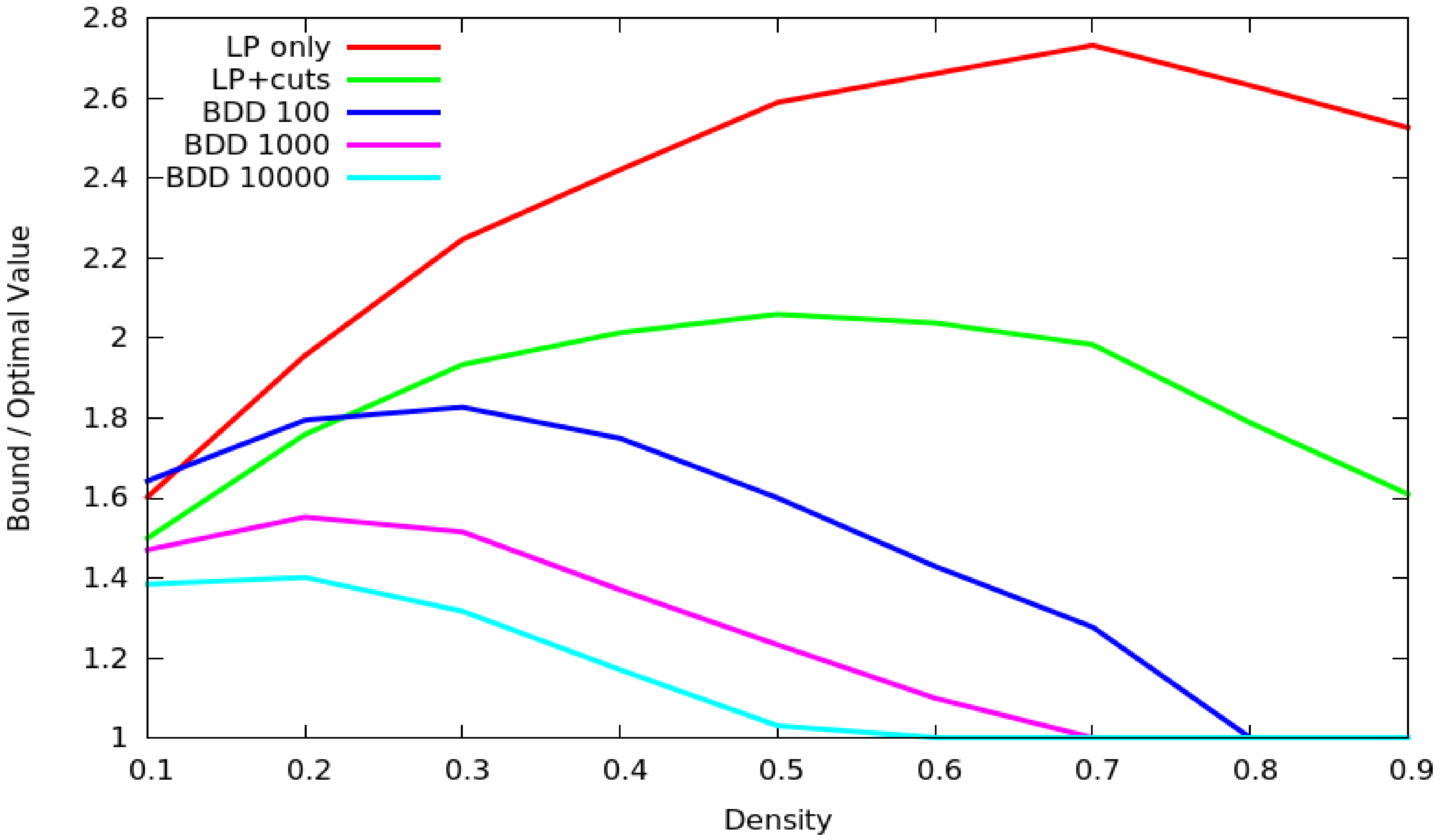
Building Relaxed Decision Diagrams



Max Width = 2



Relaxation Bound: Maximum Independent Set



Strengthening Diagram Relaxations

- Filtering operations
  - “Redundant” constraints

- Additive Bounding
  - Incorporate dual information from LP relaxations

- DD-Based Lagrangian Relaxations

Strengthening Diagram Relaxations

- Filtering operations
  - “Redundant” constraints
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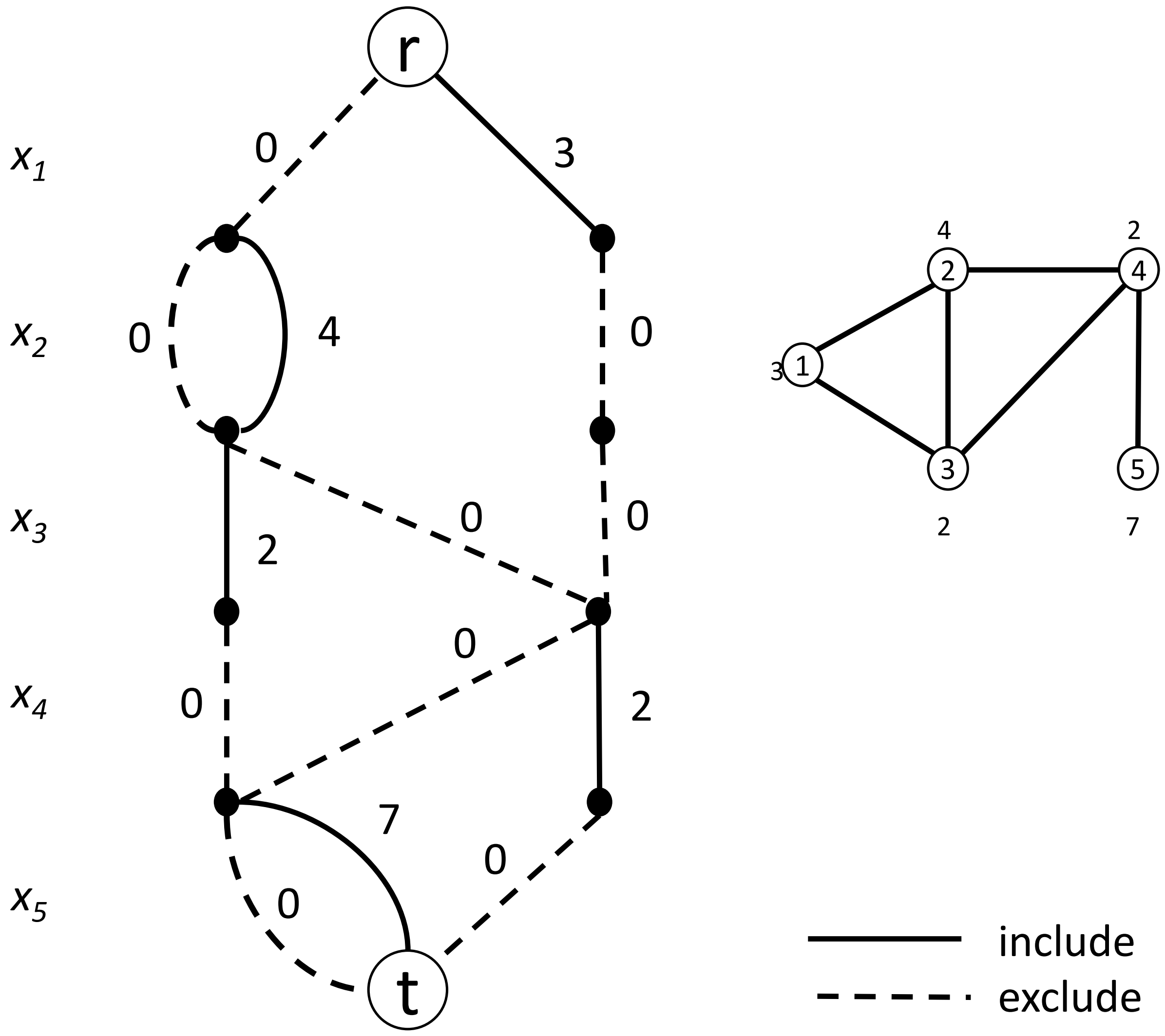
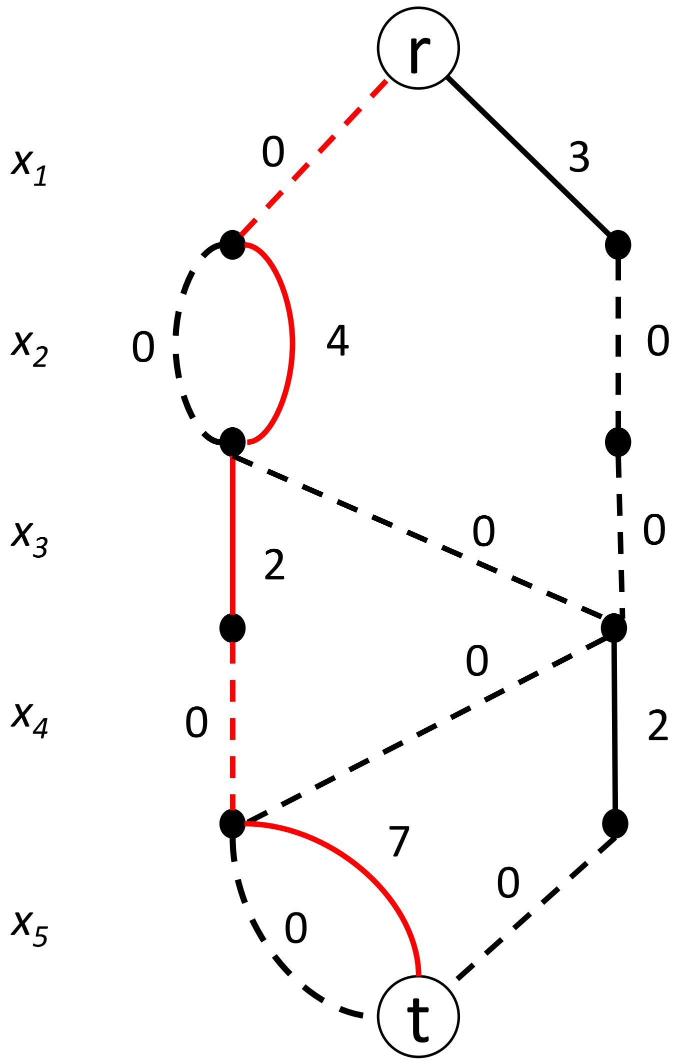
DD-Based Lagrangian Relaxation

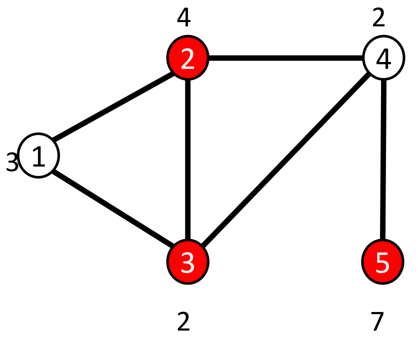


- We are solving

  max f(x)
  subject to
  x ∈ *RelaxedDD*

DD-Based Lagrangian Relaxation





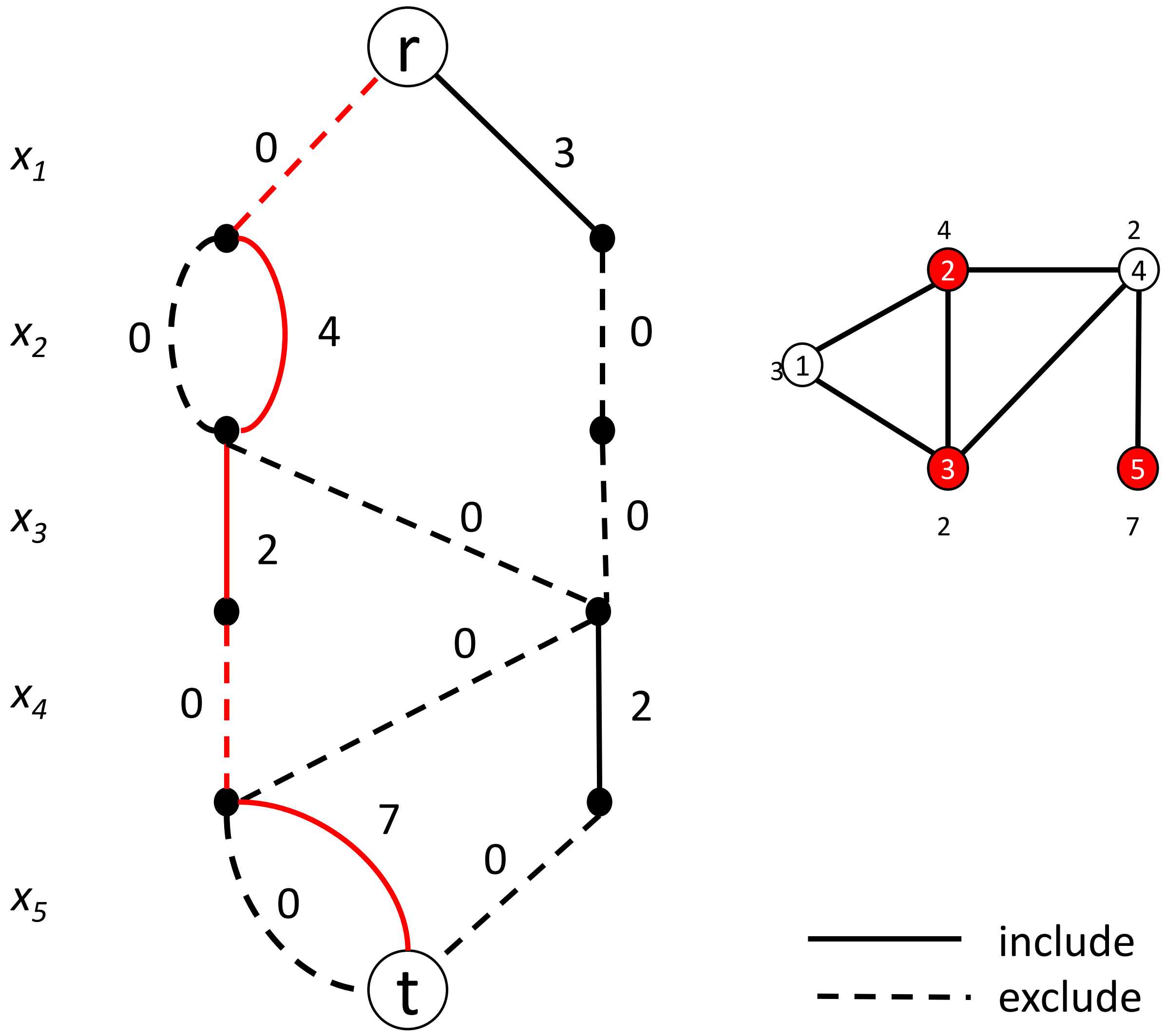
x = (0, 1, 1, 0, 1)
Upper bound = 13

include
exclude

- We are solving

  max f(x)
  subject to
  x ∈ *RelaxedDD*

DD-Based Lagrangian Relaxation



- Let A, b be such that:

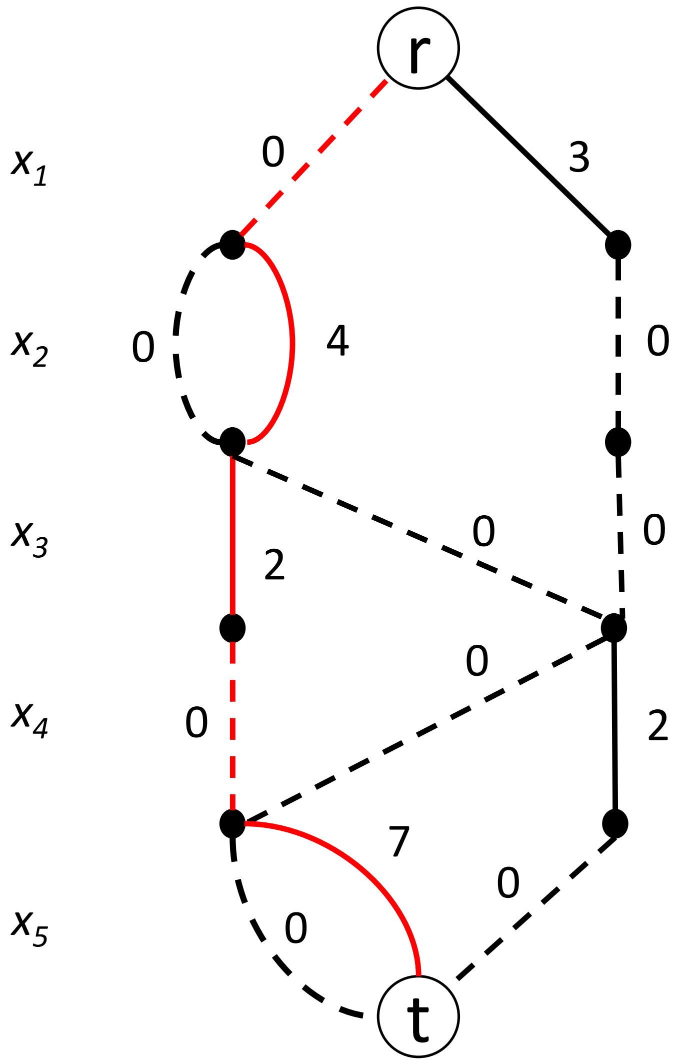
  Ax ≤ b for any **feasible** x

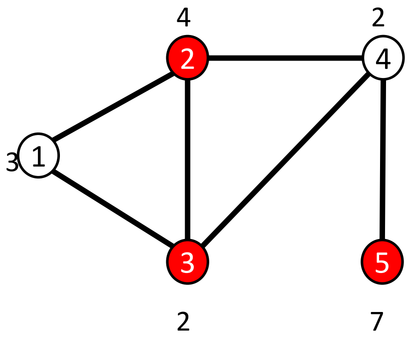
- DD-Based Lagrangian:

  max f(x) + λ(b – Ax)
  subject to
  x ∈ *RelaxedDD*

  - Gives an upper bound for any λ ≥ 0

DD-Based Lagrangian Relaxation





x = (0, 1, 1, 0, 1)
Upper bound = 13

include
exclude

- Solution (0,1,1,0,1) violates constraint

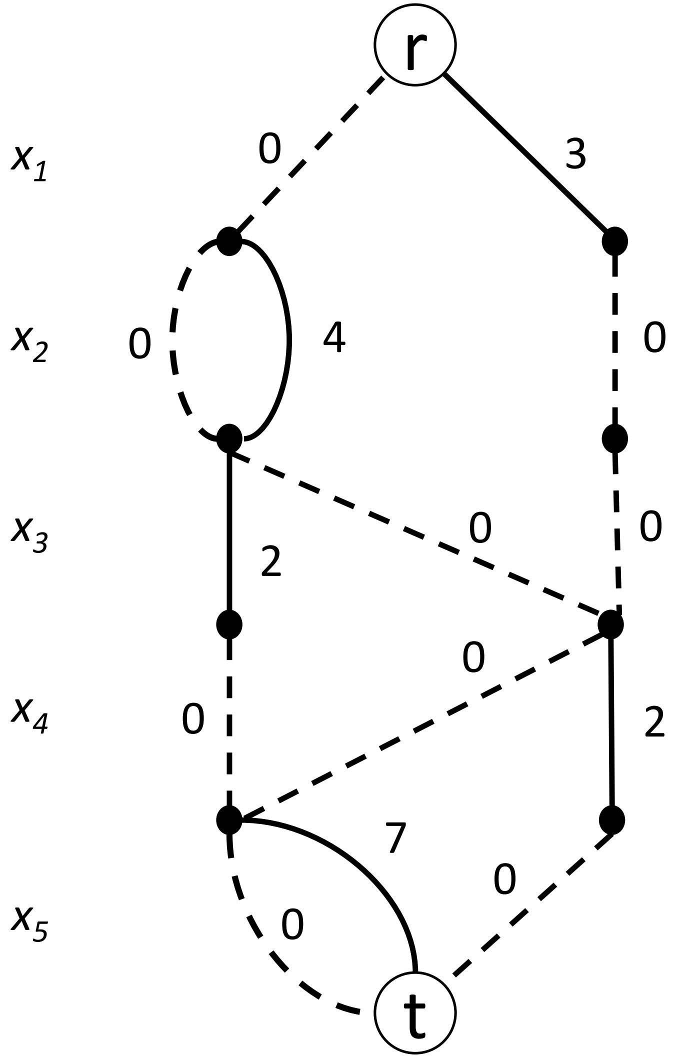
$$x_2 + x_3 \leq 1$$

- We penalize with term

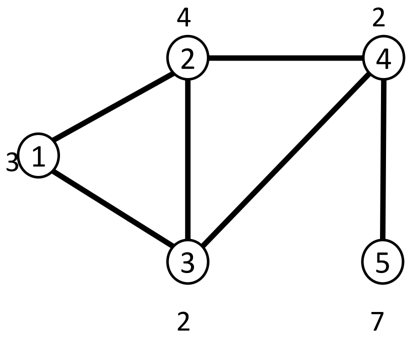
$$+ \lambda (1 - x_2 - x_3)$$

by simply changing the **cost structure** of the DD

DD-Based Lagrangian Relaxation



λ = 2



include
exclude

- Solution (0,1,1,0,1) violates constraint

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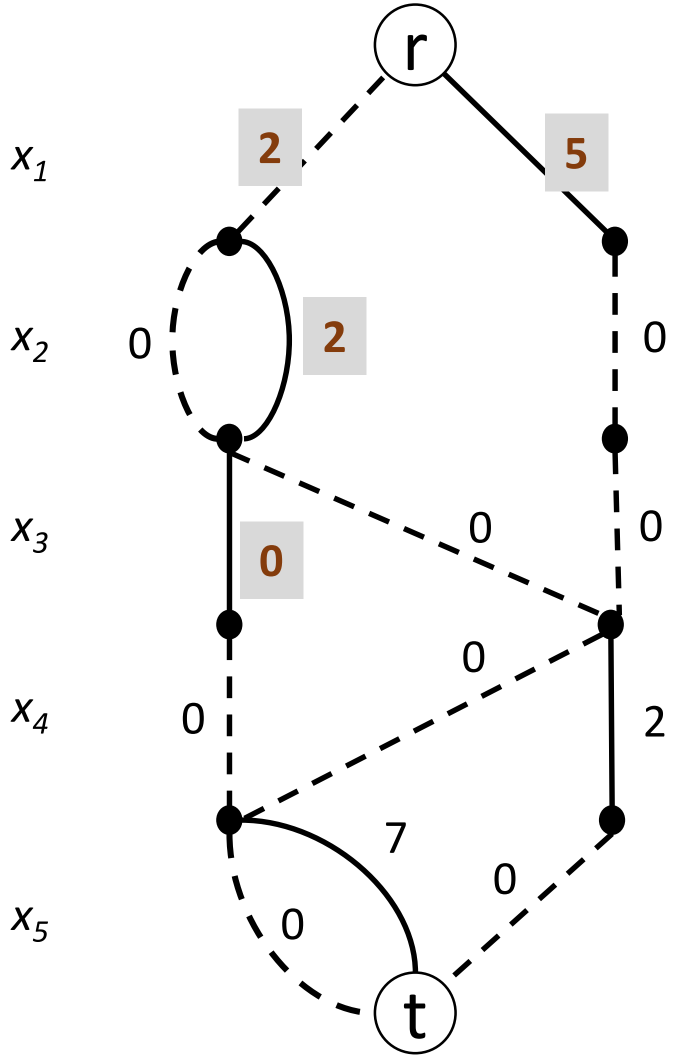
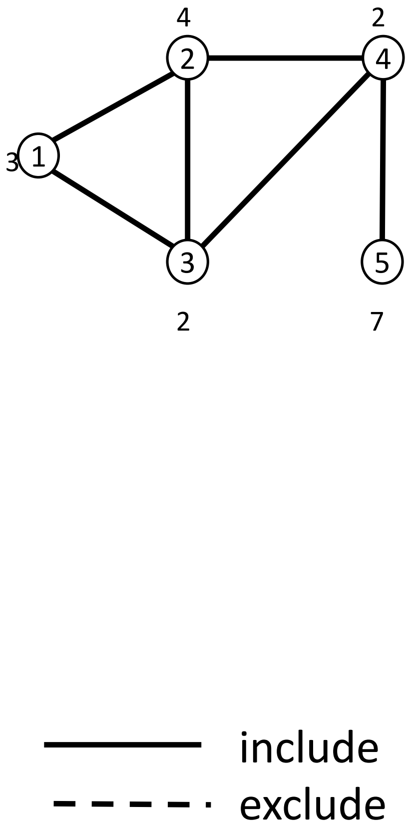
DD-Based Lagrangian Relaxation



λ = 2



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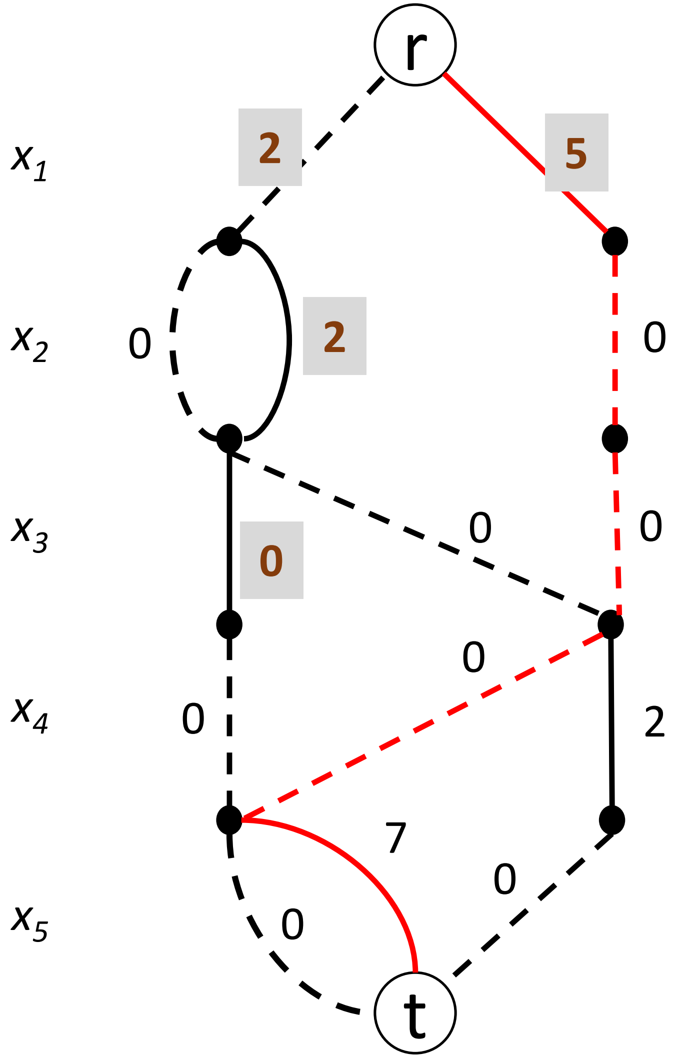
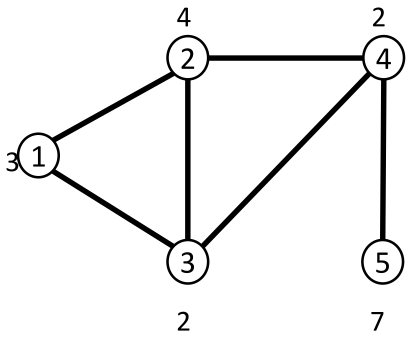
DD-Based Lagrangian Relaxation



λ = 2



Better upper bound: 12 !

—— include
- - - - exclude

- Solution (0,1,1,0,1) violates constraint

$$x_2 + x_3 \leq 1$$

- We penalize with term

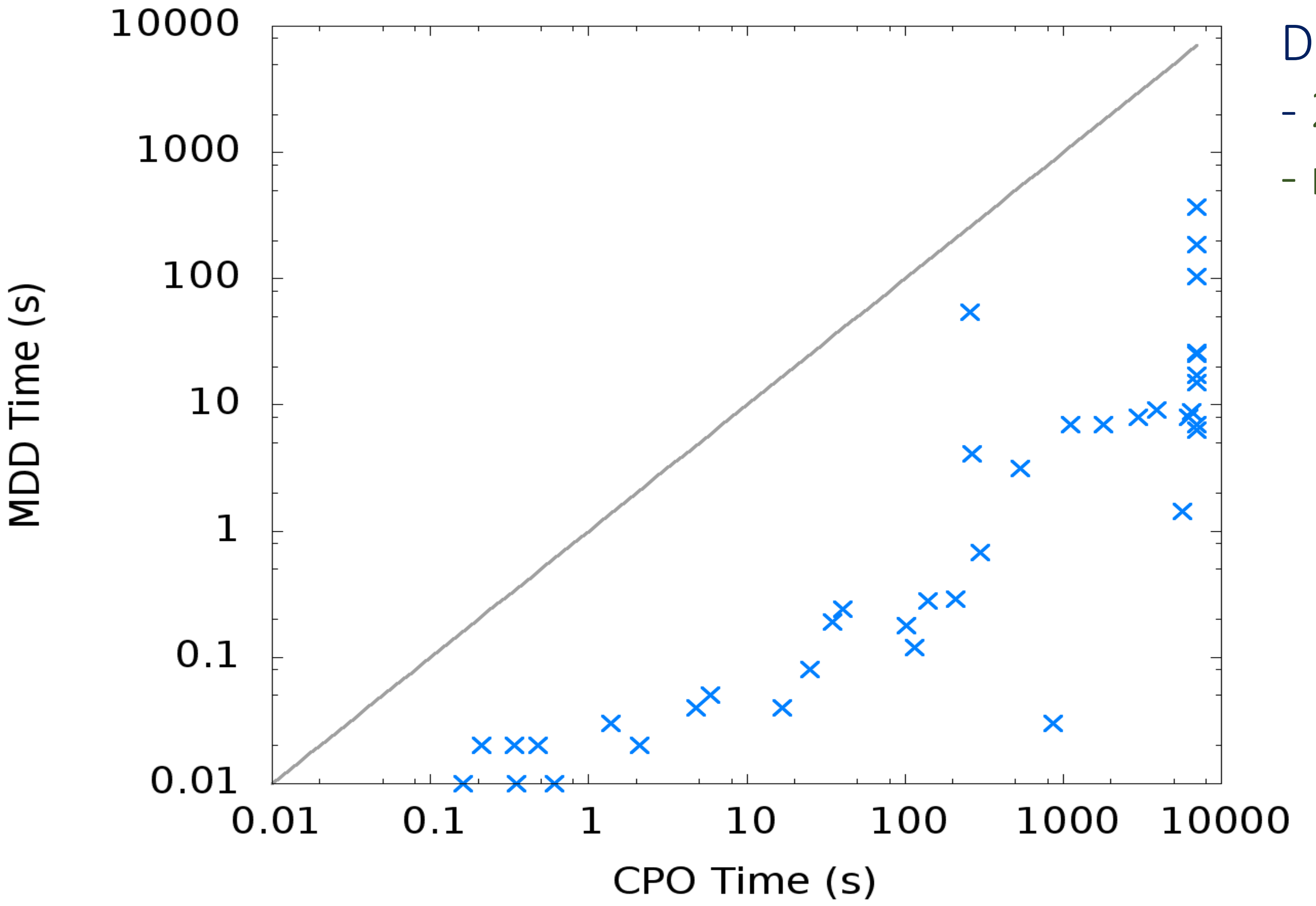
$$+ \lambda (1 - x_2 - x_3)$$

by simply changing the **cost structure** of the DD

Computational Analysis

- Incorporated into IBM ILOG CP Optimizer (CPO)
  - State-of-the-art constraint-based scheduling solver
  - Uses a portfolio of inference techniques and LP relaxations

