16.5 Curl and Divergence

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https://drive.google.com/file/d/16l1gClib9TygyW00T6S1lW1_2049N6J7/view

Curl: measures the tendency of a vector field to rotate about a pt. Curl of a vector field is a vector that pts in the direction of the axis of rotation. It has magnitude that represents the speed of rotation.

If **F** = P **i** + Q **j** + R **k** is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of **F** is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

If we rewrite Eq. 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field **F** as follow

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \begin{matrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{matrix}$$

$$= \frac{\partial R}{\partial y} i + \frac{\partial P}{\partial z} j + \frac{\partial Q}{\partial x} k - \frac{\partial R}{\partial x} j - \frac{\partial Q}{\partial z} i - \frac{\partial P}{\partial y} k$$

$$\boxed{2} \qquad \boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F}}$$

∗

∗ Add in curl F

$\int P\,dx + Q\,dy$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

Example 1: If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

$$\text{curl}\, F = \nabla \times F = \left|\begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{matrix}\right| \begin{matrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & xyz \end{matrix}$$

$$\underline{-2y}\,i + xj + yzk - 0j - \underline{xy}\,i - 0k$$

$$\boxed{(-2y - xy)\,i + xj + yzk}$$

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl.

The following theorem says that the curl of a gradient vector field is **0**.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

Example 2: Show that the vector field $\underbrace{F(x,y,z) = xz\boldsymbol{i} + xyz\boldsymbol{j} - y^2\boldsymbol{k}}$ is not conservative.

same vector field from Ex. 1

$\text{curl}\,F = (-2y - xy)i + xj + yzk$

$\text{curl}\,F \neq 0$, $\therefore$ F is not conservative

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.")

4 Theorem If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

f_x f_y f_z

Example 3: a.)Show that $\boldsymbol{F}(x,y,z) = y^2z^3\boldsymbol{i} + 2xyz^3\boldsymbol{j} + 3xy^2z^2\boldsymbol{k}$ is a conservative vector field. b.) Find a function f such that $\boldsymbol{F} = \nabla f$

a.)
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \begin{matrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2z^3 & 2xyz^3 \end{matrix}$$

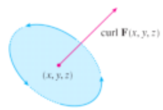
$$= \cancel{6xyz^2}\, i + \cancel{3y^2z^2}\, j + \cancel{2yz^3}\, k - \cancel{3y^2z^2}\, j - \cancel{6xyz^2}\, i - \cancel{2yz^3}\, k$$

$= 0$ ✓ ∴ conservative

b.) $\boxed{f = xy^2z^3 + C}$

The reason for the name *curl* is that the curl vector is associated with rotations.

Another occurs when **F** represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis.



If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and **F** is called **irrotational** at P.

In other words, there is no whirlpool or eddy at P.

If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

Divergence: a # that can be thought of as a measure of ROC of the density of fluid at a pt.
-yields a scalar field.

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

$$\boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Observe that curl **F** is a vector field but div **F** is a scalar field.

In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

10 $$\boxed{\operatorname{div} \mathbf{F} = \underline{\nabla \cdot} \mathbf{F}}$$

Example 4: If $\mathbf{F}(x, y, z) = \overset{P}{xz}\,\mathbf{i} + \overset{Q}{xyz}\,\mathbf{j} + \overset{R}{y^2}\,\mathbf{k}$, find div **F**.

$$\text{div}F = \nabla \cdot F = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(y^2)$$

$$\boxed{\text{div}F = z + xz}$$

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

$f_{xy} = f_{yx}$

pf: $\text{div}(\text{curl}\,F) = \nabla \cdot (\overset{\text{curl}}{\nabla \times F})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}} + \cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 R}{\partial y \partial x}} + \cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}}$$

$$= 0 \checkmark$$

→ from ex.4

Example 5: Show that the vector field $F(x, y, z) = xzi + xyzj - y^2k$ can't be written as the curl of another vector field, that is, $F \neq curl G$

$\text{div}F = z + xz$

$\therefore \text{div}F \neq 0$

If it were true that $\underline{F = \text{curl}\,G}$, then using Thm 11

$\text{div}(\text{curl}\,F) = 0$

$\text{div}.F = \underline{\text{div}\ \text{curl}\,G} = 0$, but it doesn't

$\therefore F$ is not the curl of another vector field

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

If **F**(x, y, z) is the velocity of a fluid (or gas), then div **F**(x, y, z) represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

In other words, div **F**(x, y, z) measures the tendency of the fluid to diverge

If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z).

If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.