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Existence and uniqueness of solutions to characteristic evolution in Bondi-Sachs coordinates in general relativity

Simonetta Frittelli*
*Physics Department, Duquesne University, Pittsburgh, Pennsylvania 15282
and Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260*

Luis Lehner†
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
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We show that the theorem of Duff on the existence and uniqueness of solutions to linear characteristic initial-value problems holds in the case of linearized characteristic evolution in Bondi-Sachs coordinates in general relativity. This represents the characteristic equivalent to the manifest existence and uniqueness of the case of standard Cauchy problems. This extends Sachs' original work on the characteristic approach to the Einstein equations, by considering a null-timelike approach rather than a null-asymptotic one.
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I. INTRODUCTION

The characteristic approach to general relativity was introduced by Bondi *et al.* [1] and Sachs [2] in the early 1960s, and was followed up by a number of authors in subsequent years (for a comprehensive list see [3]). Its distinguishing feature is that it considers spacetime to be foliated by a sequence of null hypersurfaces, each of which is generated by the outgoing null rays emanating from a central geodesic (or the null normals to a spacelike two-surface).

In this approach, the projections of the Einstein equations along three independent directions tangent to the null slices yield a set of six second-order partial differential equations for the six independent components of the metric in terms of coordinates adapted to the null foliation, referred to as the *main equations*. The projections of the Einstein equations along the incoming null direction (which sticks out of the null slices), and along mixed incoming-tangent directions, yield a set of four second-order partial differential equations which can be considered as conditions on the data for the other six, since they are preserved by them. For definiteness, the equations are written with respect to a specific coordinate system referred to as the Bondi-Sachs coordinates, which essentially consitute coordinates adapted to the foliation.

The characteristic approach was first introduced to study the problem of gravitational radiation emitted by isolated systems. In this context, it has been used to construct solutions from data prescribed at infinite distances from the isolated source, and a number of results have been obtained concerning the existence and uniqueness of solutions built in this fashion [4,2,5,6]. It is perhaps not completely trivial to point out that the earlier results [2,4] are based on radiative treatments of the scalar wave equation, whereas the later work by Friedrich [6,7] makes use of newly available conformal compactification techniques and the exact conformal Einstein equations.

More recently, interest has shifted to the case where data are prescribed, in a complementary fashion, both along one null slice, considered as initial, and along an interior world tube, which generates the outgoing null surfaces in the sense discussed above, in Bondi-Sachs coordinates. This case is often referred to as characteristic evolution (CE) for general relativity. Although CE has generally been viewed as a slight modification to the original asymptotic approach, to our knowledge, little of rigor can be found in the literature concerning the properties of the existence and uniqueness of solutions to such an initial-value problem.

A standard argument for the construction of a unique regular solution to a system of quasilinear partial differential equations from given initial data is provided by the Cauchy-Kowalewsky theorem. A strong condition for the Cauchy-Kowalewsky theorem to hold is that the data should be prescribed on a surface that is spacelike with respect to the system of quasilinear partial differential equations. However, surfaces that are null with respect to the spacetime metric are known to be characteristic with respect to the Einstein equations as well (for recent references, see [8–10]); thus CE cannot be accomodated within the standard Cauchy-Kowalewsky framework. It is of interest to us to point out that the Cauchy-Kowalewski theorem has been generalized by Duff [11] to generic linear characteristic initial-value problems. Here we show that CE fits the hypothesis of this adapted Cauchy-Kowalesky theorem. This means that the linearized regime of CE has a unique regular solution to every set of data. The importance of this simple observation stems from the following considerations.

Characteristic initial-value problems in general are manifestly ill posed in the following sense. Since the initial surface is characteristic with respect to the system of equations, the differential operator is internal to the surface and fails to provide the outward derivatives of a number of the variables. This is well known to be the source of the lack of uniqueness

*Email address: simo@mayu.physics.duq.edu
†Current address: Center for Relativity, The University of Texas at Austin, Austin, TX 78712.

to a generic characteristic initial-value problem. Sachs showed that the solution to the characteristic problem in general relativity in Bondi-Sachs coordinates is unique if additional data are prescribed at infinity, considered as a boundary for the spacetime. The work by Duff [11], here adapted to general relativity, on the other hand, shows that the solution exists and is unique if, in addition to the subset of free initial data, complementary data are prescribed on a noncharacteristic surface, which can be considered as a boundary. Thus the two regimes of interest, namely, local and asymptotic, are so far shown to give rise to unique regular solutions, in the context of the Einstein equations in Bondi-Sachs coordinates.

