Decision Diagrams for Constraint Programming

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Plan

What can MDDs do for Combinatorial Optimization?

- *Compact representation* of all solutions to a problem
- Limit on size gives *approximation*
- Control strength of approximation by size limit

MDDs for Constraint Programming and Scheduling

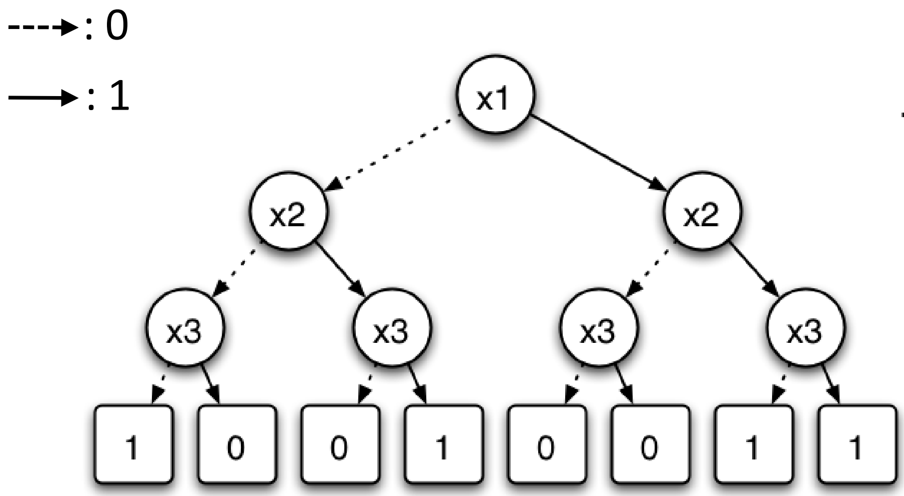
- MDD propagation natural generalization of domain propagation
- Orders of magnitude improvement possible

MDDs for Discrete Optimization

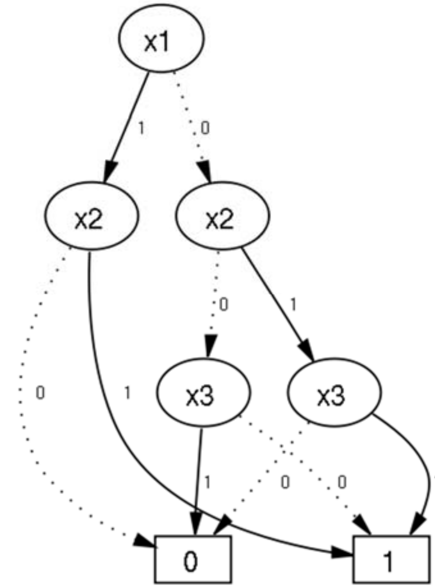
- MDD *relaxations* provide upper bounds
- MDD *restrictions* provide lower bounds
- New branch-and-bound scheme

Many Opportunities: integrated methods, theory, applications,…

Decision Diagrams



x1	x2	x3	*f*
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1



$$f(x_1, x_2, x_3) = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (x_1 \wedge x_2) \vee (x_2 \wedge x_3)$$

- Binary Decision Diagrams were introduced to compactly represent Boolean functions [Lee, 1959], [Akers, 1978], [Bryant, 1986]
- BDD: merge isomorphic subtrees of a given binary decision tree
- MDDs are multi-valued decision diagrams (i.e., for arbitrary finite-domain variables)

Brief background

- Original application areas: circuit design, verification
- Usually *reduced ordered* BDDs/MDDs are applied
  - fixed variable ordering
  - minimal exact representation
- Recent interest from optimization community
  - cut generation [Becker et al., 2005]
  - 0/1 vertex and facet enumeration [Behle & Eisenbrand, 2007]
  - post-optimality analysis [Hadzic & Hooker, 2006, 2007]
  - set bounds propagation [Hawkins, Lagoon, Stuckey, 2005]
- Interesting variant
  - relaxed MDDs

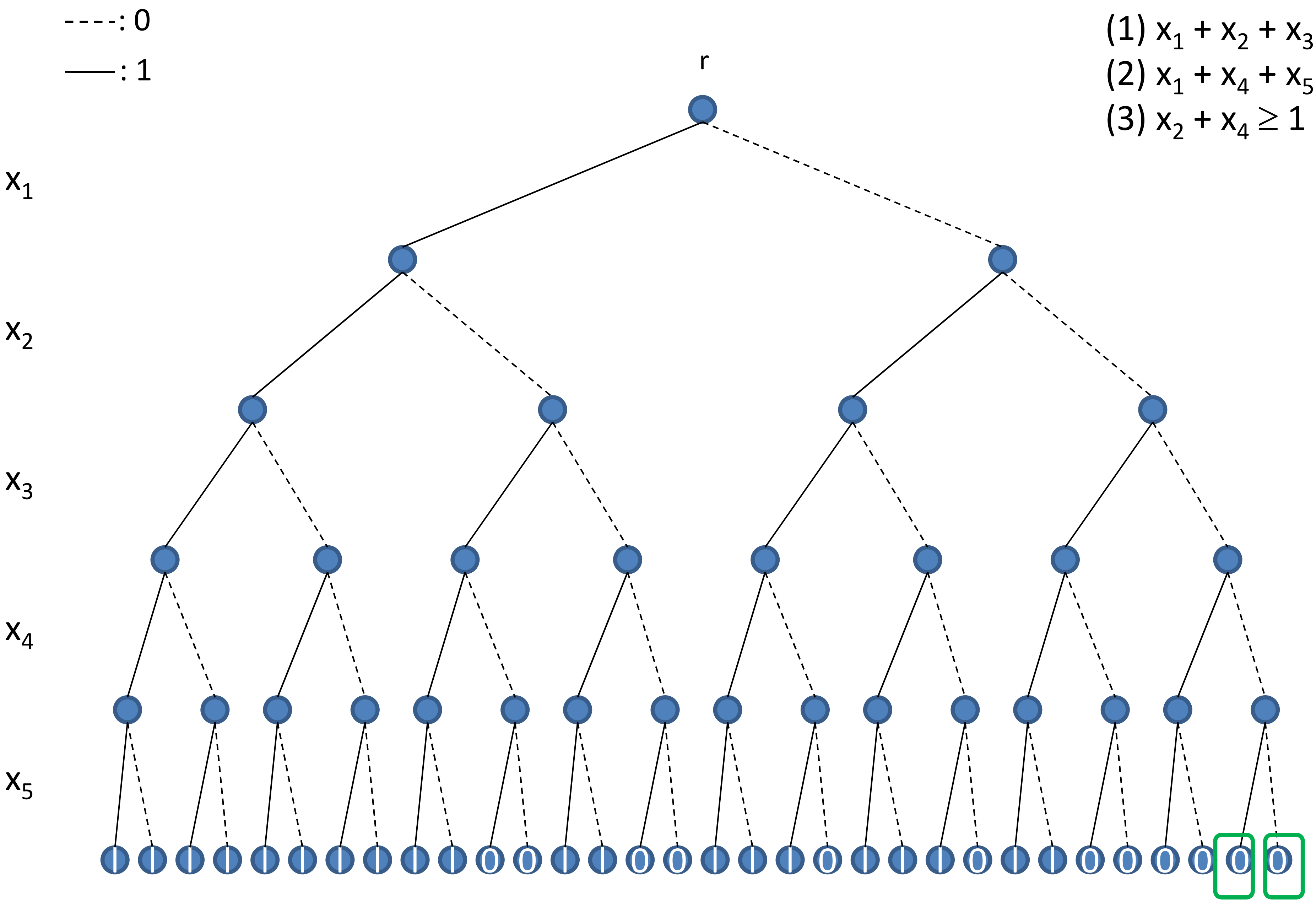
    [Andersen, Hadzic, Hooker & Tiedemann, CP 2007]