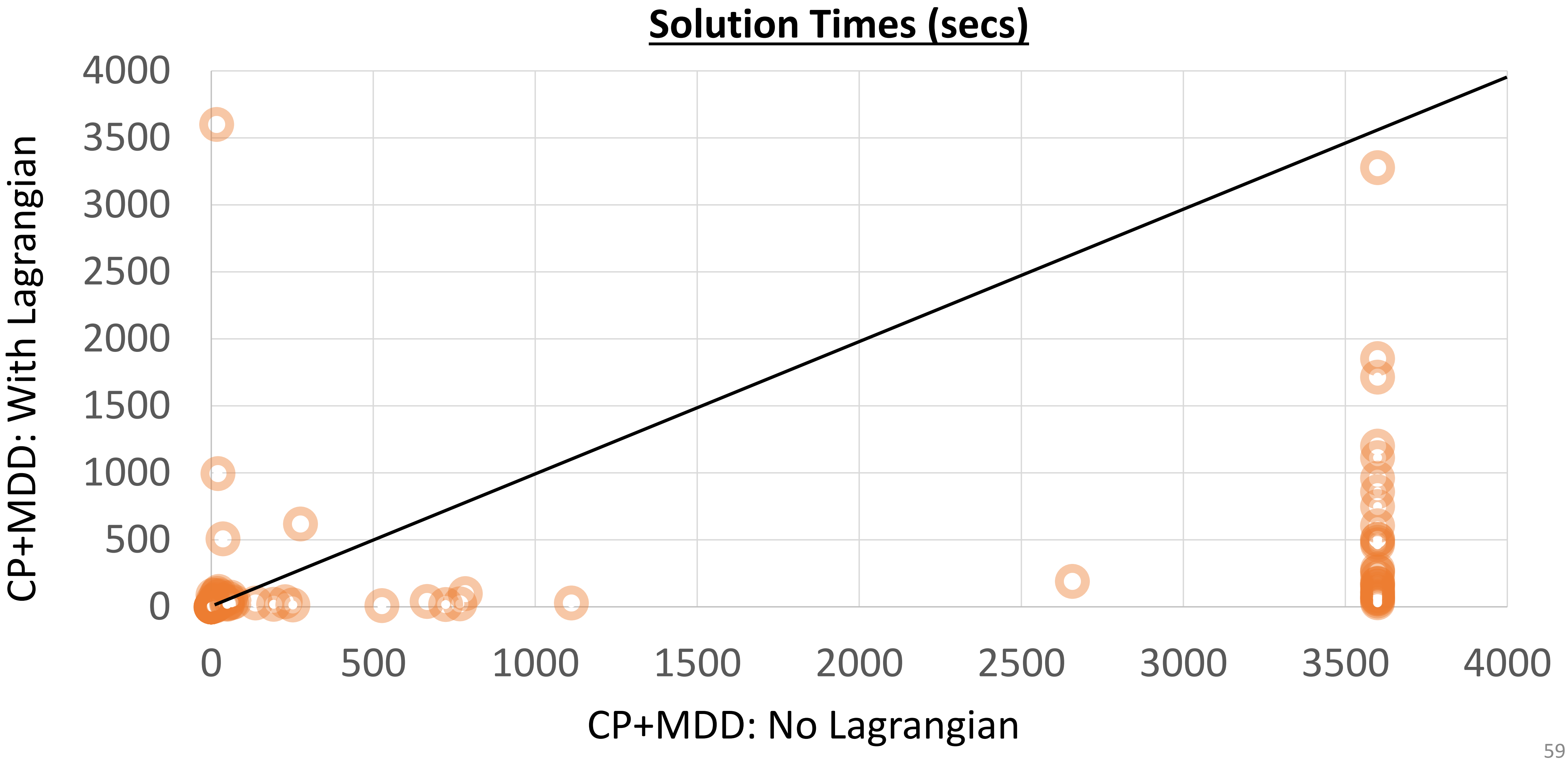
TSP with Time Windows



Dumas/Ascheuer instances

- 20-100 jobs
- maximum width: 16

DD-Based Lagrangian



Other Results

- Asymmetric TSP with Precedence Constraints
  - Closed 3 TSPLIB open instances

- Easy modeling for certain problems
  - Example: *Time-Dependent TSPs*

Relaxation Methods

E.g., Linear Programming Relaxation

Modeling Framework

E.g., Linear Inequalities

Primal Heuristics

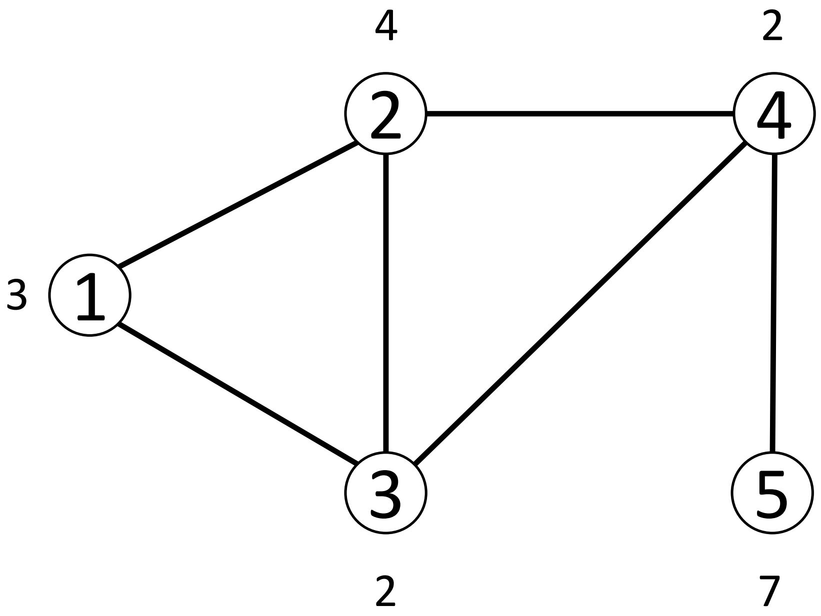
E.g., Feasibility Pump

Generic Optimization Techniques

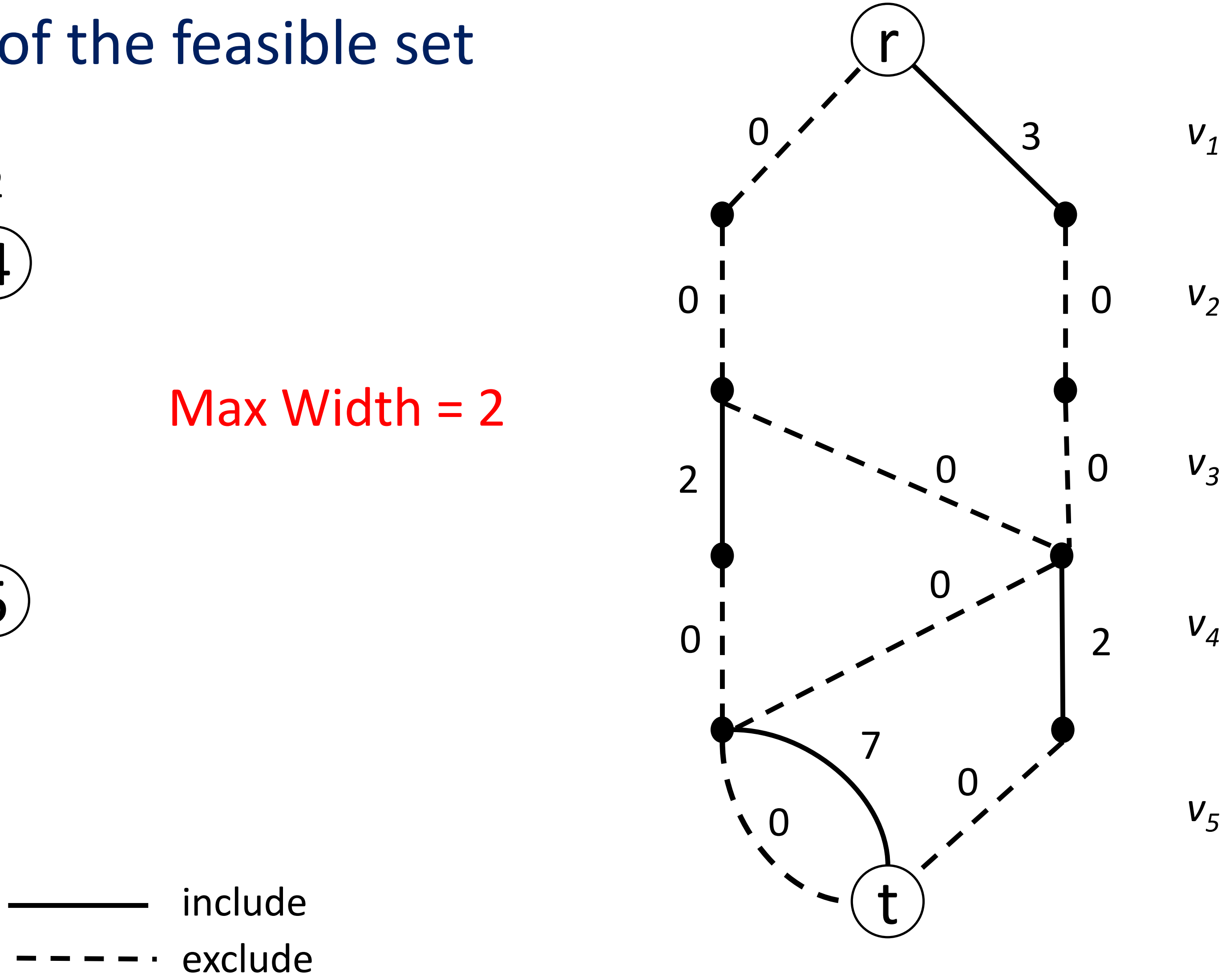
E.g., Mixed-integer Programming

Restricted Decision Diagrams

- Under-approximation of the feasible set



Max Width = 2



Restricted Decision Diagrams

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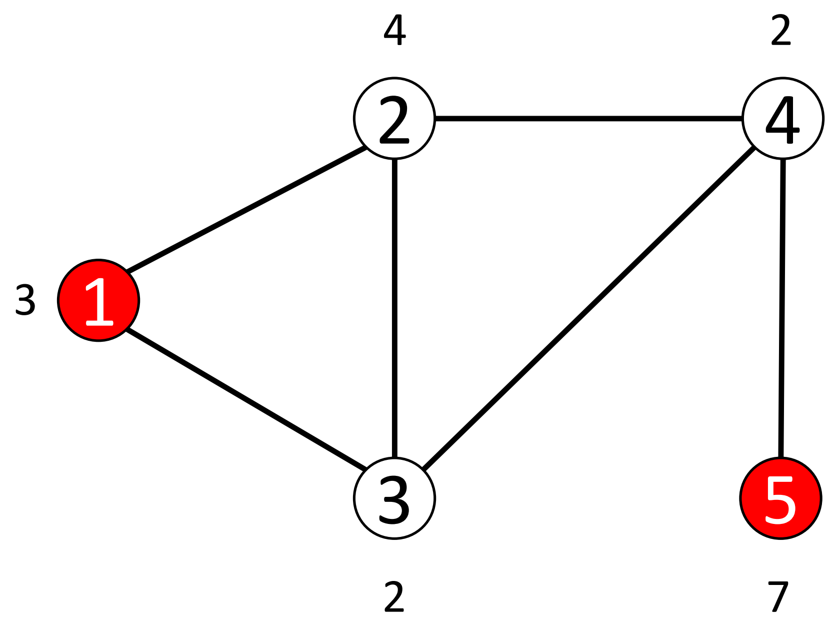

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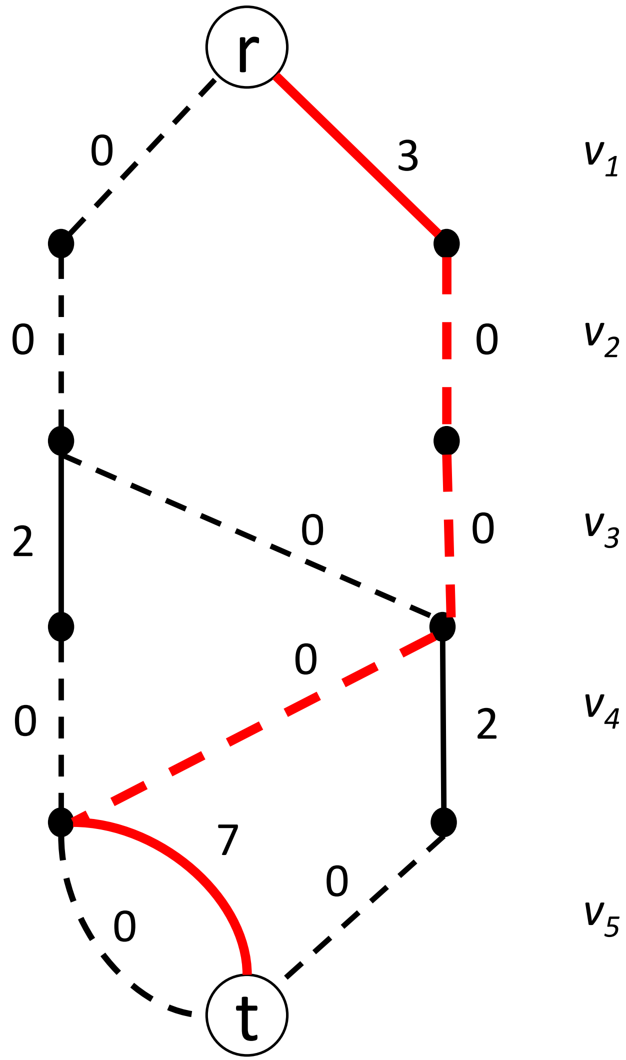
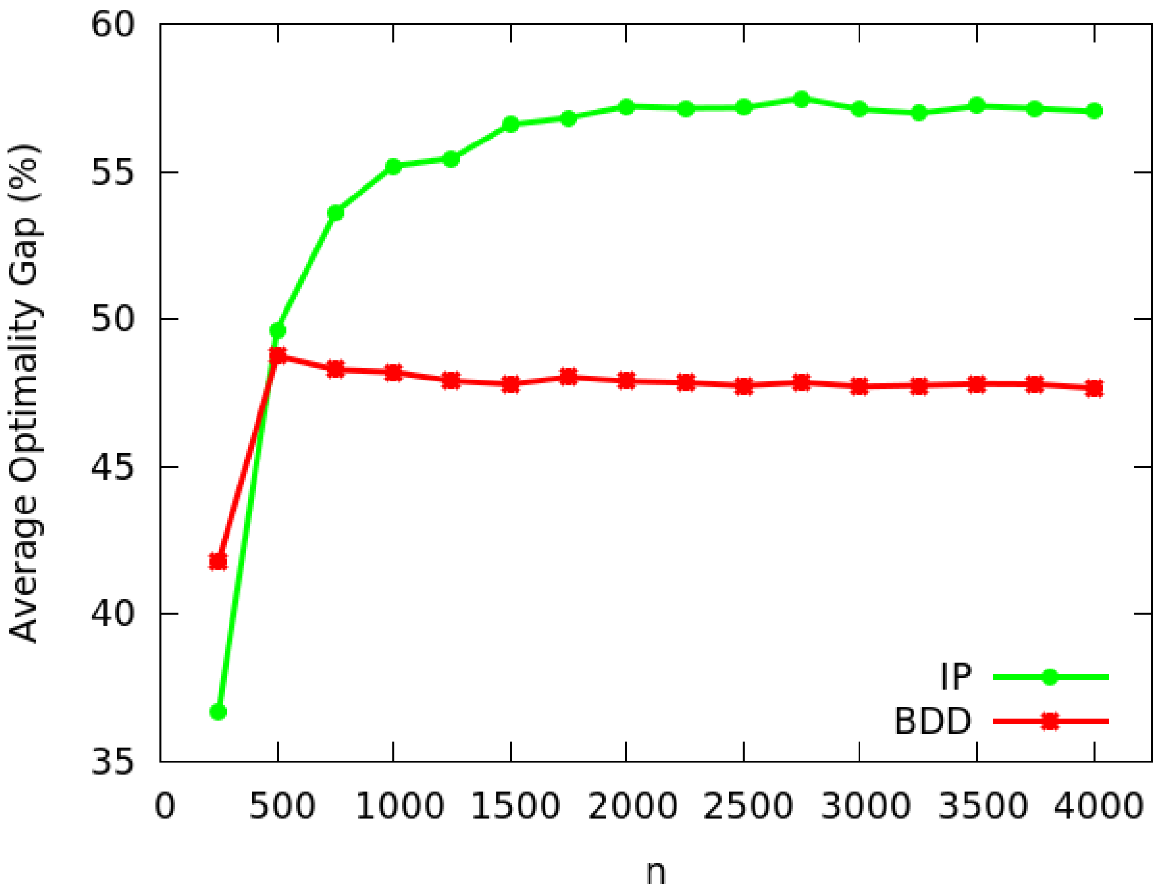
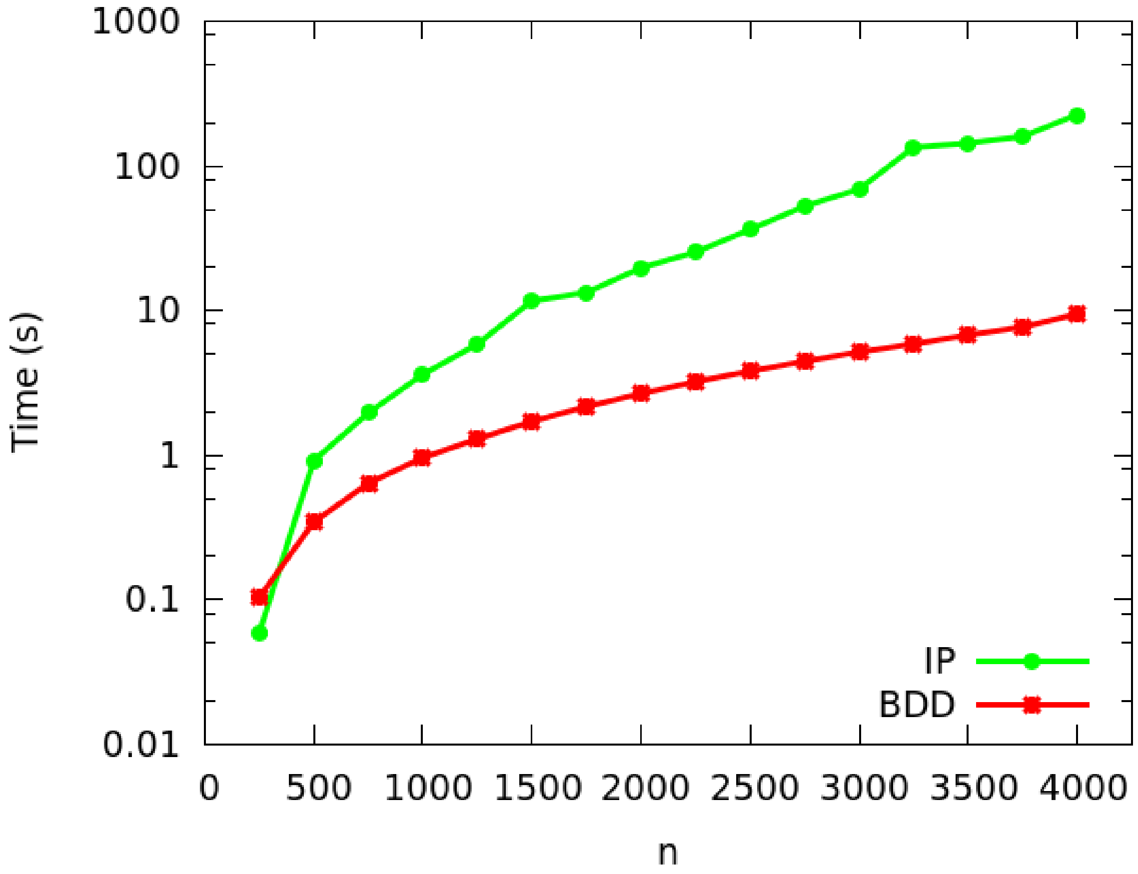


(1,0,0,0,1) → Lower bound = 10

include
exclude

Primal Bound: Set Covering





Relaxation Methods

E.g., Linear Programming Relaxation

Modeling Framework

E.g., Linear Inequalities

Primal Heuristics

E.g., Feasibility Pump

Generic Optimization Techniques

E.g., Mixed-integer Programming

Inference

E.g., valid cuts

Quick Notes on Inference

- Cut generation for MIPs
  - Several techniques from Behle'07
  - Recent: **Polar set cuts** from Relaxed Decision Diagrams
    - Talk to Christian Tjandraatmadja! (poster yesterday!)