For the sake of completeness, we reproduce the framework of CE of general relativity in Sec. II. The statement of Duff's theorem is found in Sec. III, as well as the proof that its hypotheses are satisfied in the case of CE. The argument is extended to consider the boundary at null infinity in Sec. IV. We comment on the reach and relevance of our result in Sec. V.

II. CHARACTERISTIC EVOLUTION IN GENERAL RELATIVITY

The characteristic approach to general relativity introduces a foliation based on a sequence of null cones emanating from a central timelike geodesic or, alternatively, a family of null surfaces emanating from a timelike or null world tube Γ. Each null surface runs out to infinity, covering thus the entire spacetime (outside Γ). The set of ten Einstein equations is projected down to each null slice and out of it, consequently splitting, respectively, into a set of six *main equations* which rule the evolution, and a set of four equations which are preserved by the main ones and can be thought of as conditions on the set of the, otherwise free, data on Γ.

In the present work, we concern ourselves only with the main equations, since our results apply to the entire set of data, in particular to the data restricted by the preserved conditions.

In the characteristic approach, a coordinate system adapted to the slices may be defined in the following way: suppose we foliate the spacetime with a sequence of lightlike hypersurfaces. We let u label these hypersurfaces; $x^A (A=2,3)$, label each null ray on a specific hypersurface, and r be a surface area coordinate. In the resulting $x^a=(u,r,x^A)$ coordinates, the metric takes the Bondi-Sachs form [12–15]

$$ds^2=-\left(e^{2\beta}\frac{V}{r}-r^2h_{AB}U^AU^B\right)du^2-2e^{2\beta}du\,dr -2r^2h_{AB}U^B du\,dx^A+r^2h_{AB}dx^Adx^B, \tag{1}$$

where $\det(h_{AB})=\det(q_{AB})$, with q_{AB} a unit sphere metric. The main variables are β, U^A, V, and h_{AB}. Geometrically, the tensor field h_{AB} measures the departure from spherical symmetry of the surfaces of constant u and r. β measures the expansion of the light rays. U^A measures the shift of the angular coordinates from one hypersurface to another (at constant r) and V contains the mass aspect of the system.

The main equations are [13]

$$\begin{aligned}r(rh_{AB,u})_{,r}-\frac{1}{2}(rVh_{AB,r})_{,r}={}&2e^{\beta}D_AD_Be^{\beta}-r^2h_{AC}D_BU^C_{,r}-\frac{r^2}{2}h_{AB,r}D_CU^C+\frac{r^4}{2}e^{-2\beta}h_{AC}h_{BD}U^C_{,r}U^D_{,r}-r^2U^CD_Ch_{AB,r}\\&-2rh_{AC}D_BU^C+r^2h_{AC,r}(h_{BE}h^{CD}D_DU^E-D_BU^C)+\frac{1}{2}h_{AB}\left[-r^2h^{CD}_{,r}\left(h_{CD,u}-\frac{V}{2r}h_{CD,r}\right)\right.\\&\left.-2e^{\beta}D_CD^Ce^{\beta}+D_C(r^2U^C)_{,r}-\frac{1}{2}r^4e^{-2\beta}h_{CD}U^C_{,r}U^D_{,r}\right],\end{aligned} \tag{2a}$$

$$\beta_{,r}=\frac{1}{16}rh^{AC}h^{BD}h_{AB,r}h_{CD,r}, \tag{2b}$$

$$(r^4e^{-2\beta}h_{AB}U^B_{,r})_{,r}=2r^4(r^{-2}\beta_{,A})_{,r}-r^2h^{BC}D_Ch_{AB,r}, \tag{2c}$$

$$2e^{-2\beta}V_{,r}=\mathcal{R}-2D^AD_A\beta-2D^A\beta D_A\beta+r^{-2}e^{-2\beta}D_A(r^4U^A)_{,r}-\frac{1}{2}r^4e^{-4\beta}h_{AB}U^A_{,r}U^B_{,r}, \tag{2d}$$

where D_A is the covariant derivative associated with $h_{AB}(D_C h_{AB}=0)$ and $\mathcal{R}$ the curvature scalar of the two-metric h_{AB}. We have defined h^{AB} via $h^{AB}h_{BC}=\delta^A_C$ and use it to raise indices $A,B,\ldots$. These equations represent a real version of the actual equations used in CE, which use two complex stereographic coordinates on the spacelike sections at constant u and r. Equations (2b)–(2d) are exactly Eqs. (9)–(11) in [15]. Equation (2a) can be reobtained by expressing Eq. (25) of [15] in terms of our current variables.