Exact MDDs for discrete optimization

(1) $x_1 + x_2 + x_3 \geq 1$

(2) $x_1 + x_4 + x_5 \geq 1$

(3) $x_2 + x_4 \geq 1$



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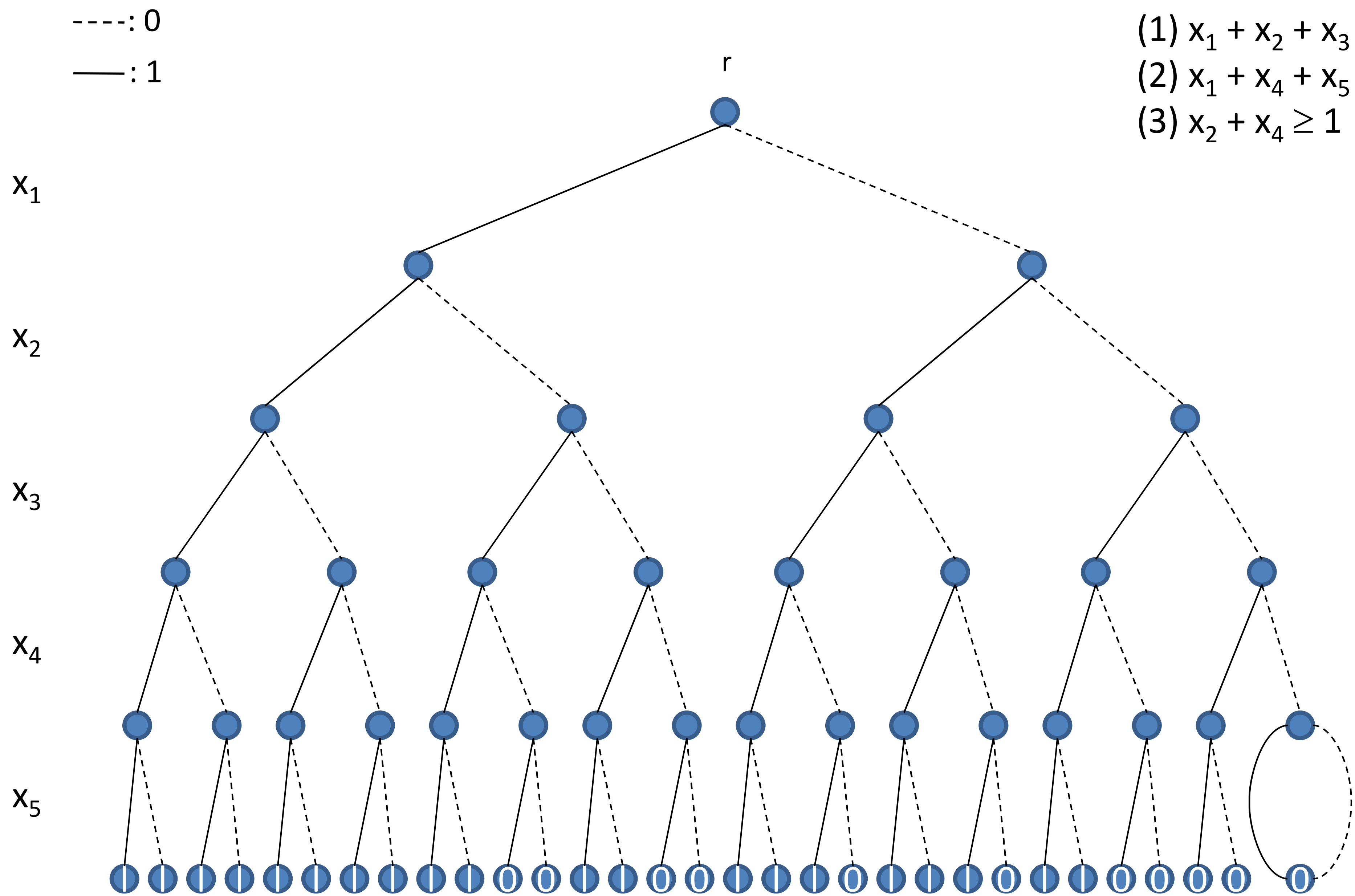

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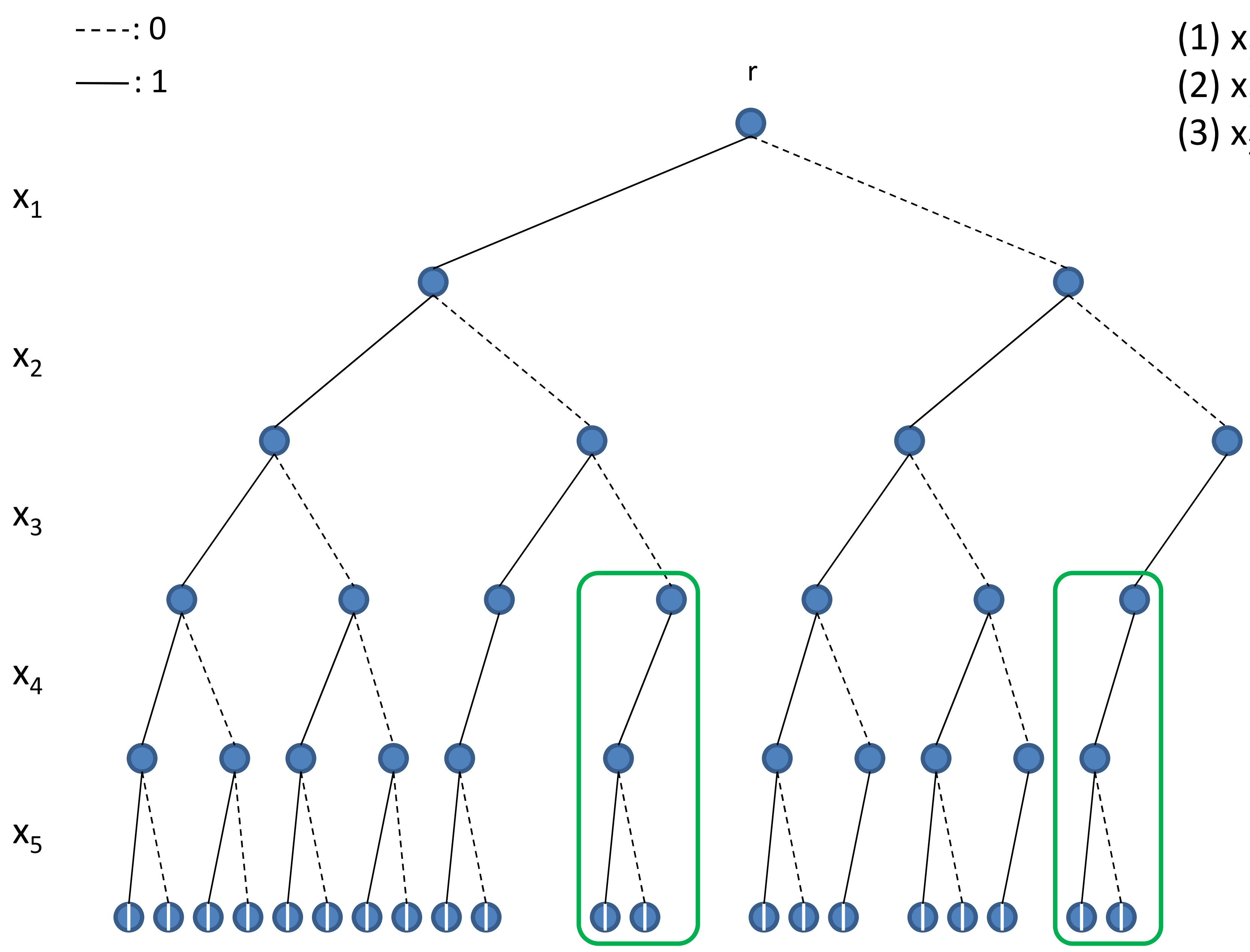
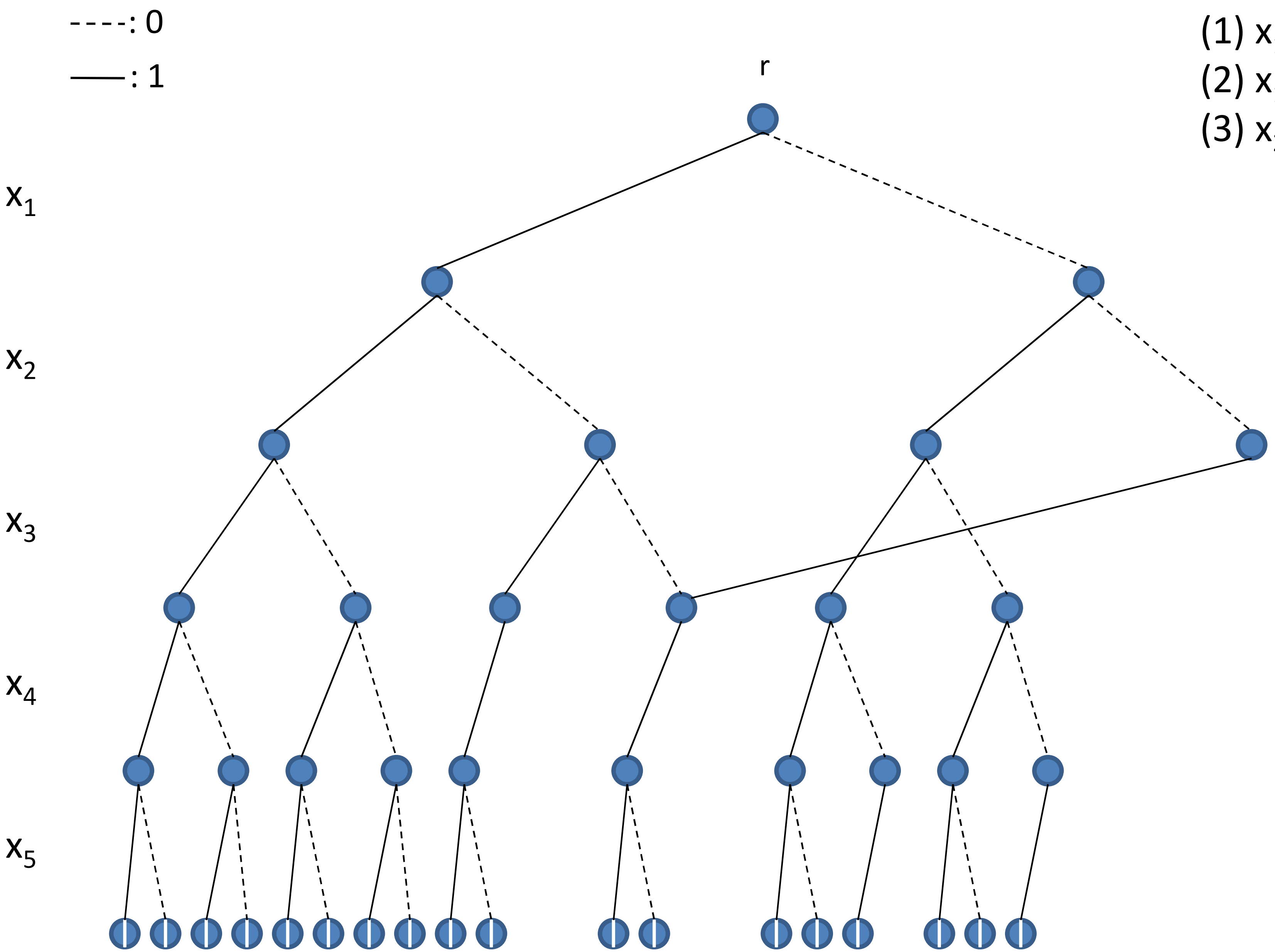