- Highly-structured Cuts
  - Precedence relations that must hold in scheduling problems

- We are still exploring notion of *decision diagram separation*
  - Cire & Hooker, ISAIM 2014

Relaxation Methods

E.g., Linear Programming Relaxation

Modeling Framework

E.g., Linear Inequalities

Primal Heuristics

E.g., Feasibility Pump

Generic Optimization Techniques

E.g., Mixed-integer Programming

Inference

E.g., valid cuts

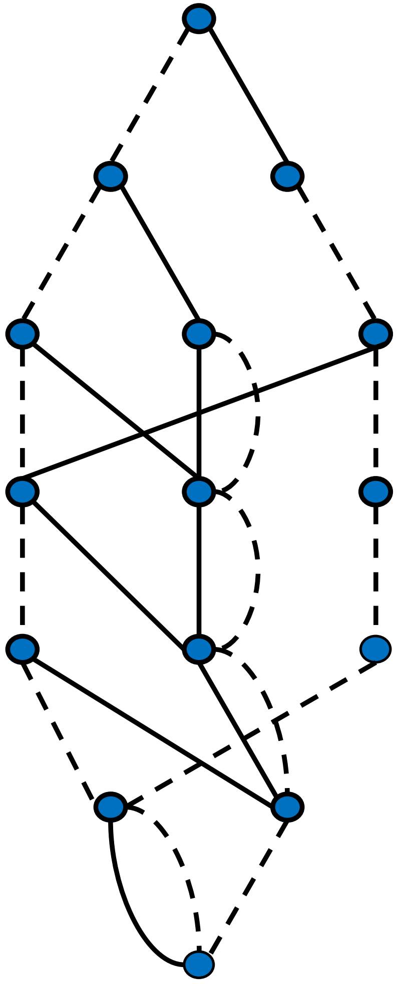
Search

E.g., Branch and bound

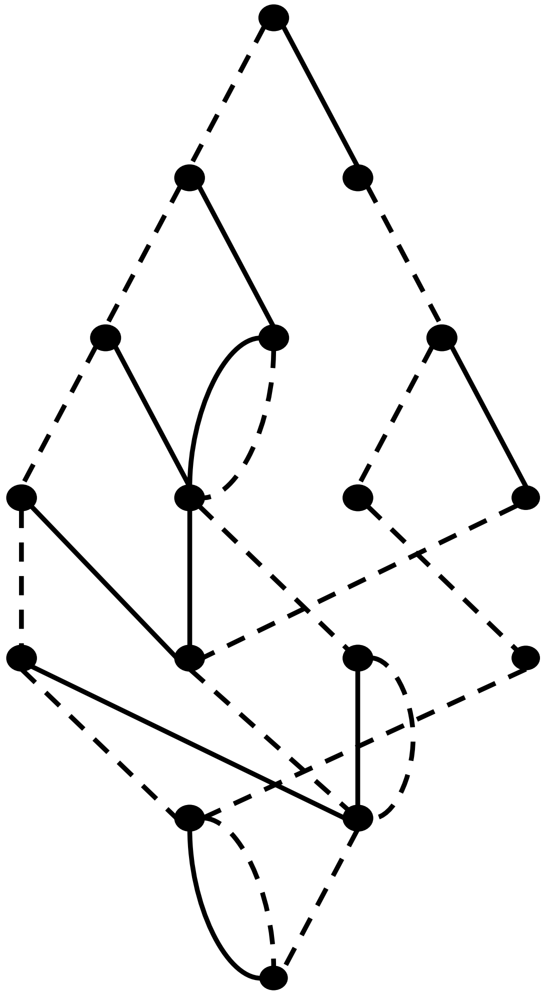
Exact Method

- Novel decision diagram branch-and-bound scheme
  - Relaxed diagrams play the role of the LP relaxation
  - Restricted diagrams are used as primal heuristics

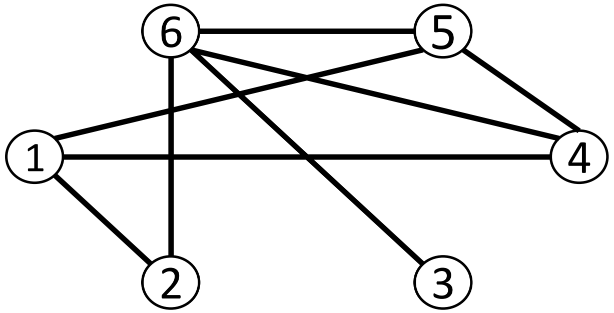
- Branching is done on the **nodes** of the diagram
  - Branching on **pools** of partial solutions
  - Eliminate search symmetry

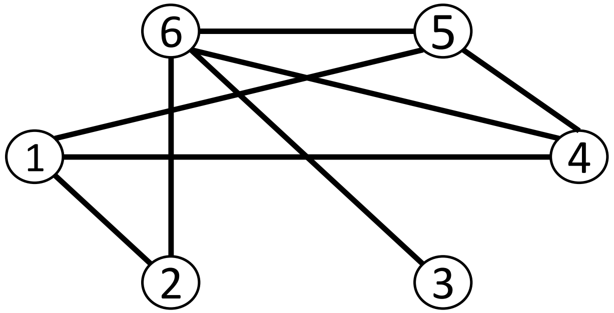


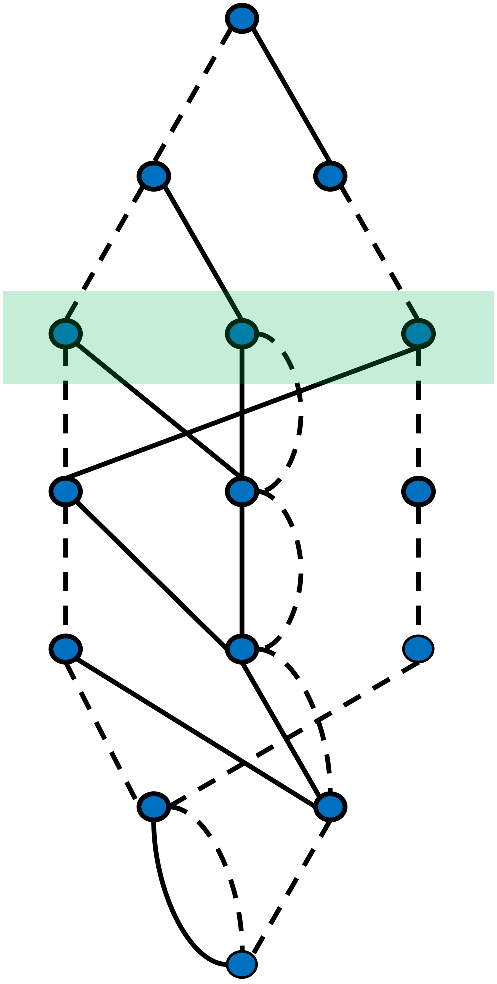
Relaxed



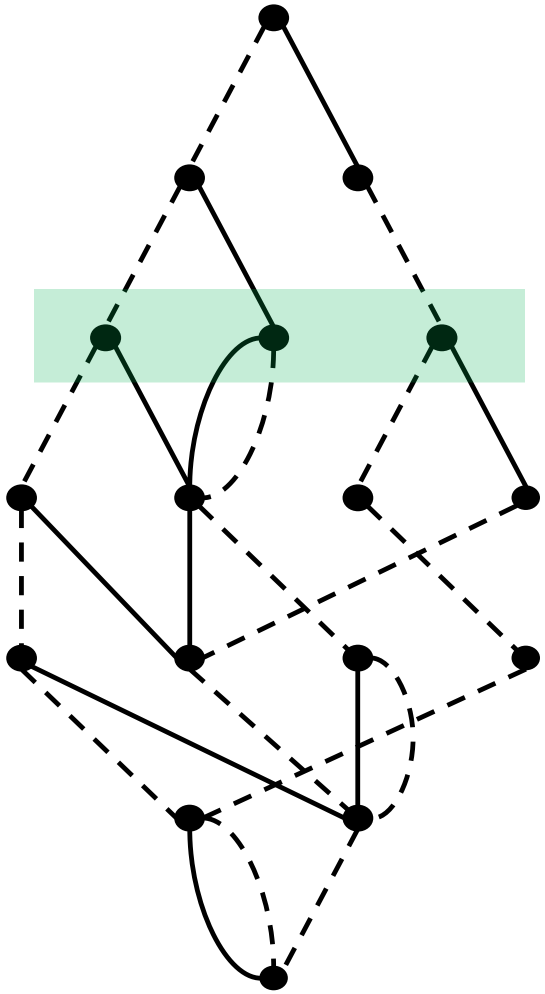
Exact





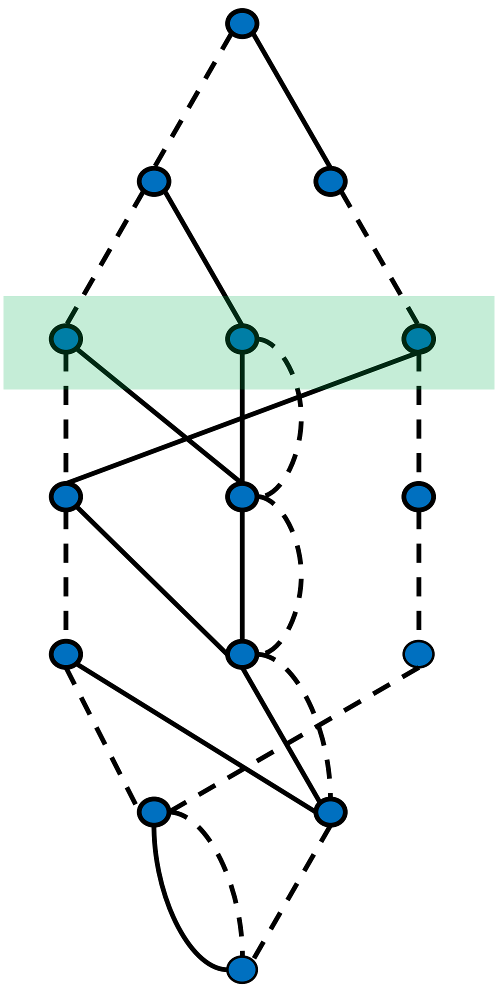


Relaxed

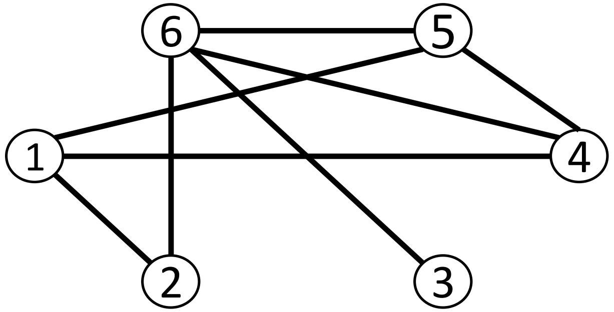


Exact

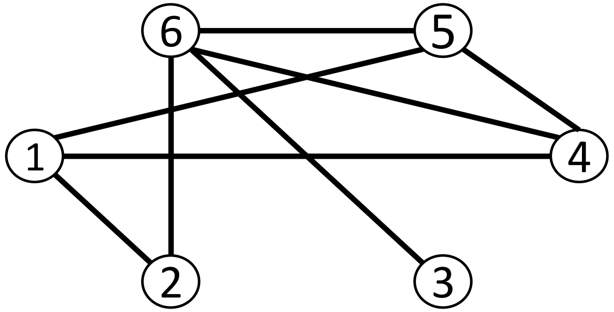
Up to a certain layer, the diagrams are the same (i.e., one layer before you start forcefully merging)