In their standard form, the characteristic equations (2) have the following properties. In the first place, from the point of view of partial differential equations, they constitute a second-order system (irrespective of the fact that they do not display second derivatives with respect to the null coordinate u).

Second, and most important, adopting the viewpoint that u acts like a time for evolution, one recognizes the presence of only two evolution equations (2a), which are first order in time. Hence, the other equations can be regarded as constraints that the field variables must satisfy on each null surface. Consequently, the set of characteristic equations is conventiently organized in a hierarchical manner [16], allowing for a straightforward procedure to construct a solution.

Given initial data consisting of h_{AB} on a characteristic hypersurface $\mathcal{N}_0$ at some initial time $u=u_0$ and sufficiently smooth boundary data, β is obtained by integrating Eq. (2b) radially outwards starting from Γ. Notably, since h_{AB} is known on the entire surface, Eq. (2b) can be regarded as an ordinary equation for β. Then, U^A, which follows next in the hierarchy (2c), is analogously integrated since both h_{AB} and β are now known on $\mathcal{N}_0$. Finally, V is obtained by integrating Eq. (2d), which completely determines the metric on $\mathcal{N}_0$.

The metric information on the first hypersurface can now be used to obtain $(rh_{AB})_{,ur}$ from Eq. (2a). In the last step of the evolution cycle, the $f^{lm}_{AB,u}$ are integrated in time, hence furnishing h_{AB} on a new hypersurface $\mathcal{N}_{u_0+du}$ at time $u=u_0+du$.

The simplicity of this hierarchy has been exploited in the realm of numerical relativity [13,17,14,18], obtaining excellent results.

III. DUFF'S THEOREM APPLIED TO CHARACTERISTIC EVOLUTION IN GENERAL RELATIVITY

In this section we consider systems of R first-order linear partial differential equations for R variables u_s in N spacetime dimensions, of the form

$$a^a{}_{rs}\frac{\partial u_s}{\partial x^a}+b_{rs}u_s=f_r,$$

$$r,s=1,\ldots,R \quad \text{and} \quad a=1,\ldots,N, \tag{3}$$

where summation over repeated indices is understood, or, in matrix notation,

$$\boldsymbol{A}^a\frac{\partial \boldsymbol{u}}{\partial x^a}+\boldsymbol{B}\boldsymbol{u}=\boldsymbol{f}. \tag{4}$$

A surface $\phi(x^a)=0$ is referred to as characteristic with respect to Eq. (4) if and only if the determinant of the characteristic matrix [19], namely, the matrix obtained by contracting $\boldsymbol{A}^a$ with $\phi_{,a}$, vanishes:

$$\det(\boldsymbol{A}^a\phi_{,a})=0. \tag{5}$$

The characteristic surface $\phi(x^a)=0$ is said to be of multiplicity μ if the rank of $\boldsymbol{A}^a\phi_{,a}$ is $R-\mu$.

If $\phi=0$ is a characteristic surface, then the characteristic matrix $\boldsymbol{A}^a\phi_{,a}$ has a set of μ linearly independent unit right null vectors $\boldsymbol{z}^{(\alpha)}$ and a set of μ linearly independent unit left null vectors $\boldsymbol{y}^{(\alpha)}$ in the sense that

$$\boldsymbol{A}^a\phi_{,a}\boldsymbol{z}^{(\alpha)}=0, \tag{6}$$

$$\boldsymbol{y}^{(\alpha)}\boldsymbol{A}^a\phi_{,a}=0, \quad \alpha=1,\ldots,\mu. \tag{7}$$

In this context, Duff proved the following theorem.

Theorem. Let G given by $\phi(x^a)=0$ *be a characteristic surface of multiplicity* μ *relative to the analytic linear system*

$$\boldsymbol{A}^a\frac{\partial \boldsymbol{u}}{\partial x^a}+\boldsymbol{B}\boldsymbol{u}=\boldsymbol{f}$$

of R first-order linear equations. Let T given by $\psi(x^a)=0$ *be noncharacteristic, intersecting G in an edge C such that*

$$\det(\boldsymbol{y}^{(\alpha)}\boldsymbol{A}^a\psi_{,a}\boldsymbol{z}^{(\beta)})\neq 0. \tag{8}$$

Then there exists a unique analytic solution which satisfies $R-\mu$ *initial conditions on G and* μ *boundary conditions on T. The initial conditions are the* $R-\mu$ *linearly independent values of*

$$\boldsymbol{A}^a\phi_{,a}\boldsymbol{u} \quad \text{on} \quad \phi=0 \tag{9}$$

and the boundary conditions are the μ *values of*

$$\boldsymbol{y}^{(\alpha)}\boldsymbol{A}^a\psi_{,a}\boldsymbol{u} \quad \text{on} \quad \psi=0. \tag{10}$$