Exact MDDs for discrete optimization



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Exact MDDs for discrete optimization

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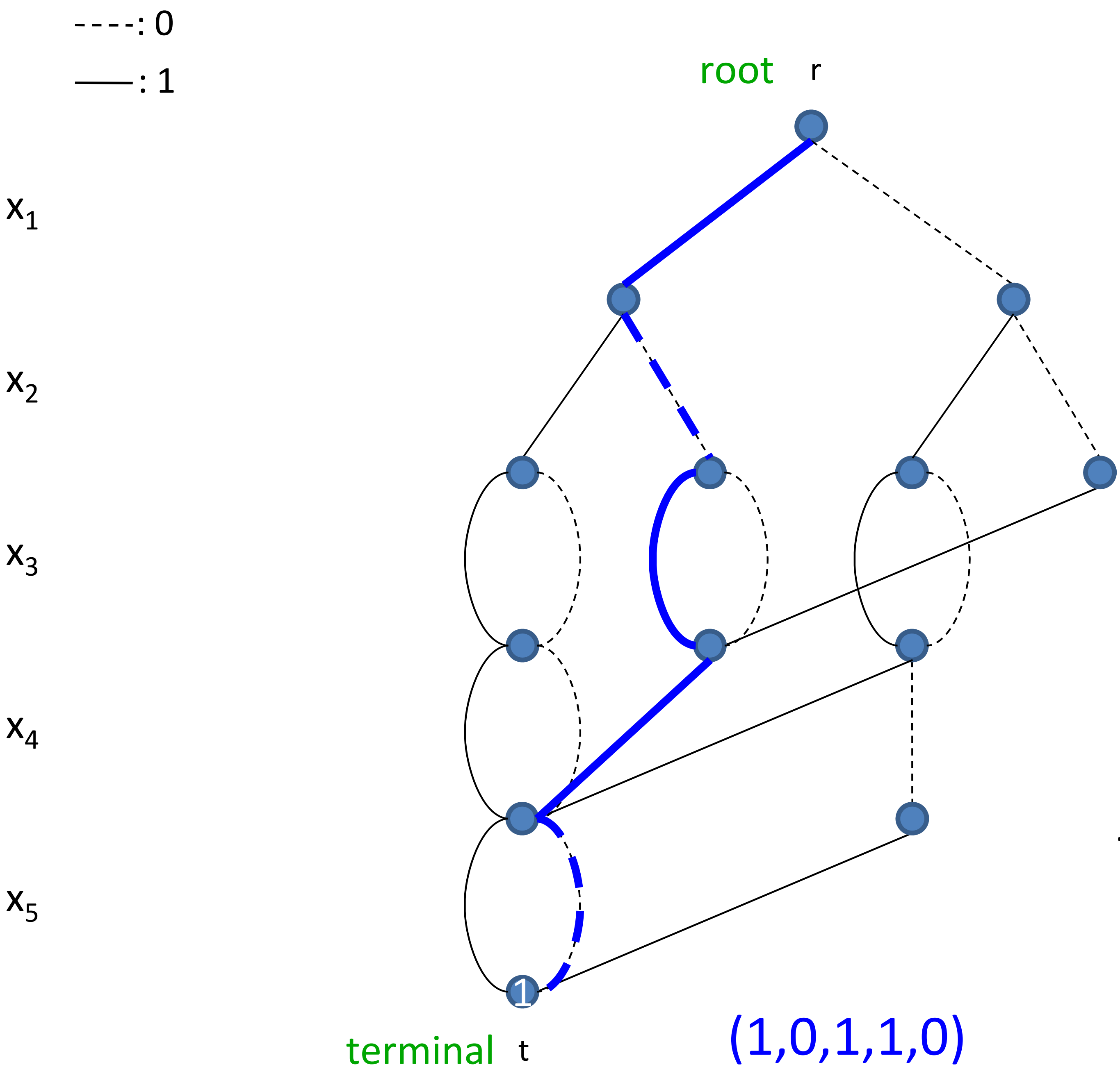


(1,0,1,1,0)