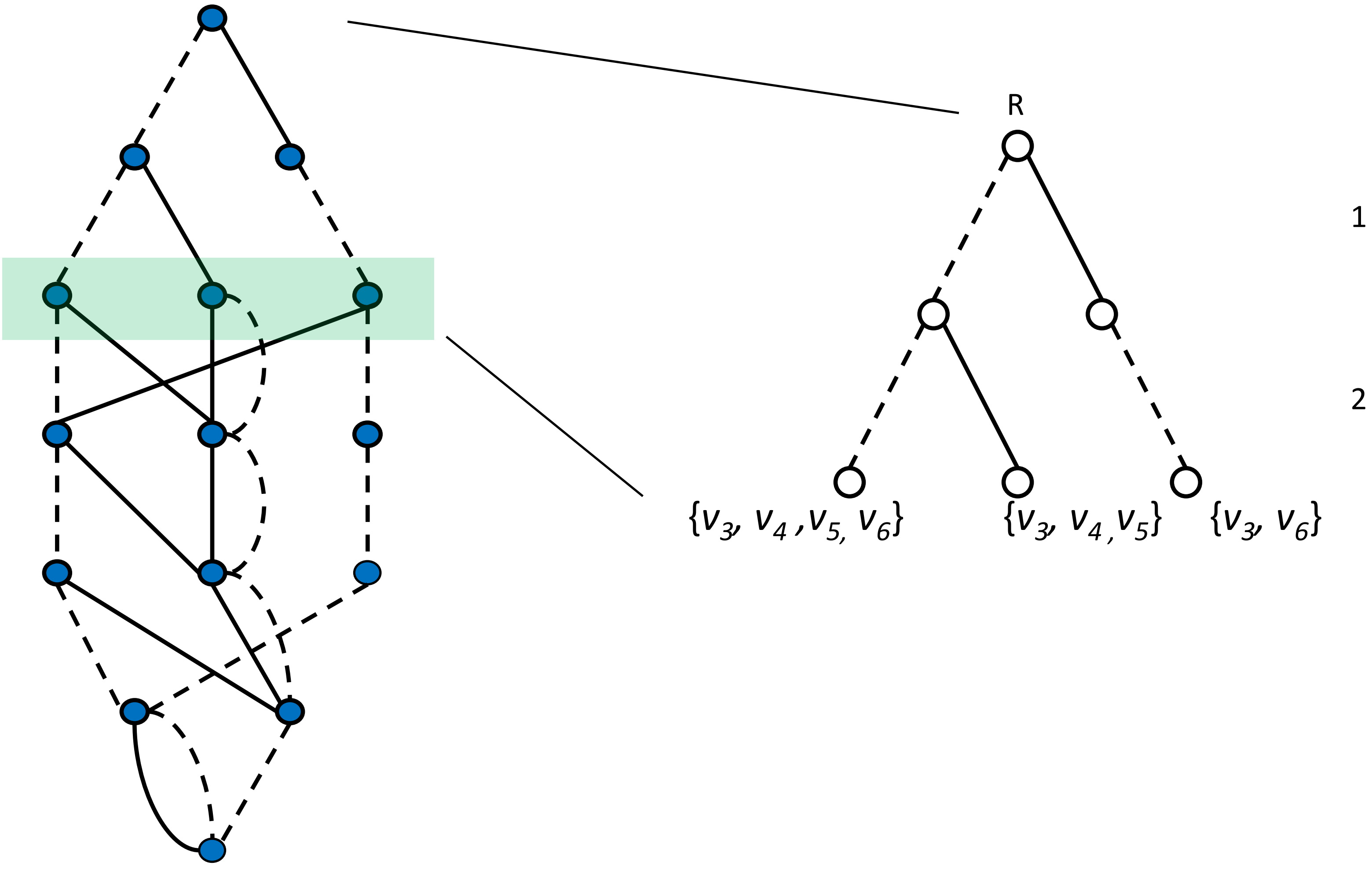


Relaxed

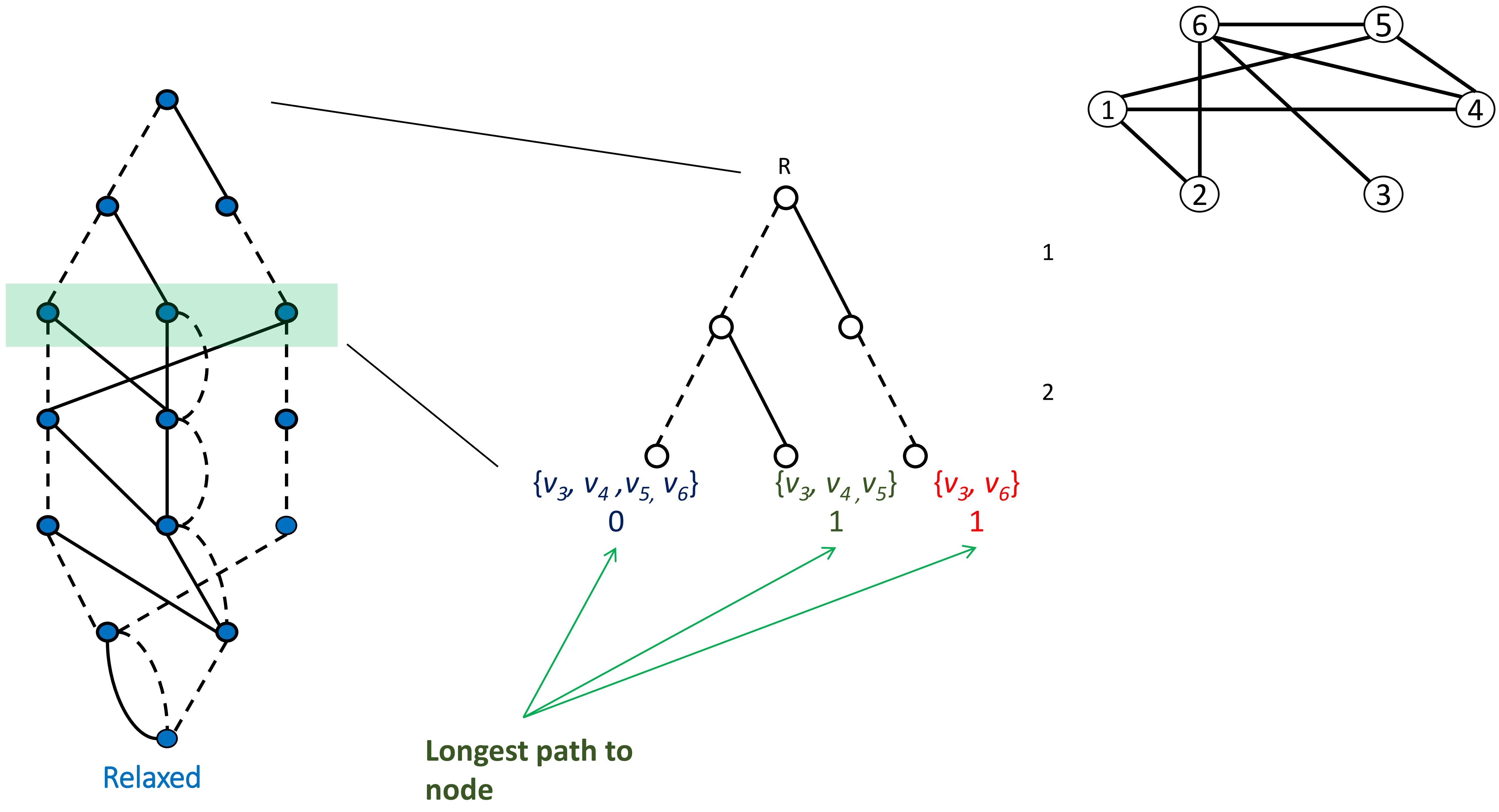


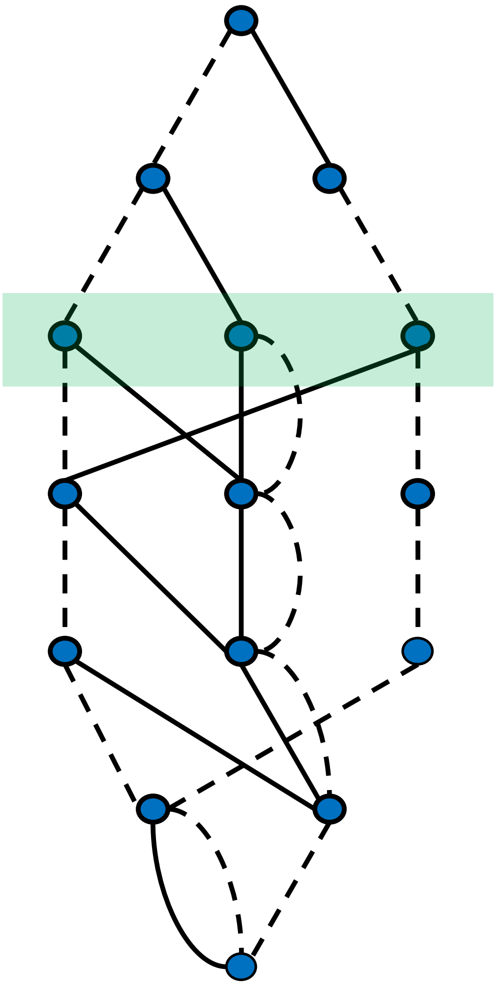
Thus, an optimum solution must necessarily pass through one of these nodes



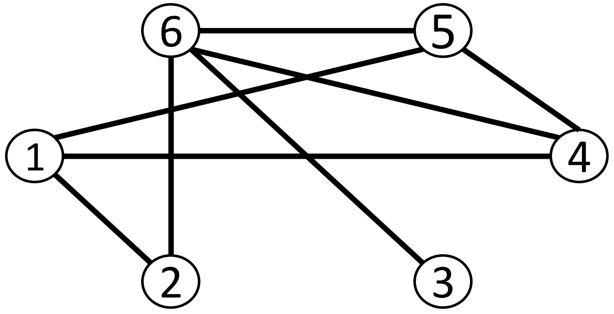


Relaxed


R
1
2
{v_3, v_4, v_5, v_6}
0
{v_3, v_4, v_5}
1
{v_3, v_6}
1
Longest path to node
Relaxed
6
5
1
4
2
3



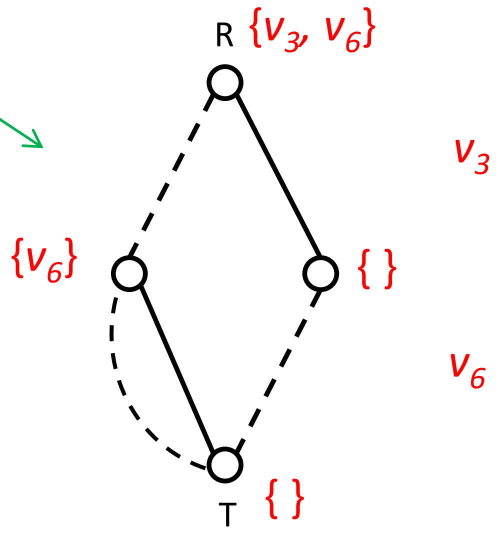
Relaxed

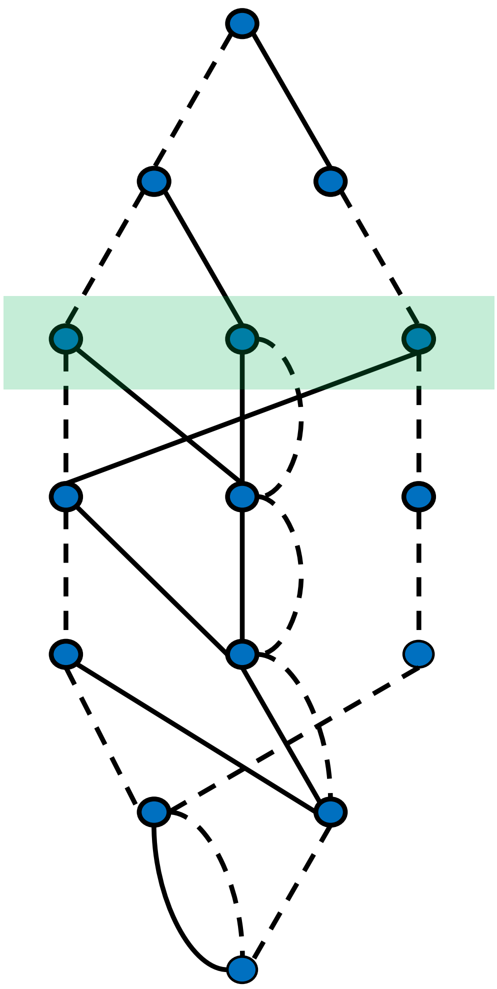


$\{v_3, v_4, v_5, v_6\}$
0

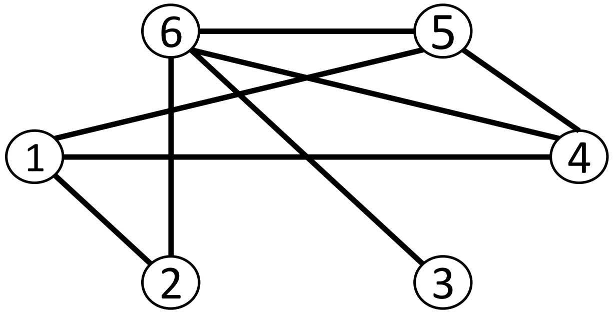
$\{v_3, v_4, v_5\}$
1

$\{v_3, v_6\}$
1





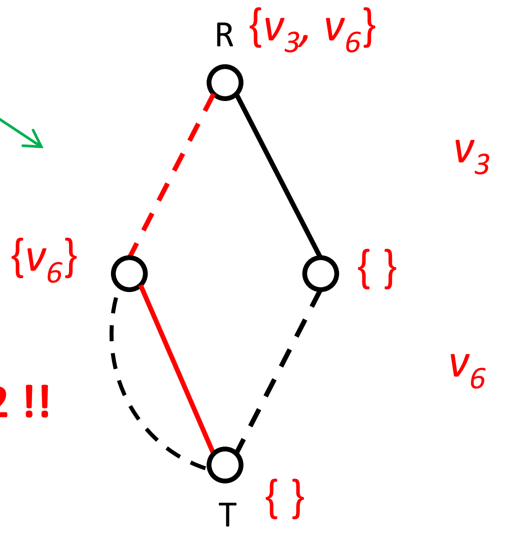
Relaxed



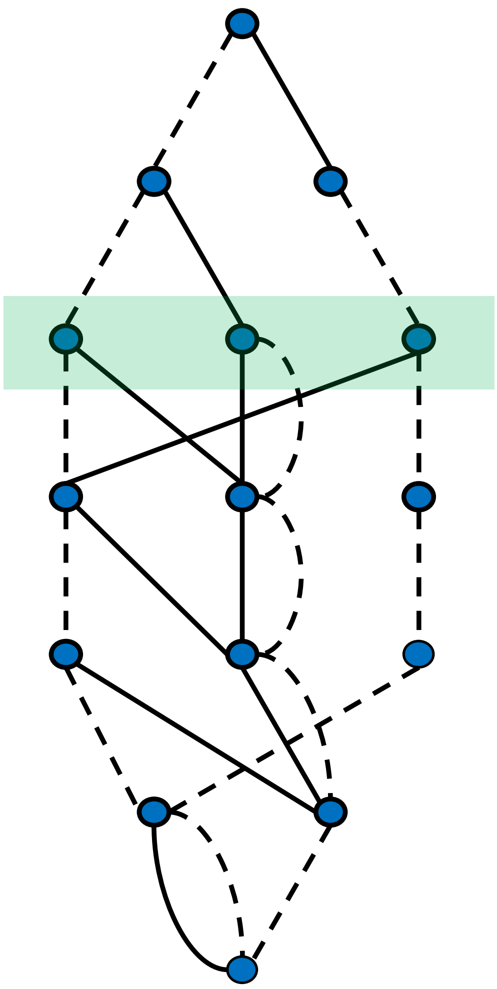
$\{v_3, v_4, v_5, v_6\}$ 0

$\{v_3, v_4, v_5\}$ 1

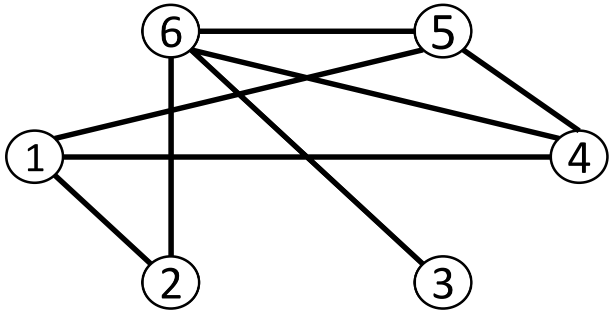
$\{v_3, v_6\}$ 1



Solution of value 2 !!



Relaxed



$\{v_3, v_4, v_5, v_6\}$ 0

$\{v_3, v_4, v_5\}$ 1

$\{v_3, v_6\}$ 1

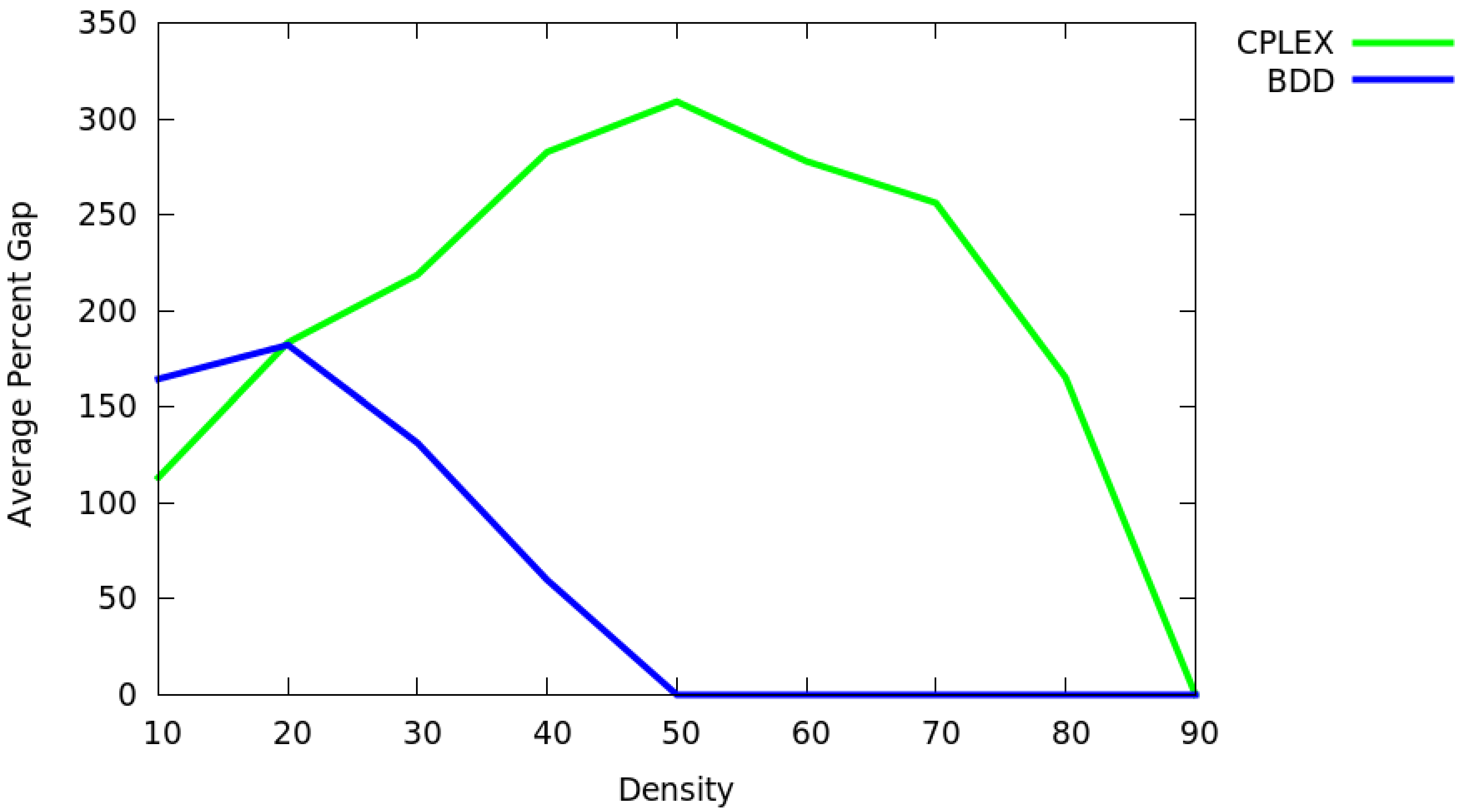
Explore each separately, saving the best solution/bound found

Maximum Cut

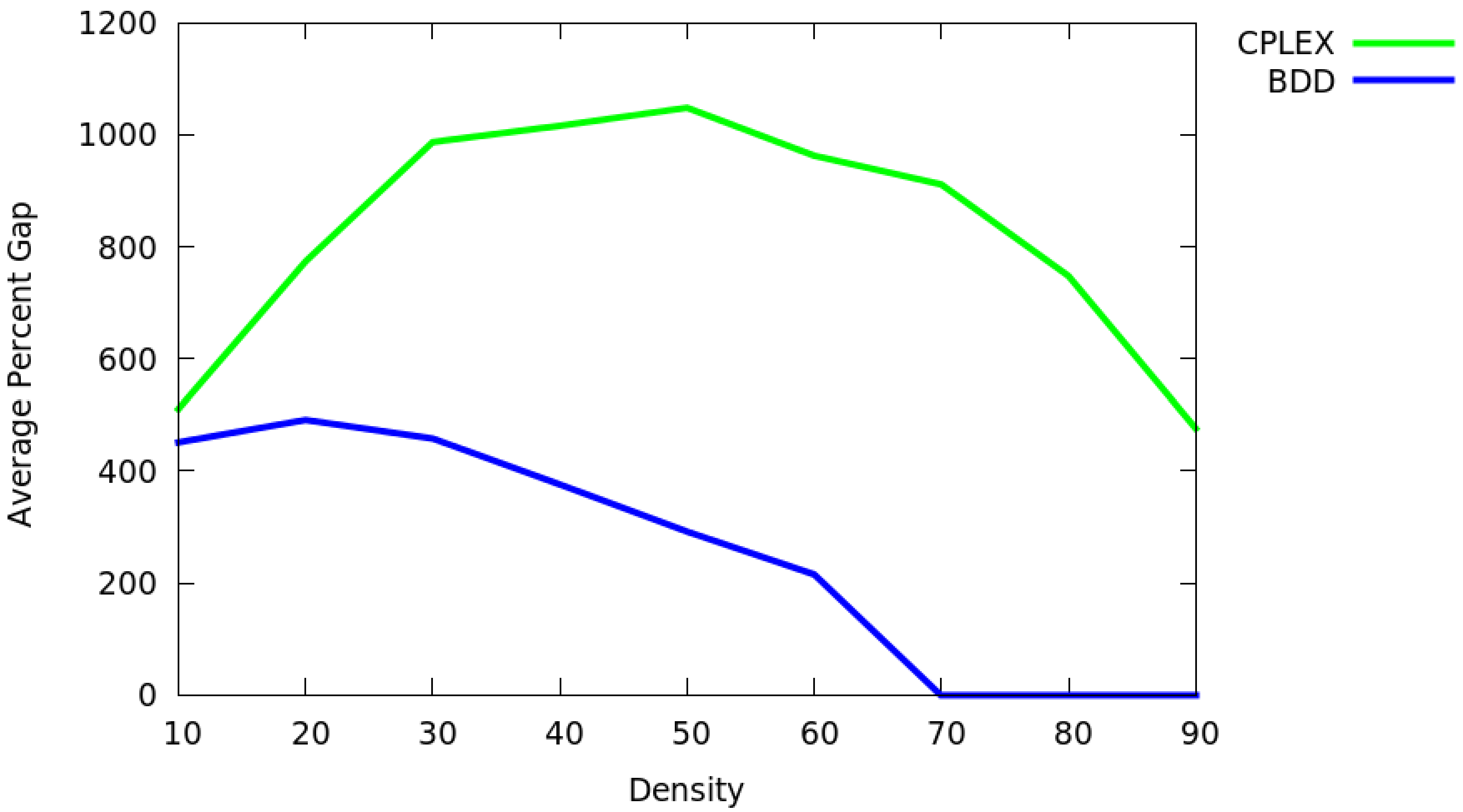
- Reduced certain optimality gaps

instance	old % gap	new % gap	% reduction
g11	11.17	0.53	95.24
g50	1.84	0.32	82.44
g32	11.59	10.64	8.20
g12	11.69	10.79	7.69
g33	11.70	11.30	3.39
g34	12.32	11.99	2.65