Each path corresponds to a solution

Approximate MDDs

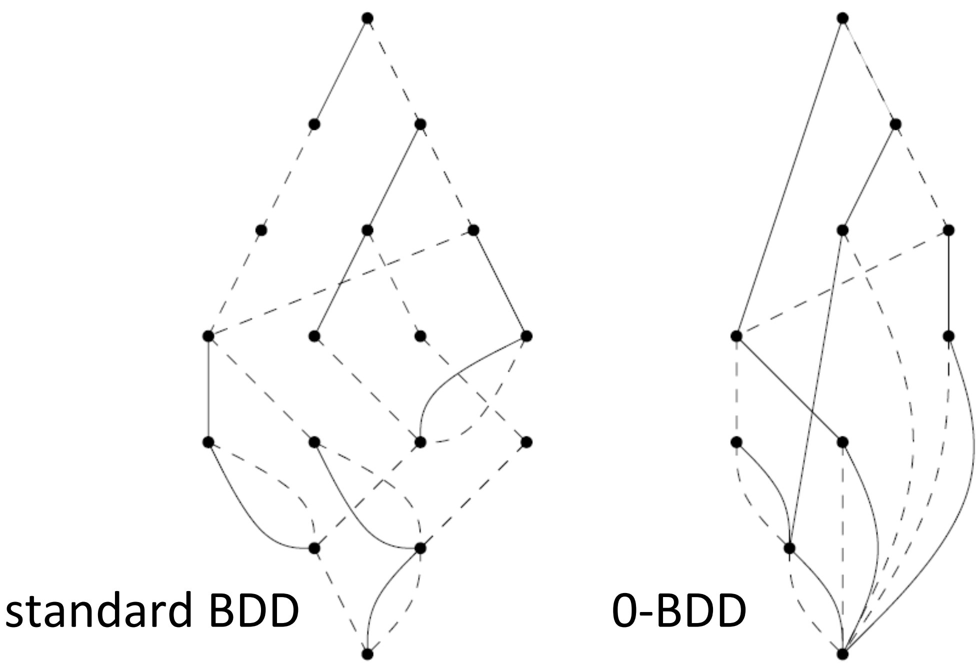
- Exact MDDs can be of exponential size in general
- Can we limit the size of the MDD and still have a meaningful representation?
  - Yes, first proposed by Andersen et al. [2007] :

    Limit the *width* of the MDD (the maximum number of nodes on any layer)

- Limited-width MDDs: main focus of this tutorial

Suppressed Decision Diagrams

- Zero-suppressed BDD (0-BDD or ZDD)
  - arc skips layers for which variables will take value 0
- One-suppressed BDD (1-BDD)
  - arc skips layers for which variables will take value 1
- Zero/one-suppressed BDD (0/1-BDD)
  - arc skips layers for which variables will take value 0/1



- Similarly suppressed MDDs can be defined
- Will not be discussed in detail, but methodology can be extended

MDDs for Constraint Programming

Motivation

Constraint Programming applies

- systematic search and
- inference techniques

to solve combinatorial problems

Inference mainly takes place through:

- Filtering provably inconsistent values from variable domains
- Propagating the updated domains to other constraints

$$x_1 > x_2$$

$$x_1 + x_2 = x_3$$

$$\textit{alldifferent}(x_1, x_2, x_3, x_4)$$

$$x_1 \in \{\cancel{1},2\},\ x_2 \in \{\cancel{0},1,\cancel{2},\cancel{3}\},\ x_3 \in \{\cancel{2},3\},\ x_4 \in \{0,\cancel{1}\}$$

Characterization of propagation

- Let C(X) be a constraint on variables X. Let D(x) denote the domain of possible values for x in X.
- Constraint C(X) is domain consistent if for each x in X, each v in D(x) belongs to a solution to C.

$x_1 \neq x_2$

$x_1 \neq x_3$

$x_2 \neq x_3$

alldifferent(x_1,x_2,x_3)

$x_1 \in \{2,3\}$, $x_2 \in \{1,2,3\}$, $x_3 \in \{2,3\}$

- *Establish domain consistency*: Remove all inconsistent values from the variable domains.

Illustrative example

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alldifferent(x_1,x_2,x_3,x_4)	(1)
$x_1 + x_2 + x_3 \geq 9$	(2)
$x_i \in \{1,2,3,4\}$	

(1) and (2) both domain consistent

List of all solutions to *alldifferent*:

x_1	x_2	x_3	x_4
~~1~~	~~2~~	~~3~~	~~4~~
~~1~~	~~2~~	~~4~~	~~3~~
~~1~~	~~3~~	~~2~~	~~4~~
	…		
4	3	2	1

Suppose we could evaluate (2) on this list

projection: $D(x_i) = \{1,2,3,4\}$

Illustrative example

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alldifferent(x_1,x_2,x_3,x_4)	(1)
$x_1 + x_2 + x_3 \geq 9$	(2)
$x_i \in \{1,2,3,4\}$	

List of all solutions to *alldifferent*:

	x_1	x_2	x_3	x_4
✓	2	3	4	1
✓	2	4	3	1
✓	3	2	4	1
		…		
✓	4	3	2	1

← Suppose we could evaluate (2) on this list

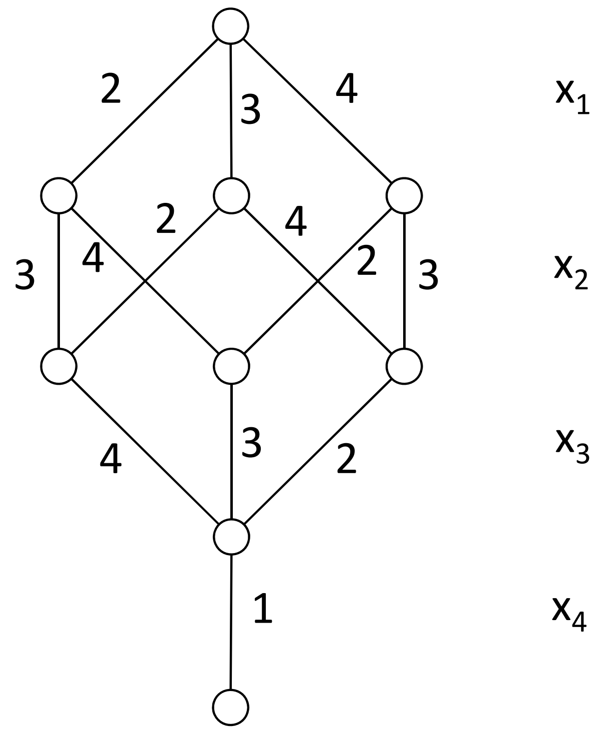
→ projection: $D(x_4) = \{1\}$

Illustrative example (cont'd)

$$alldifferent(x_1,x_2,x_3,x_4) \quad (1)$$
$$x_1 + x_2 + x_3 \geq 9 \quad (2)$$
$$x_i \in \{1,2,3,4\}$$

List of all solutions: use MDDs

x_1	x_2	x_3	x_4
2	3	4	1
2	4	3	1
3	2	4	1
	…		
4	3	2	1



Motivation for MDD propagation

- All structural relationships among variables are projected onto the domains
- Potential solution space implicitly defined by Cartesian product of variable domains (very coarse relaxation)

We can communicate more information between constraint using MDDs [Andersen et al. 2007]

- Explicit representation of more refined potential solution space
- Limited width defines *relaxed* MDD
- Strength is controlled by the imposed width

MDD-based Constraint Programming

- Maintain limited-width MDD
  - Serves as relaxation
  - Typically start with width 1 (initial variable domains)
  - Dynamically adjust MDD, based on constraints
- Constraint Propagation
  - Edge filtering: Remove provably inconsistent edges (those that do not participate in any solution)
  - Node refinement: Split nodes to separate edge information
- Search
  - As in classical CP, but may now be guided by MDD

Characterization of MDD Propagation

Domain consistency generalizes naturally to MDDs:

- Let C(X) be a constraint on variables X and let M be an MDD on X
- Constraint C is MDD consistent if for each arc in M, there is at least one path in M that represents a solution to C

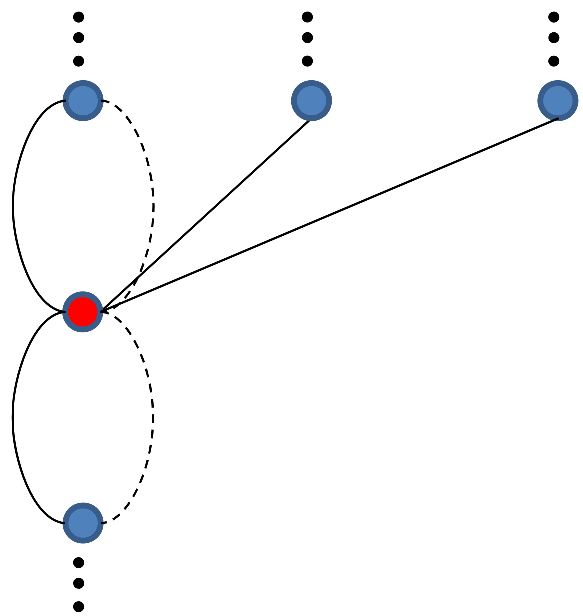
Equivalent to domain consistency for MDD of width 1

Specific MDD propagation algorithms

- Linear equalities and inequalities [Hadzic et al., 2008] [Hoda et al., 2010]
- *Alldifferent* constraints [Andersen et al., 2007]
- *Element* constraints [Hoda et al., 2010]
- *Among* constraints [Hoda et al., 2010]
- Disjunctive scheduling constraints [Hoda et al., 2010] [Cire & v.H., 2011, 2013]
- *Sequence* constraints (combination of *Amongs*) [Bergman et al., 2014]
- Generic re-application of existing domain filtering algorithm for any constraint type [Hoda et al., 2010]

Constraint Representation in MDDs

- For a given constraint type we maintain specific 'state information' at each node in the MDD
- Computed from incoming arcs (both from top and from bottom)
- State information is basis for MDD *filtering* and for MDD *refinement*



First example: Among constraints

- Given a set of variables X, and a set of values *S*, a lower bound *l* and upper bound *u*,

$$Among(X, S, l, u) := \; l \leq \sum_{x \in X} (x \in S) \leq u$$

"among the variables in X, at least *l* and at most *u* take a value from the set *S*"

- Applications in, e.g., sequencing and scheduling

Example: Nurse Rostering

- Set of nurses N, who can work evening, day, or night shift, or can have a day off: {e,d,n,o}
- Rules: Each nurse works
  - at most 2 night shifts out of every 8 consecutive days,
  - at least 22 work shifts out of every 30 consecutive days,
- Planning horizon: 80 days
- For each day i, we have fixed demand $D_{i,s}$ for each shift s = e,d,n.
- Goal: Find a solution that meets demand and satisfies all constraints.

CP model

- Variables $x_{i,n}$: shift of nurse n on day i

  $D(x_{i,n}) = \{e,d,n,o\}$

- Constraints

meet demand: $\sum_n (x_{i,n}=s) \geq D_{i,s}$ for all i,s

at most 2/8: $\mathit{Among}(\{x_{i,n},\ldots,x_{i+7,n}\}, \{n\}, 0, 2)$

for all i=1,…,73, and n

at least 22/30: $\mathit{Among}(\{x_{i,n},\ldots,x_{i+29,n}\}, \{e,d,n\}, 22, 30)$

for all i=1,…,51, and n

First example: Among constraints

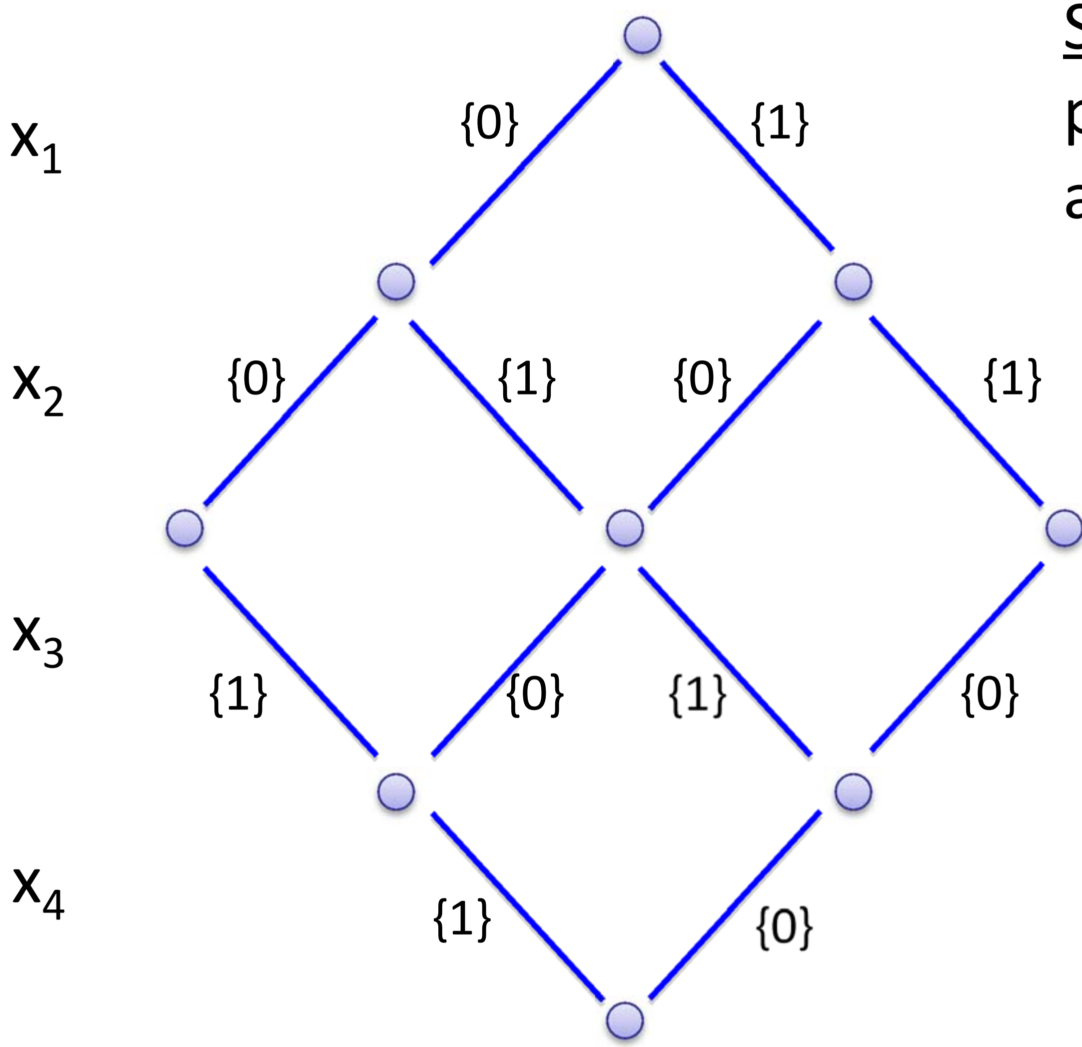
- Given a set of variables X, and a set of values *S*, a lower bound *l* and upper bound *u*,

$$\mathit{Among}(X, S, l, u) := \ l \leq \sum_{x \in X} (\ x \in S\ ) \leq u$$

"among the variables in X, at least *l* and at most *u* take a value from the set *S*"

- Applications in, e.g., sequencing and scheduling
- WLOG assume here that X are binary and *S* = {1}
- Let's develop an MDD propagation algorithm

Example MDD for Among



State information:
path length from top and from bottom

Exact MDD for *Among*($\{x_1,x_2,x_3,x_4\},\{1\},2,2$)

MDD Filtering for Among

Goal: Given an MDD and an *Among* constraint, remove *all* inconsistent edges from the MDD