Maximum Independent Set: **500 variables**



Maximum Independent Set: **1500 variables**



Parallel Search with Decision Diagrams

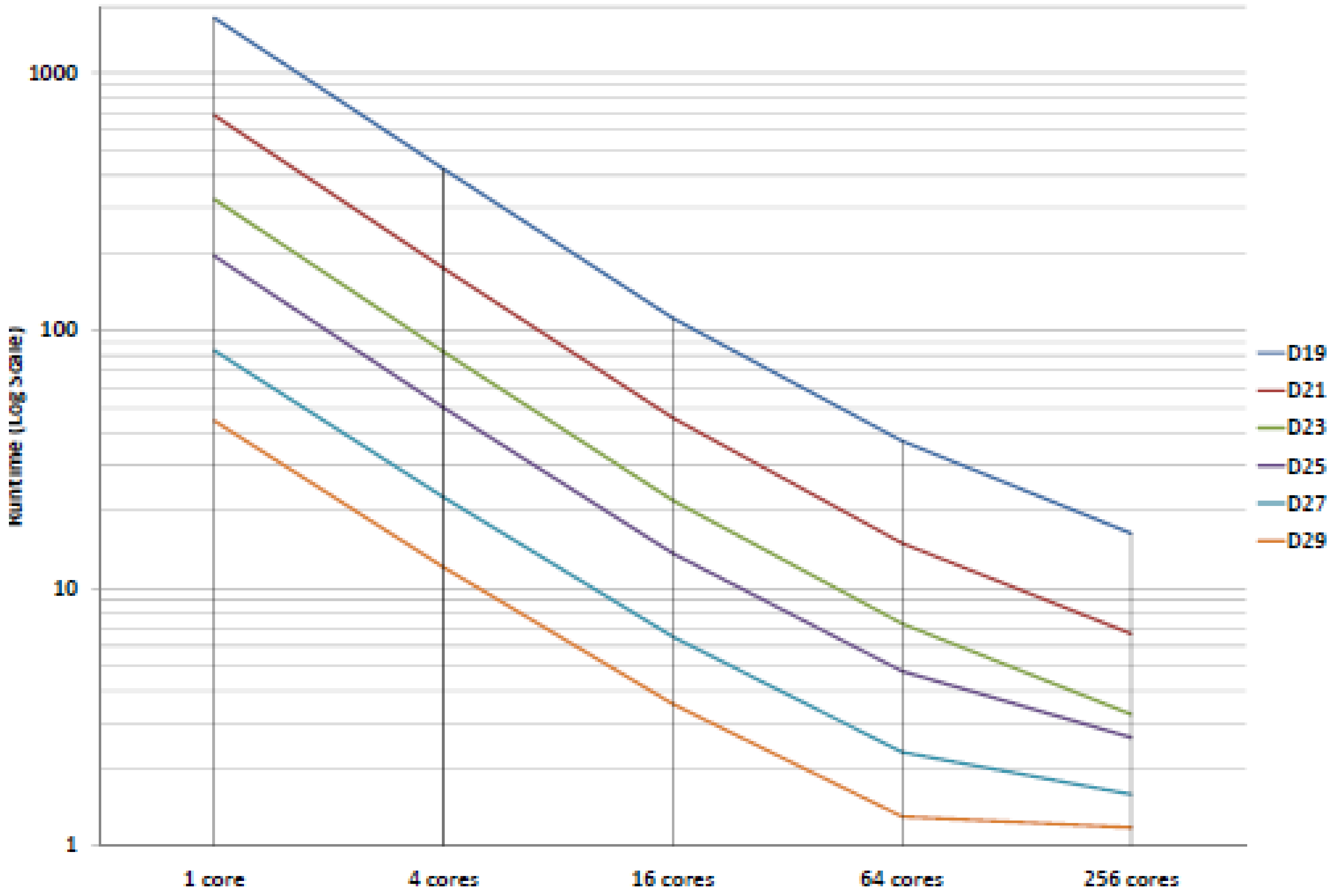
- New branching scheme is very suitable to parallelism
- Idea: explore **DP States** in different cores
  - Relatively little information needs to be shared
  - Most of the computational work involves computing relaxations/restrictions, done locally by each computer core
  - Easier to distribute load
- Joint work with Horst Samulowitz, Vijay Saraswat (IBM Research), and Ashish Sabharwal (Allen Inst.)

Parallel Search: Why bother?

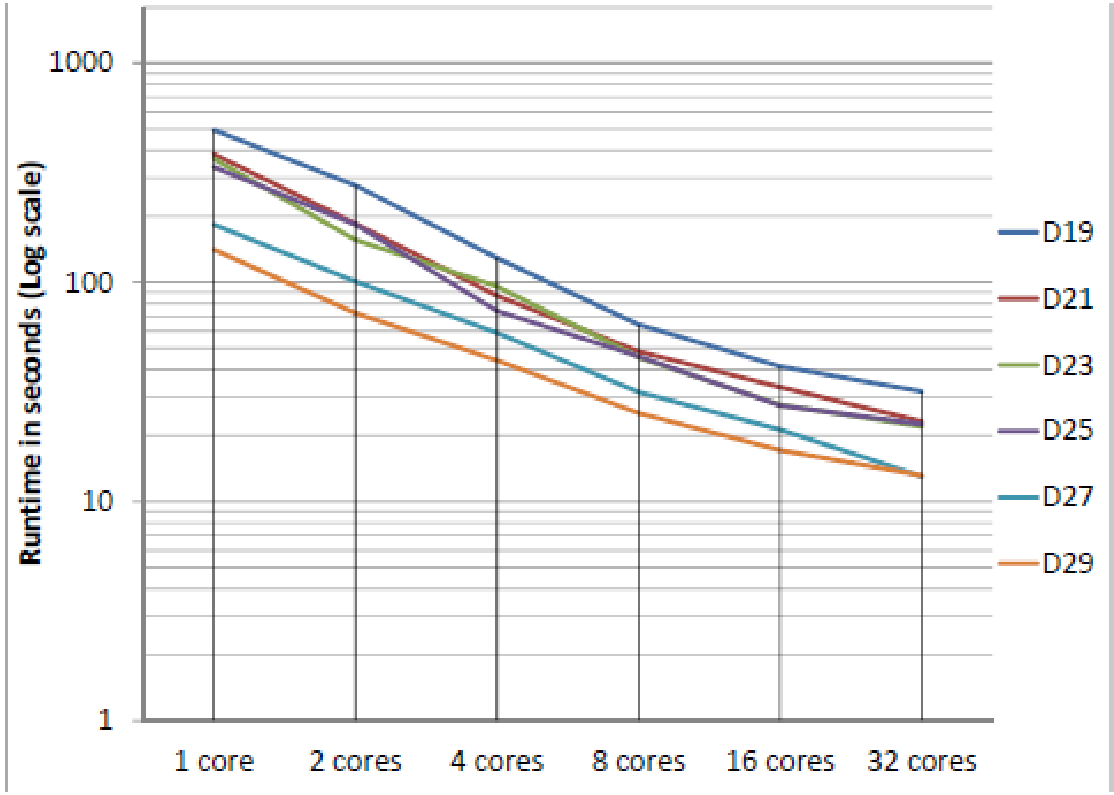
- Current technology
  - Integer Programming
    - Gurobi: Average speedup factor (Gu, 2013)
      - 1.7x on 5 cores
      - 1.8x on 25 cores
    - CPLEX (Mittleman, 2009)
      - 1.67x on 4 cores
  - SAT
    - 2013 SAT competition
      - 8x on 32 cores
  - Constraint Programming
    - Only focus on infeasible instances/finding all solutions

Parallel Search with Decision Diagrams

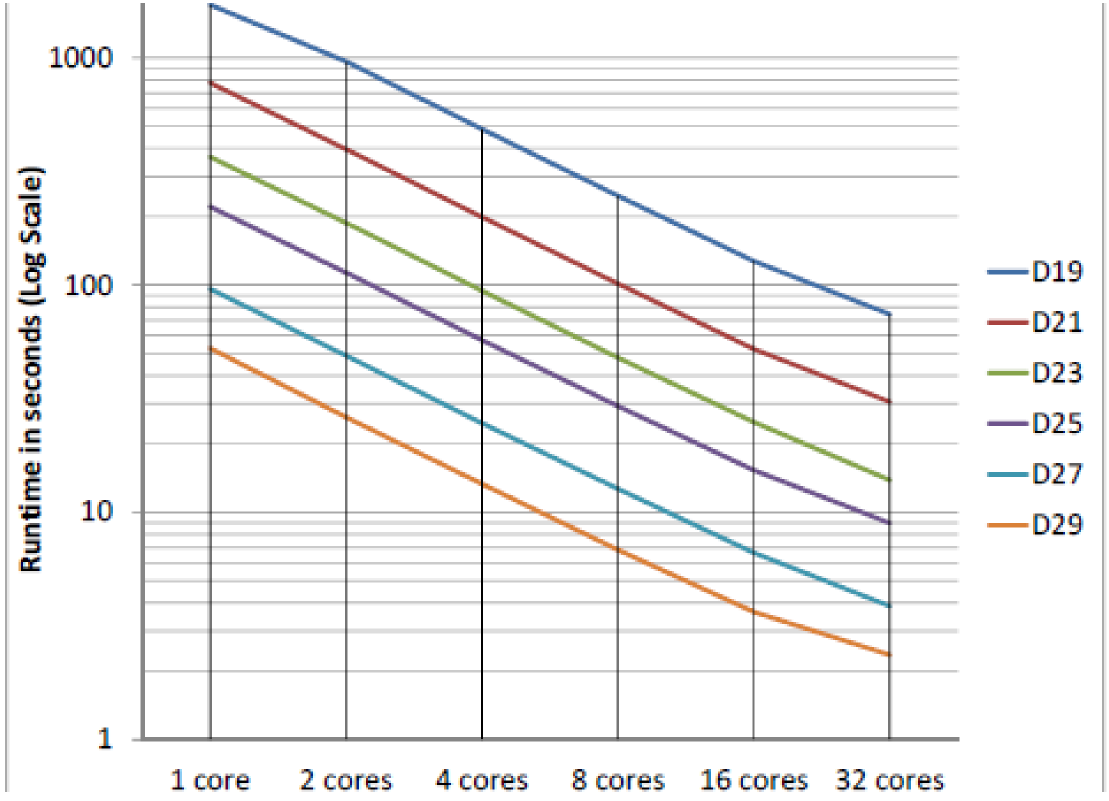
C125.9	**1 core**	**4 cores**	**16 cores**	**64 cores**	**256 cores**
Time to solve (s)	1100.91	277.07	70.74	19.53	8.07
Speedup	-	*3.97x*	*15.56x*	*56.37x*	*136.42x*



Runtime (Log Scale)
1000
100
10
1
1 core
4 cores
16 cores
64 cores
256 cores
D19
D21
D23
D25
D27
D29



CPLEX



Decision Diagrams

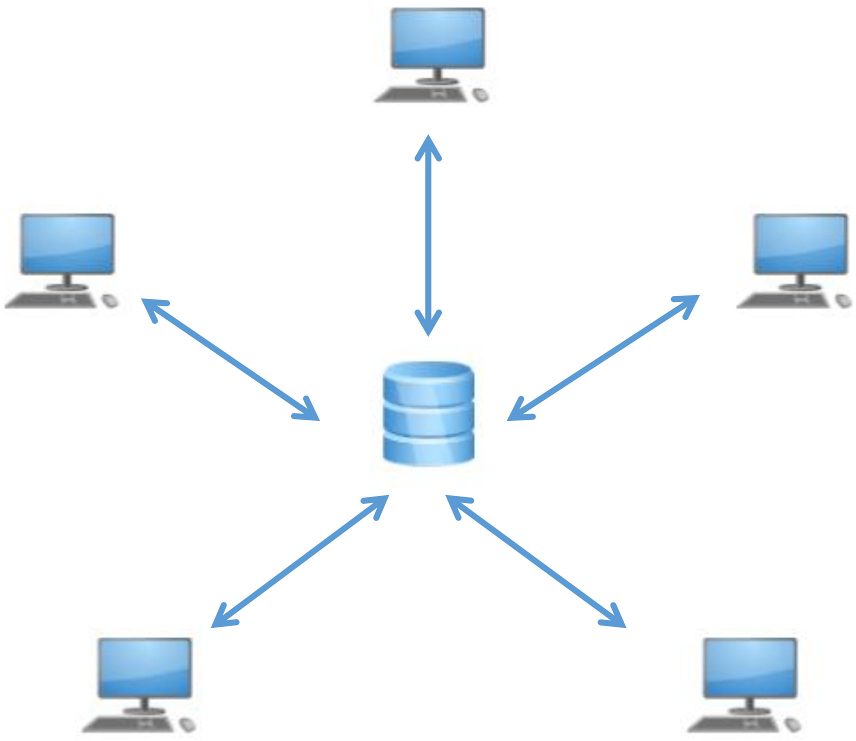
Thank you!

Decision Diagram Page:
http://www.andrew.cmu.edu/user/vanhoeve/mdd/

acire@utsc.utoronto.ca

Parallel Architecture

- We consider a **centralized architecture**
  - **Master** maintains a **pool of states** to process
  - **Workers** receive states, generate relaxed diagrams, and send new states to master
- Suitable to small architectures (up to 256 cores)



Master & Workers Pools

- Master keeps a **priority queue** of states
  - States with better optimization bounds have a higher priority of being explored
- Workers also keep a local priority queue
  - Relaxed (and restricted) decision diagrams are computed very quickly
  - Reduce communication to master
- Key issue: large memory consumption
  - Pools may grow quickly for very large problems
  - If memory is almost exceeded, priority queue becomes a regular queue (depth-first search)

Load Balancing

- Crucial question in many parallelization scheme
- In our case: How to distribute states among workers?
  - Too many nodes at once: many workers will be idle
  - Too few nodes: communication becomes bottleneck

Load Balancing

$$nodes\ to\ send = \min\left(c.\frac{size\ of\ pool}{number\ of\ cores}, \frac{avg\ states\ added}{c'}\right)$$

where **c** and **c’** are some constants (in our experiments, c = c’ = 2)