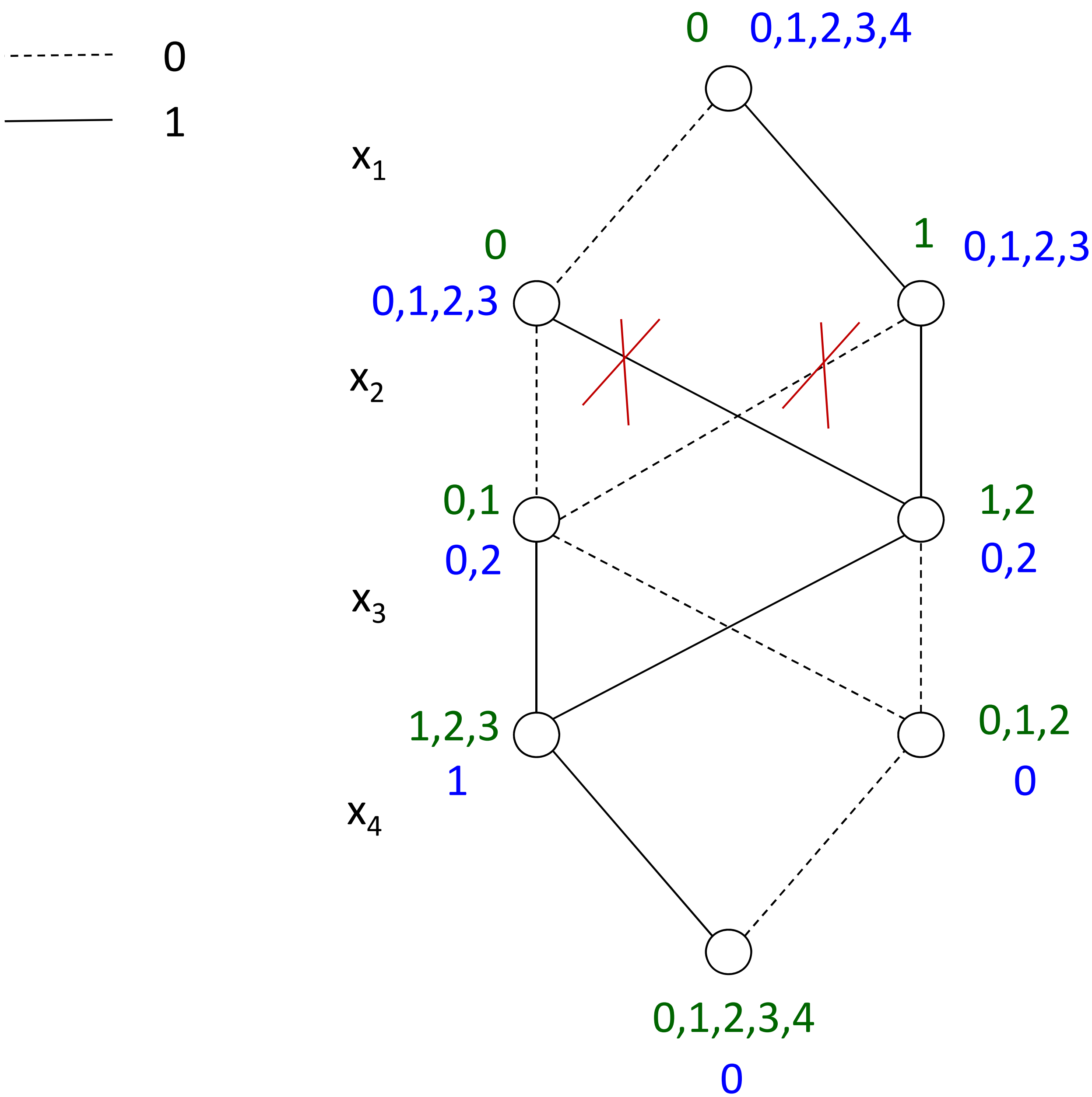
(establish MDD-consistency)

Approach:

- Compute path lengths from the root and from the sink to each node in the MDD
- Remove edges that are not on a path with length between lower and upper bound

[Hoda et al., CP 2010]

Example



State: T U

path lengths from top (T)
and from bottom (U)

What happens if we only maintain bounds instead of all path lengths?

Among$(\{x_1,x_2,x_3,x_4\},\{1\},2,2)$

MDD Filtering for Among

Goal: Given an MDD and an *Among* constraint, remove *all* inconsistent edges from the MDD

(establish MDD-consistency)

Approach:

- Compute path lengths from the root and from the sink to each node in the MDD
- Remove edges that are not on a path with length between lower and upper bound
- Complete (MDD-consistent) version
  - Maintain all path lengths; quadratic time
- Partial version (may not remove all inconsistent edges)
  - Maintain and check bounds (longest and shortest paths); linear time

Node refinement for Among

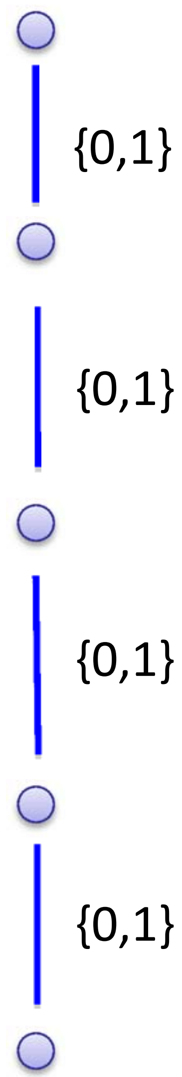
For each layer in MDD, we first apply edge filter, and then try to refine

- consider incoming edges for each node
- split the node if there exist incoming edges that are not equivalent (w.r.t. path length)
- in other words, need to identify *equivalence classes*

Example:

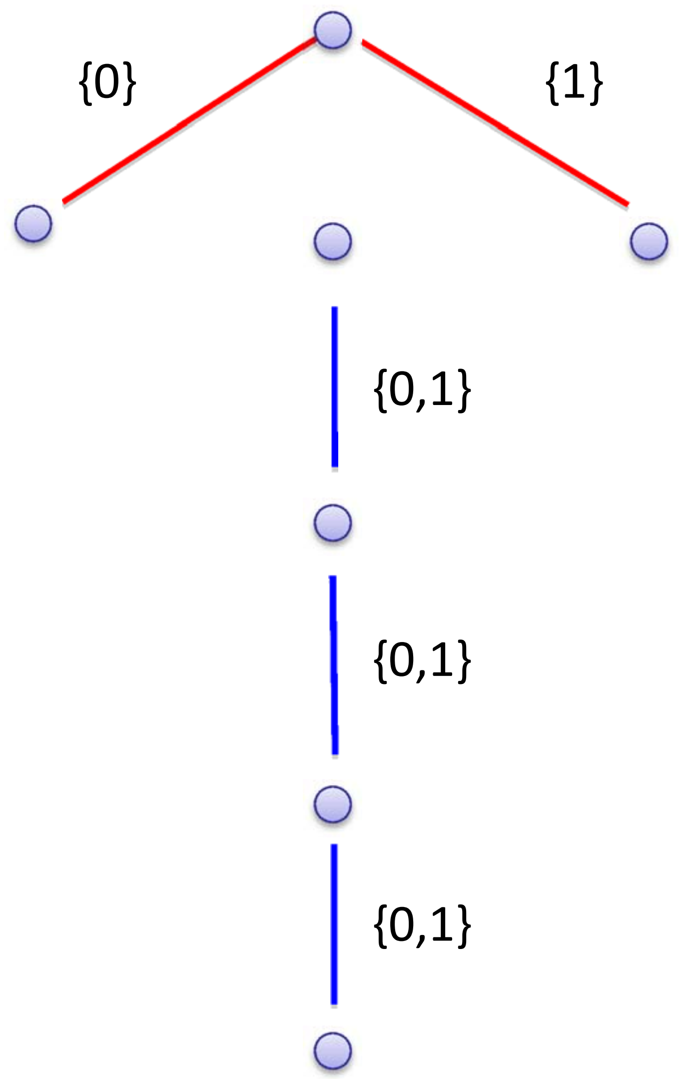
- We will propagate *Among*($\{x_1,x_2,x_3,x_4\}$,{1},2,2) through a BDD of maximum width 3

Example



$Among(\{x_1,x_2,x_3,x_4\},\{1\},2,2)$

Example



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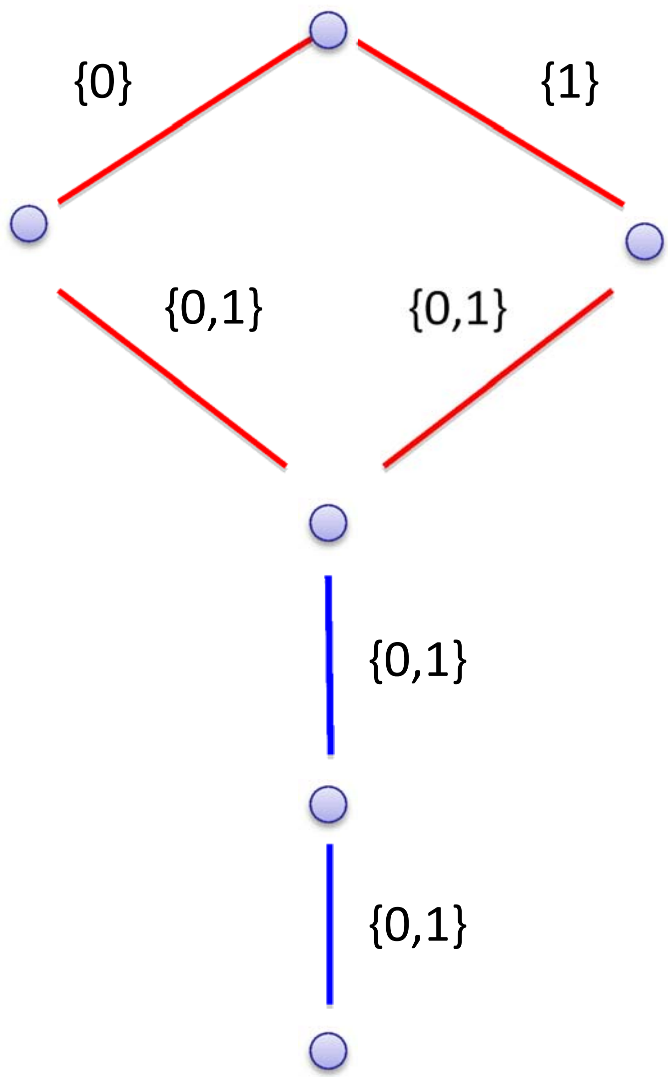
Example



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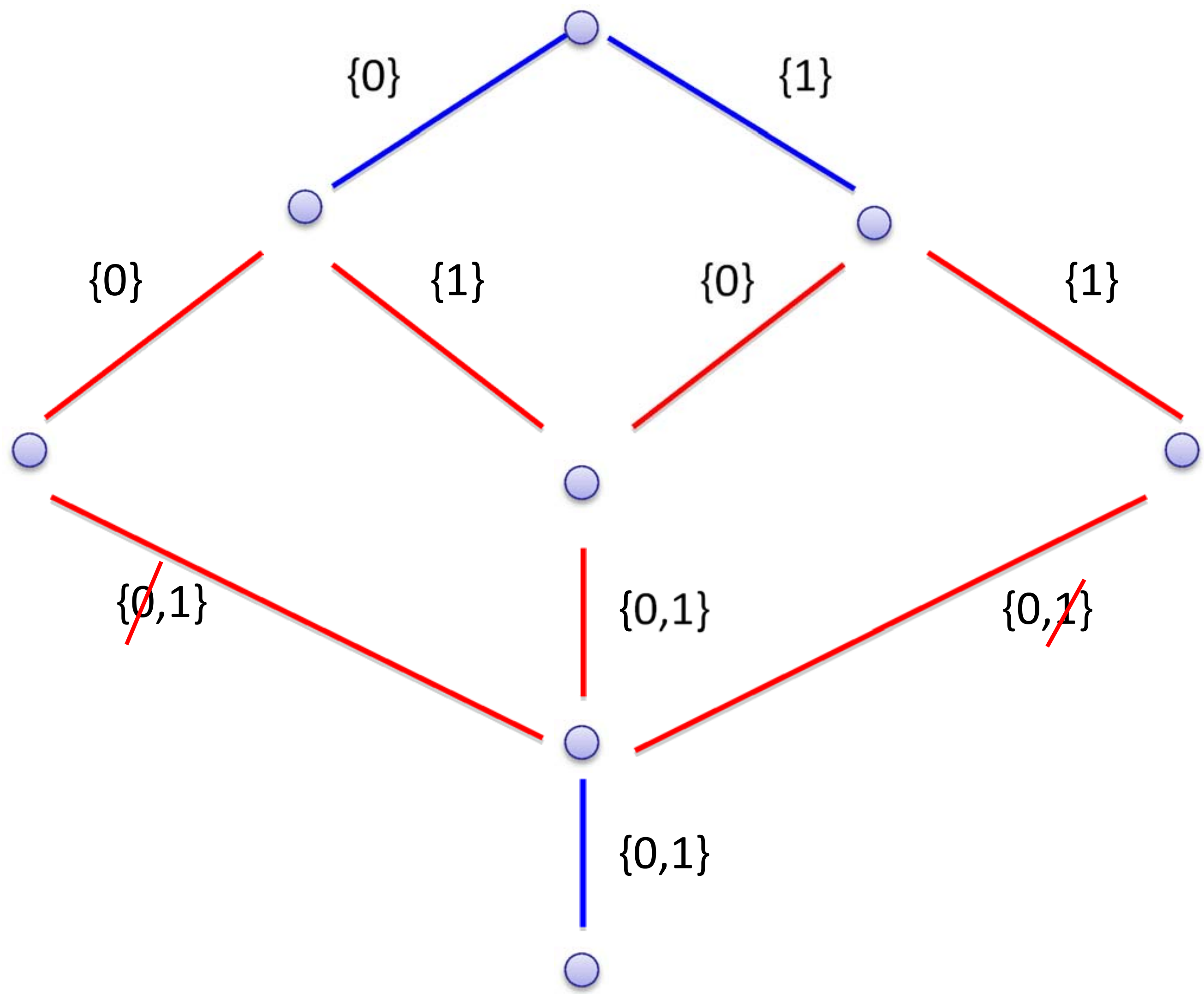
Example



Among$(\{x_1,x_2,x_3,x_4\},\{1\},2,2)$

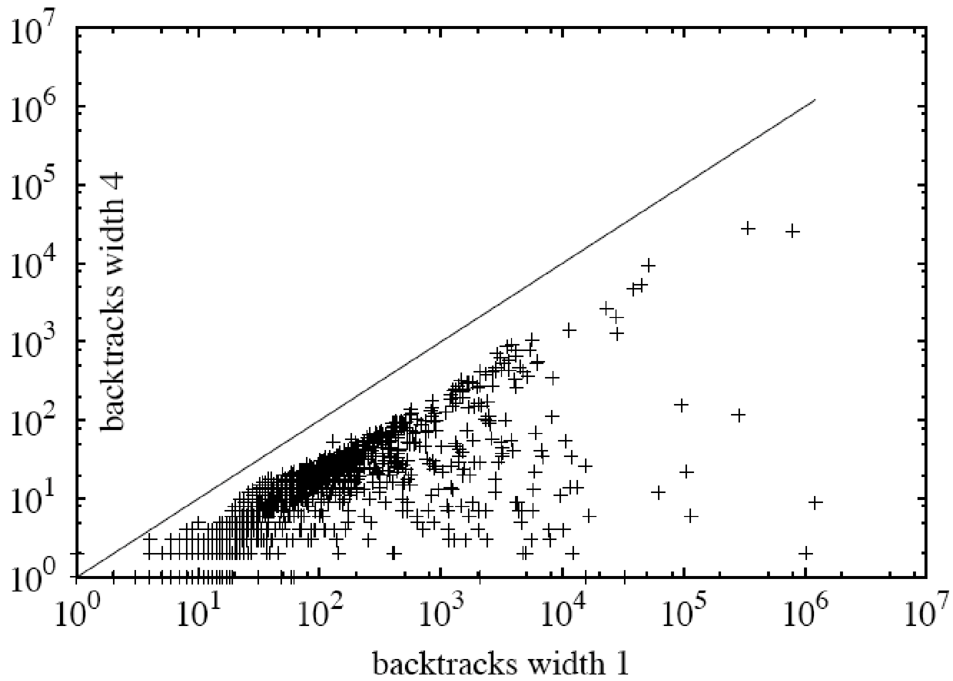
Experiments

- Multiple among constraints
  - 50 binary variables total
  - 5 variables per among constraint, indices chosen from normal distribution with uniform-random mean in [1..50] and stdev 2.5, modulo 50 (i.e., somewhat consecutive)
  - Classes: 5 to 200 among constraints (step 5), 100 instances per class