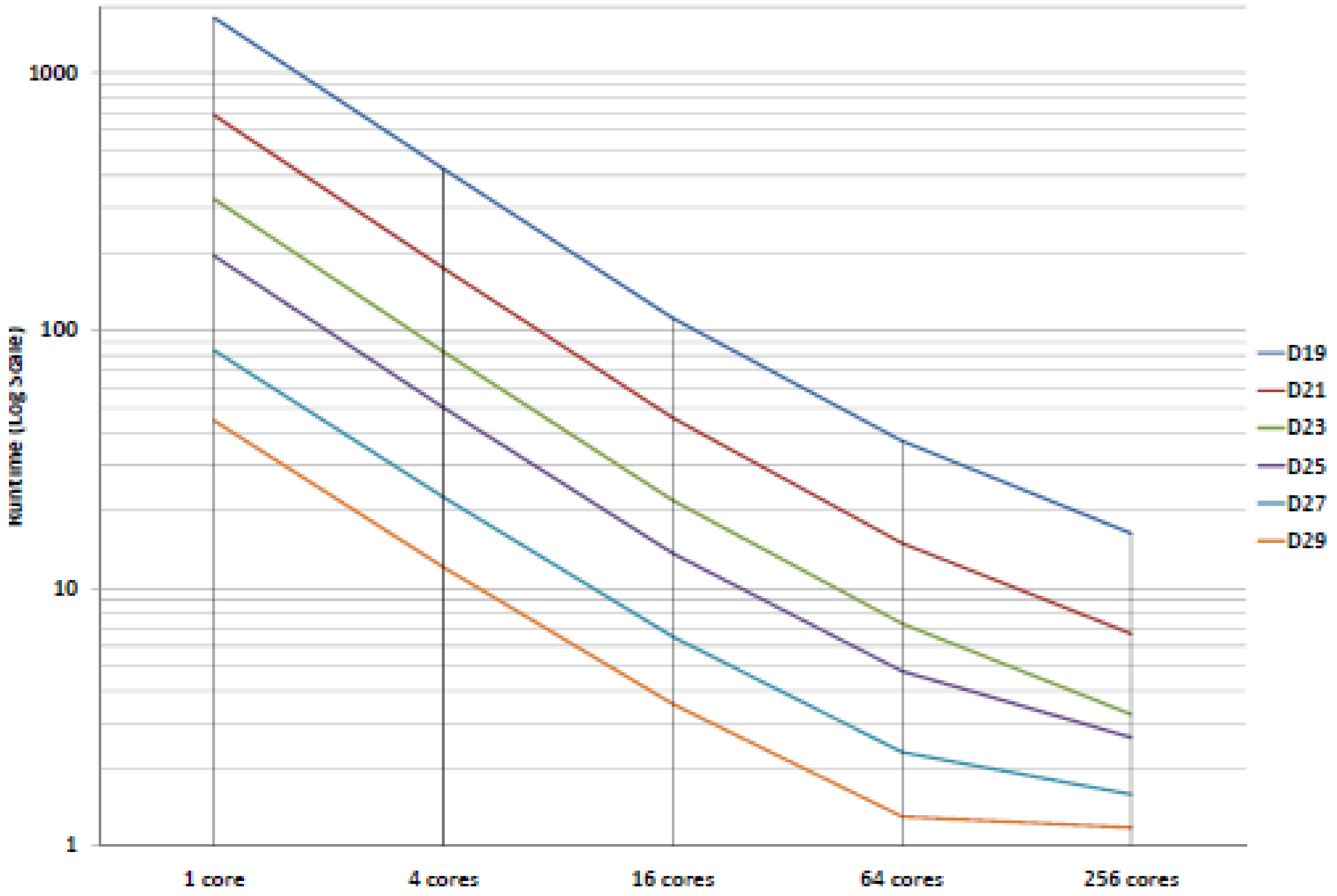
Load Balancing

75% of nodes with best optimization bounds

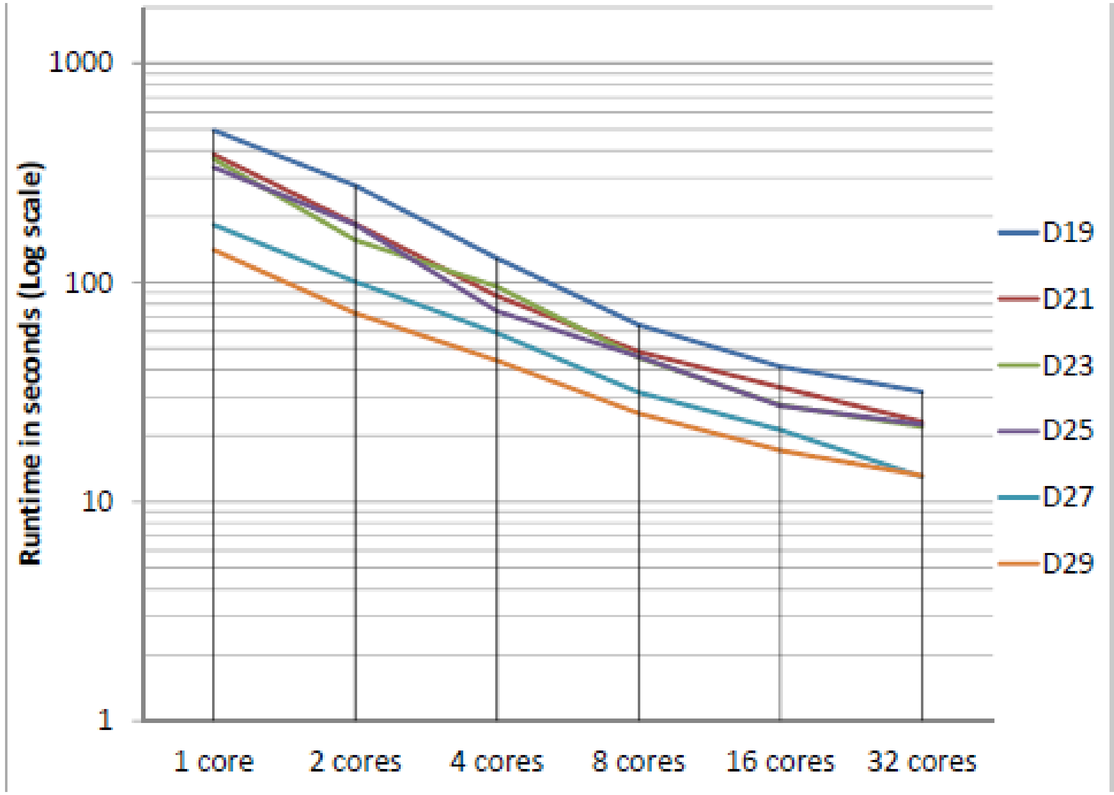
- Speed up the processing of promising nodes

Computational Results

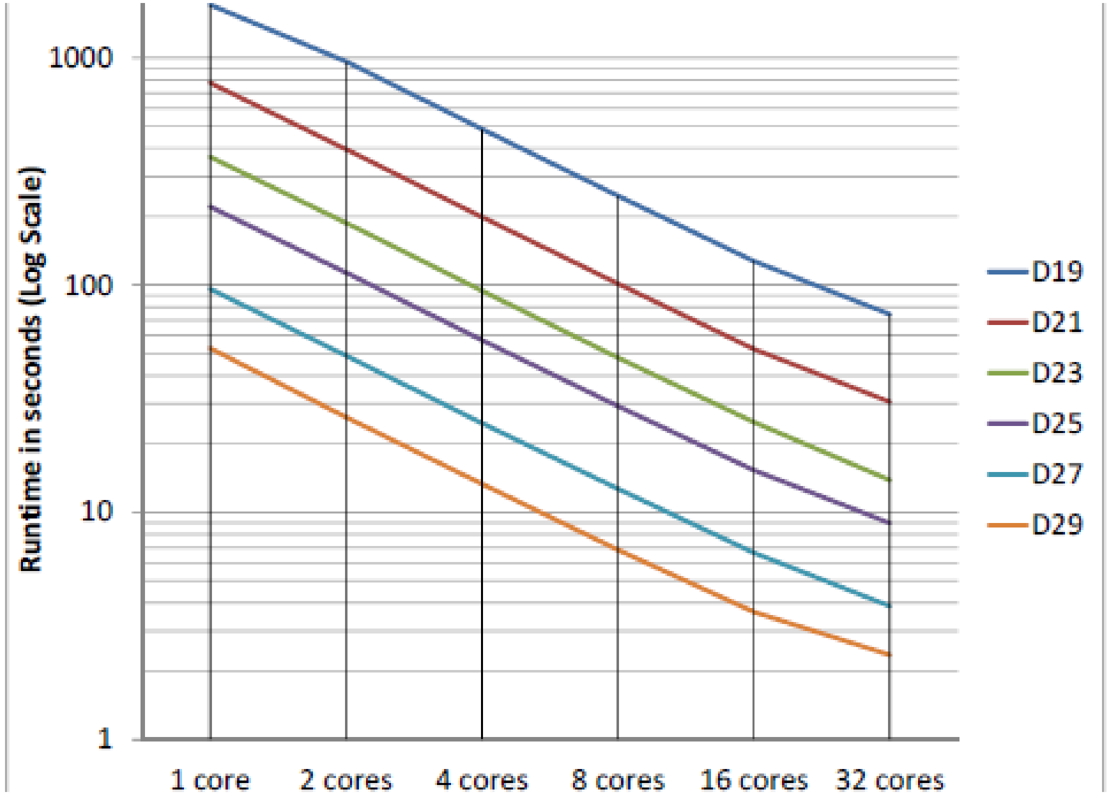
- Relaxed decision diagrams implemented in **C++**
- Parallel architecture implemented in **X10**
  - IBM X10 Team: Vijay Saraswat et al
  - x10-lang.org
- Tested in a computer cluster with 256 cores
  - 16 computers, each with 32 cores, 64 GB RAM



1000
100
10
1
Runtime (Log Scale)
1 core
4 cores
16 cores
64 cores
256 cores
D19
D21
D23
D25
D27
D29



CPLEX



Decision Diagrams

Other results

- Also observe same behaviour for other problem classes
  - Proved optimality for some maxcut instances for the first time
  - Testing on some variations of constrained TSP

- Other architectures
  - Work-stealing models

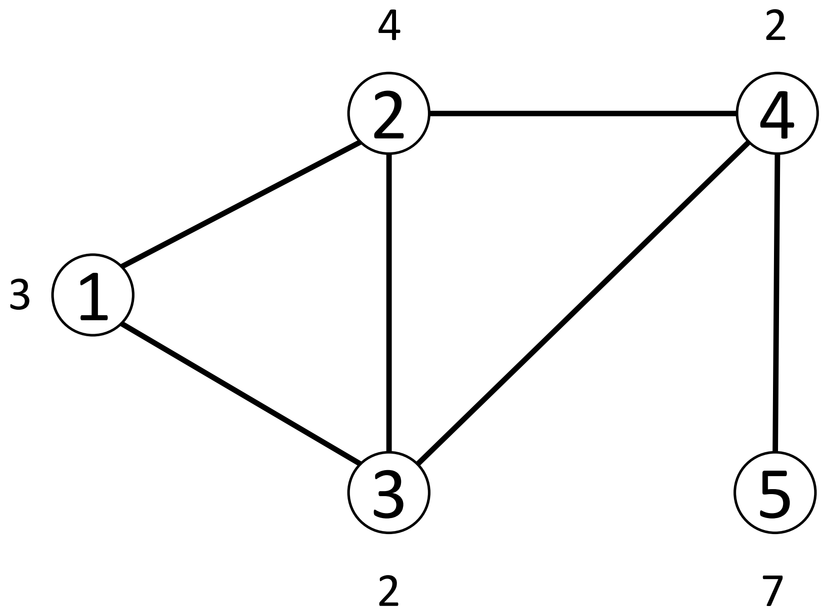
Thank you!

Relaxed Decision Diagrams

- Computational study on the max. independent set problem
  - Able to provide tighter bounds than integer programming models
- Application on Single-Machine Scheduling Problems
  - Closed open TSPLib instances, orders of magnitude improvement over constraint programming models, plus theoretical properties
- Application on Timetabling Problems
  - Orders of magnitude speed up in solving times compared to state-of-the-art approaches, plus theoretical properties

Modeling Framework

Ex.: Maximum independent set problem

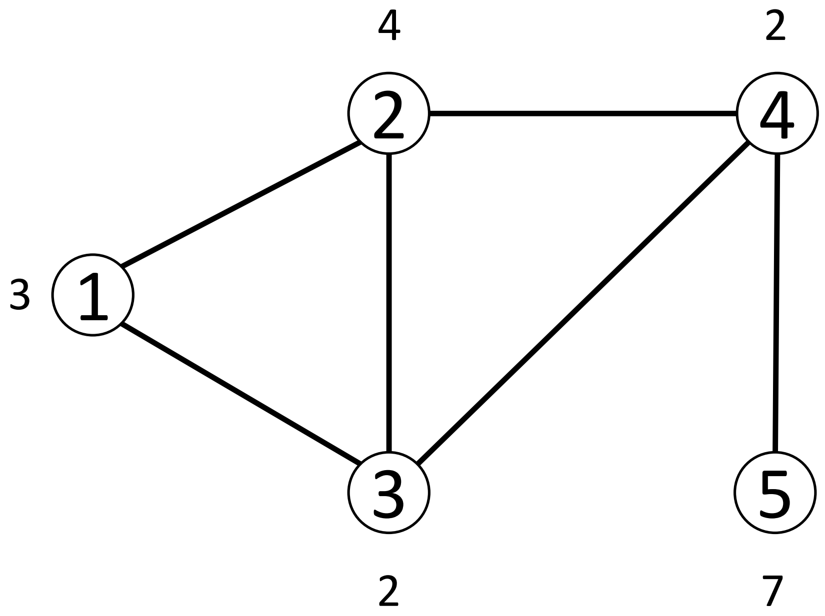


- Our model: **Dynamic Programming**
  - Exploit recursiveness
  - Solved by **stages**
  - Passing from one stage to another corresponds to transitioning from a **state** to another

- Decision diagram: **State-Transition Graph**
  - **Nodes** corresponds to **states**
  - **Arcs** are **state transitions**
  - **Arc weights** are **transition costs**

Modeling Framework

Ex.: Maximum independent set problem



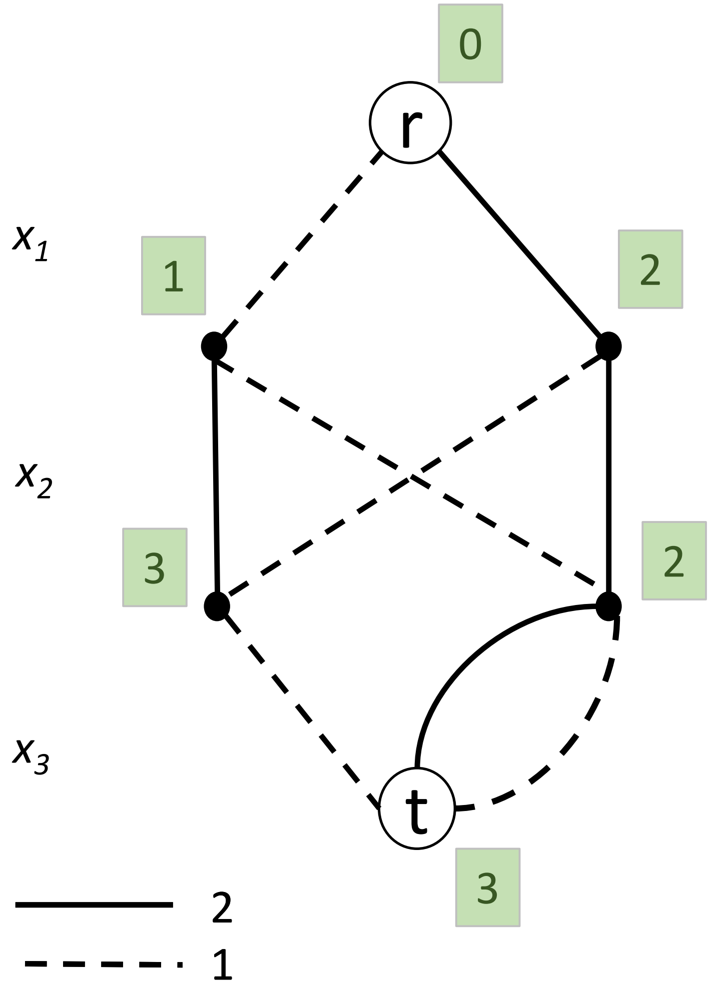
- DP model for the maximum independent set:

$$V_i(S) = \begin{cases} max\,\{V_{i-1}(S \setminus \{i\}), V_{i-1}(S \setminus N(i)) + 1\}, & i \in S \\ V_{i-1}(S \setminus N(i)), & o.w. \end{cases}$$

$$V_i(\emptyset) = 0, \qquad i = 1, \ldots, 5$$

- **Highlights:**
  - Stage i: **select** vertex i
  - State: set of **eligible** vertices