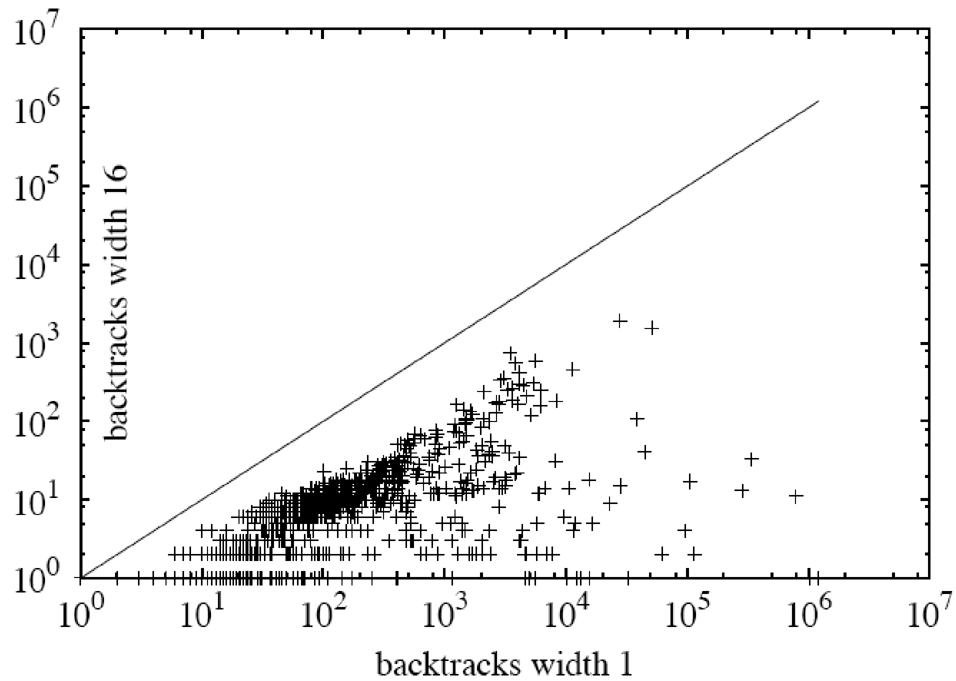
- Nurse rostering instances (horizon *n* days)
  - Work 4-5 days per week
  - Max A days every B days
  - Min C days every D days
  - Three problem classes

- Compare width 1 (traditional domains) with increasing widths

Multiple Amongs: Backtracks

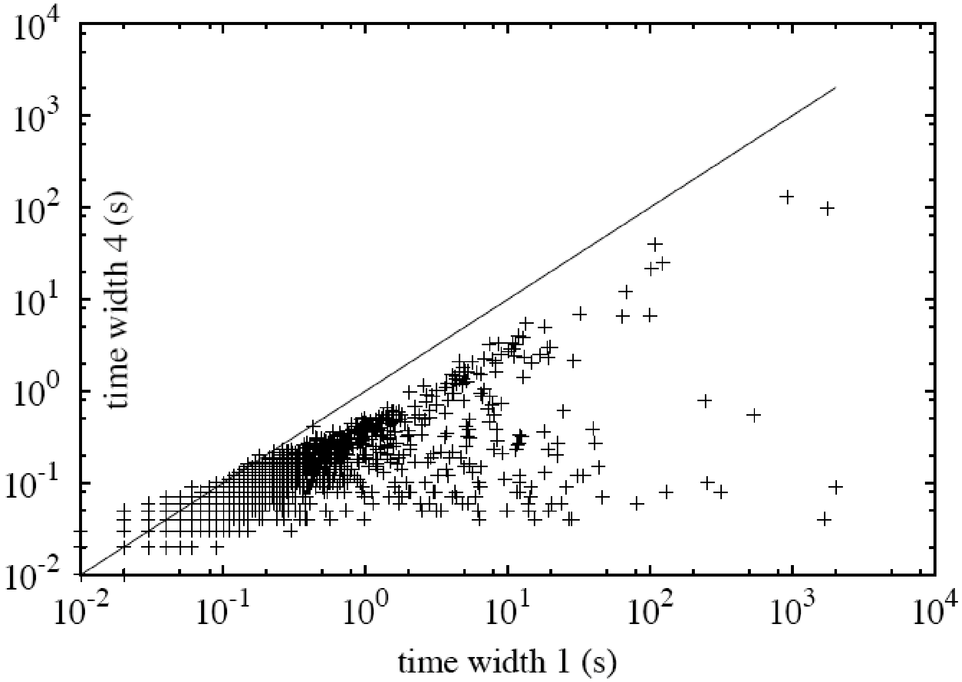


width 1 vs 4

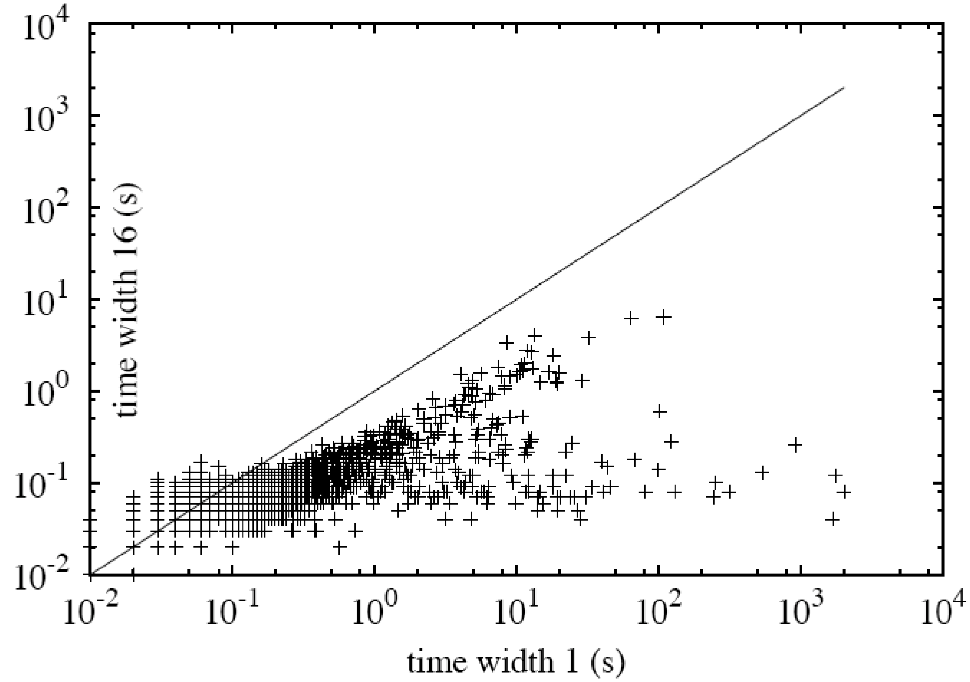


width 1 vs 16

Multiple Amongs: Running Time



width 1 vs 4



width 1 vs 16

Nurse rostering problems

	Size	Width 1 BT	Width 1 CPU	Width 4 BT	Width 4 CPU	Width 32 BT	Width 32 CPU
Class 1	40	61,225	55.63	8,138	12.64	3	0.09
	80	175,175	442.29	5,025	44.63	11	0.72
Class 2	40	179,743	173.45	17,923	32.59	4	0.07
	80	179,743	459.01	8,747	80.62	2	0.32
Class 3	40	91,141	84.43	5,148	9.11	7	0.18
	80	882,640	2,391.01	33,379	235.17	55	3.27

Exercises

1. Consider the constraint $x \neq y$ for two finite-domain variables x and y. Assume that x and y belong to a set X of variables for which we are given a relaxed MDD. Design an MDD propagator for $x \neq y$.

2. Consider the following CSP:

   $x_1 \in \{0,1\}$, $x_2 \in \{0,1,2\}$, $x_3 \in \{1,2\}$

   $x_1 \neq x_2$, $x_2 \neq x_3$, $x_1 \neq x_3$

   Apply filtering and refinement (using the propagator from Exercise 1), starting from a width-1 MDD, until the MDD represents all solutions to the CSP.

More exercises

3. Design an MDD propagator for the constraint

$$\sum_{i=1}^{n} c_i x_i \leq b$$

where c_i , b are constants and x_i are finite-domain variables. Can we establish MDD consistency in polynomial time for this constraint (for an arbitrary MDD defined on x_i) ?

4. Suppose we have a system of linear constraints:

$$\sum_{i=1}^{n} c_{ij} x_i \leq b_j \qquad \text{for j = 1,...,m}$$

How would you use the propagator from exercise 3 to handle this system?