Filtering



max $4x_1 + 4x_2 + x_3$

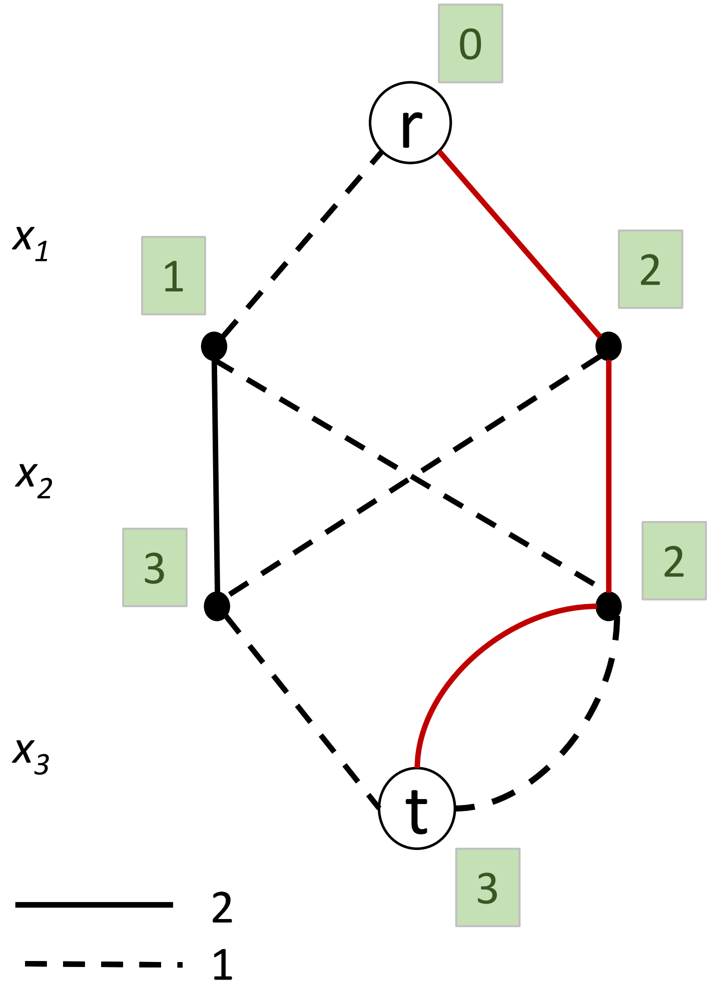
subject to

$$x_1 + x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \in \{1,2\}$$

- **Max Width** = 2
- **State**: left-hand side of constraint

Filtering



max $4x_1 + 4x_2 + x_3$

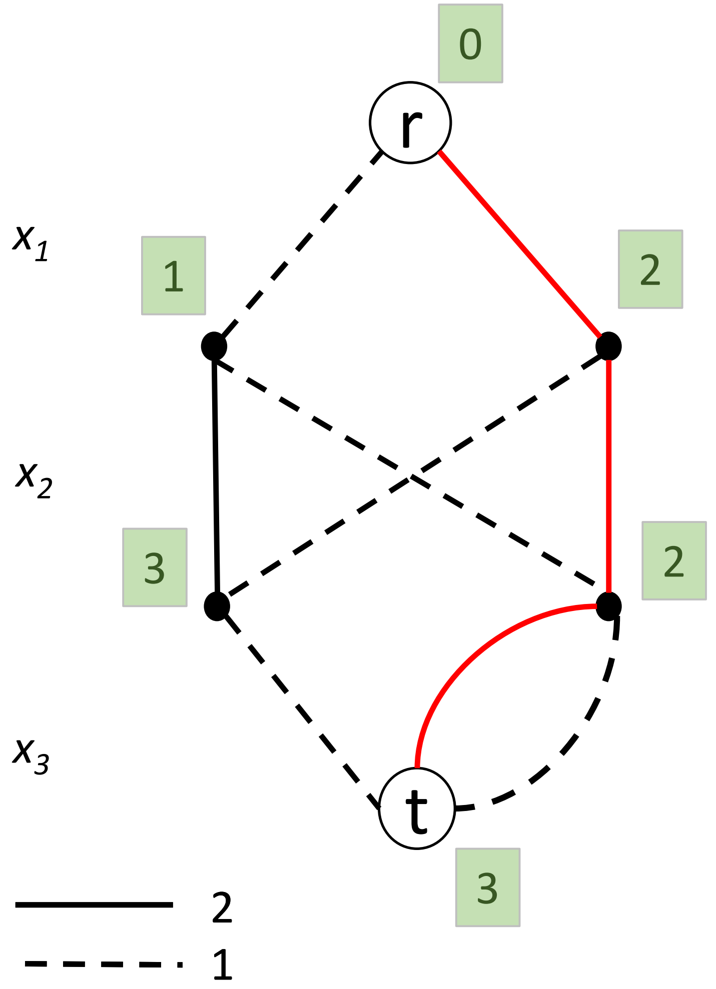
subject to

$$x_1 + x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \in \{1,2\}$$

- **Max Width** = 2
- **State**: left-hand side of constraint
- Longest path: $x_1 = x_2 = x_3 = 1$

Filtering

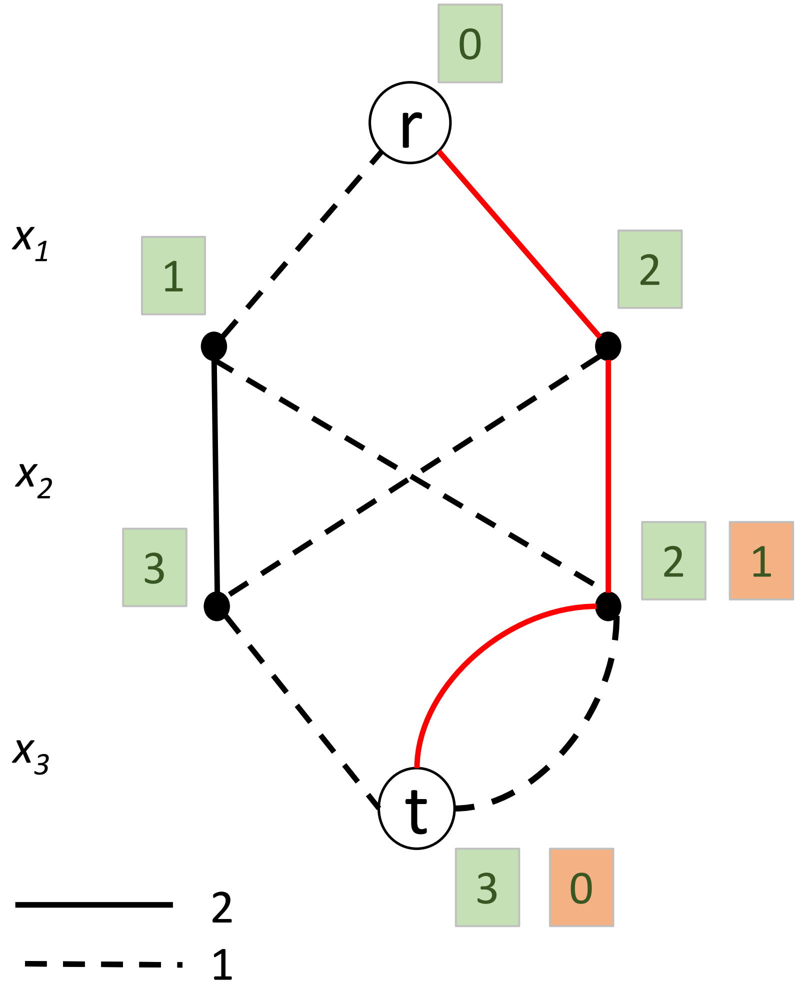


max $4x_1 + 4x_2 + x_3$
subject to
$x_1 + x_2 + x_3 \leq 4$
$x_1, x_2, x_3 \in \{1,2\}$

- Note that **top-down** is a **forward recursion:**

$$V_i(\ldots) = V_{i-1}(\ldots) + \ldots$$

Filtering



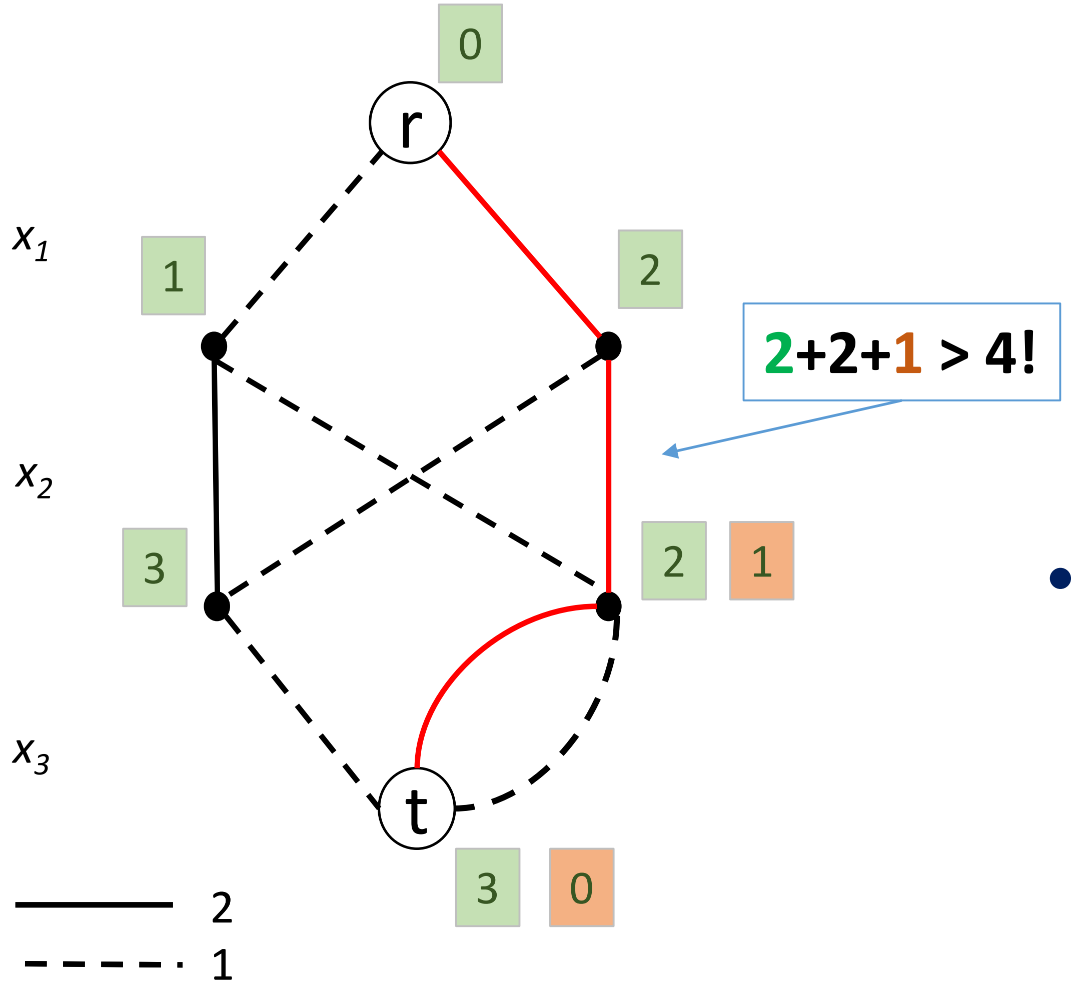
$$\max\ 4x_1 + 4x_2 + x_3$$

subject to

$$x_1 + x_2 + x_3 \leq 4$$
$$x_1, x_2, x_3 \in \{1,2\}$$

- But what happens when we do a **backward recursion**?

Filtering



max $4x_1 + 4x_2 + x_3$

subject to

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- But what happens when we do a **backward recursion**?

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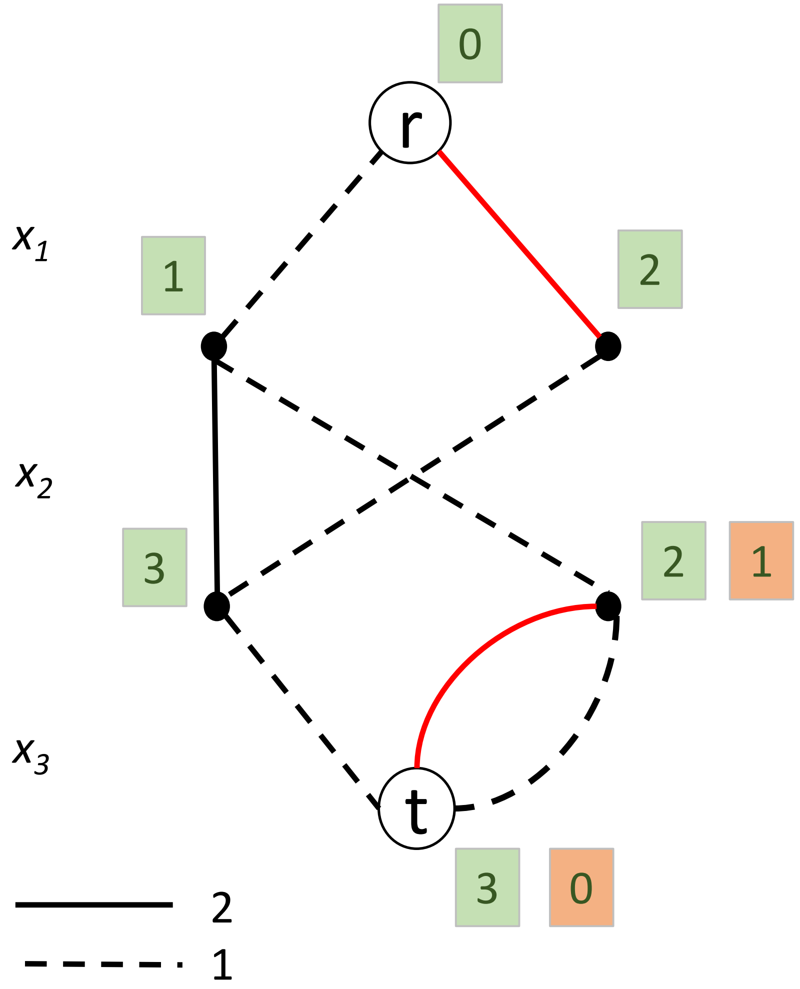


max $4x_1 + 4x_2 + x_3$

subject to

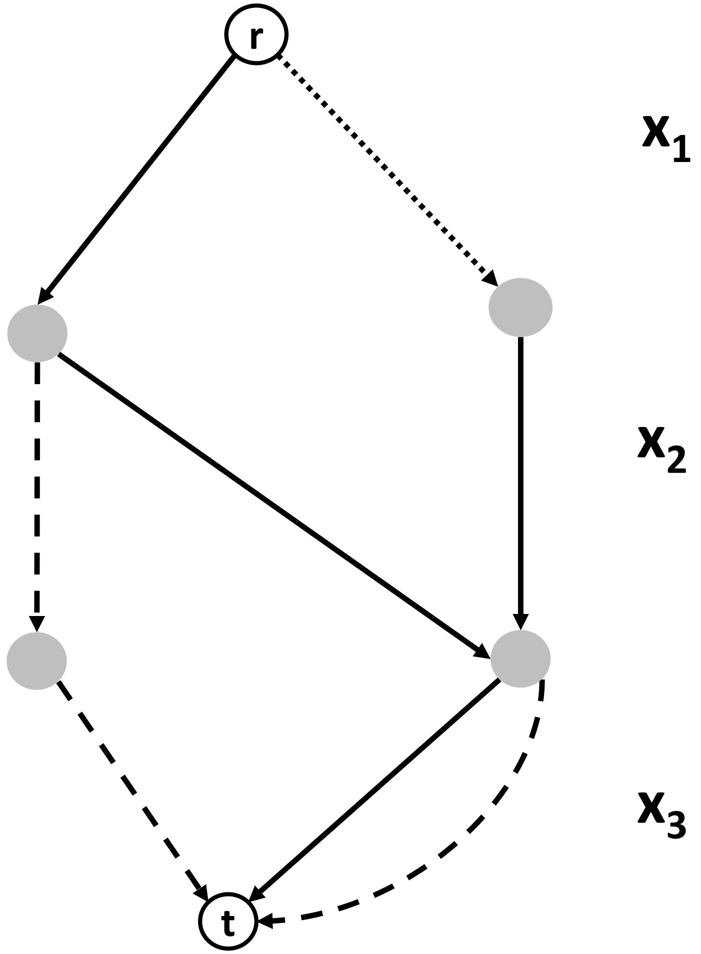
$$x_1 + x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \in \{1,2\}$$

- Underlying concept: Use **“redundant” DP formulations** to remove arcs, e.g.:

$$\mathbf{V'_i(\ldots) = V'_{i-1}(\ldots) + V'_{i+1}(\ldots) + \ldots}$$

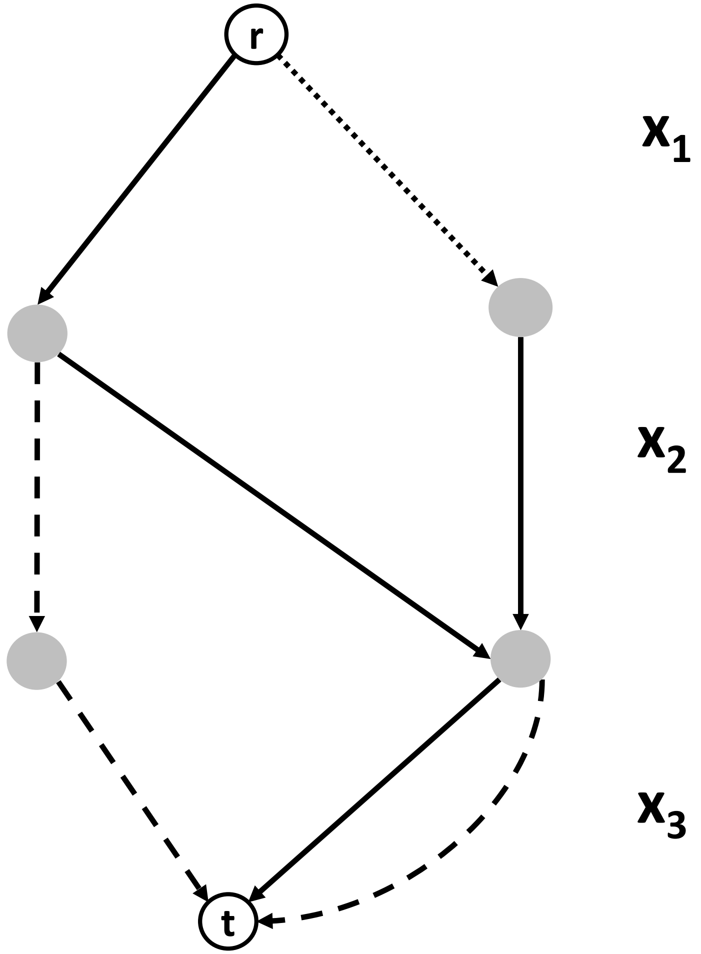
Some theoretical insights



- Let **X** the set of solutions represented by an MDD
- Optimizing a linear function **f** over the MDD is equivalent to solving the **LP problem**:

Minimize cx **subject to** **x is a flow from r to t**	=	**Minimize cx** **subject to** **x ∈ conv(X)**

Some theoretical insights



- Let **Ax ≥ b** be a set of constraints that we dualize over the MDD.
- If z* is the optimal shortest path after dualization, then

$$\mathbf{z^* =} \begin{array}{l} \textbf{Minimize cx} \\ \textbf{subject to} \\ \mathbf{Ax \geq b} \\ \mathbf{x \in conv(X)} \end{array}$$