In the remainder of this work, we restrict our attention to the linearization of the characteristic equations (2) around Schwarzschild spacetime. Linearization around flat spacetime is obtained in the usual manner, by setting $m=0$, and is therefore contained in what follows. The line element of this spacetime can be written as

$$ds^2=-\left(\frac{r-2m}{r}\right)du^2-2dudr+r^2q_{AB}dx^Adx^B, \tag{11}$$

which corresponds to the choice $\beta=0, U^A=0, V=r-2m$, and $h_{AB}=q_{AB}$ (with q_{AB} the unit sphere metric).

In spacetimes where the departure from Schwarzschild spacetime can be considered small, the quantities β and U^A are of first order, as well as the departures $V-r+2m$ and $h_{AB}-q_{AB}$, which we consider as our variables. To examine the solutions in this regime, we introduce a notation related to that of [15]

$$W\equiv V-r+2m, \tag{12}$$

$$J_{AB}\equiv h_{AB}-q_{AB}. \tag{13}$$

In order to formulate the problem as a first-order system, we define additional first-order variables, which are of first order in the departure from Schwarzschild space as well, of the form

$$B_A\equiv\beta_{,A}, \tag{14}$$

$$M_{ABC}\equiv J_{AB,C}, \tag{15}$$

$$Q^A\equiv U^A{}_{,r}, \tag{16}$$

$$P_{AB}\equiv J_{AB,r}. \tag{17}$$

In terms of these variables, the linearized characteristic equations become

$$r^2P_{AB,u}-\frac{1}{2}r(r-2m)P_{AB,r}-D_AB_B+\frac{1}{2}r^2q_{AC}D_BK^C+rq_{AC}D_BU^C+\frac{1}{2}q_{AB}D^CB_C-\frac{1}{2}rq_{AB}D_CU^C$$
$$-\frac{1}{4}r^2q_{AB}D_CK^C-(r-m)P_{AB}+rJ_{AB,u}=0, \tag{18}$$

$$\beta_{,r}=0, \tag{19}$$

$$r^4q_{AB}Q^B{}_{,r}+r^2q^{BC}D_CP_{AB}+4rB_A+4r^3q_{AB}Q^B=0, \tag{20}$$

$$2W_{,r}+2D^AB_A-D_AQ^A-4rD_AU^A-2q^{AB}q^{CD}D_DM_{ABC}=0, \tag{21}$$

and the following two equations are satisfied as well, as a consequence of Eqs. (14), (15), (17), and (19):

$$M_{ABC,r}=P_{AB,C}, \tag{22}$$

$$B_{A,r}=0. \tag{23}$$

The system consisting of Eqs. (18), (19), (20), (21), (16), (17), (22), and (23) constitutes a linear first-order system of 16 equations for 16 variables, namely, the metric (J_{AB},β,W,U^A) and those of its first spatial derivatives (P_{AB},M_{ABC},B_A,Q^A) necessary to put the system into first-order form. In matrix notation, the system (16)–(23) has the form

$$
\left[\begin{pmatrix}
r^2 & 0 & 0 & 0 & r & 0 & 0 & 0\\
 & 0 & & & & & & \\
 & & 0 & & & & & \\
 & & & 0 & & & & \\
 & & & & 0 & & & \\
 & & & & & 0 & & \\
 & & & & & & 0 & \\
 & & & & & & & 0
\end{pmatrix}\frac{\partial}{\partial u}
+\begin{pmatrix}
-r(r-2m)/2 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & r^4 q_{AB} & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 2 & \\
 & & & & & & & 1
\end{pmatrix}\frac{\partial}{\partial r}+\boldsymbol{L}^B\frac{\partial}{\partial x^B}+\boldsymbol{B}\right]
\begin{pmatrix} P_{AB}\\ M_{ABC}\\ B_A\\ Q^A\\ J_{AB}\\ \beta\\ W\\ U^A \end{pmatrix}=0. \tag{24}
$$

In relation to the system (24), our claim is that (i) a surface $u=0$ is characteristic of multiplicity $\mu=14$, (ii) a surface $r=r_0(r_0>2m)$ is noncharacteristic, and (iii) there exists a unique analytic solution for every set of 2 pieces of data P_{AB} on $u=0$ and 14 pieces of data $(M_{ABC},B_A,Q^A,J_{AB},\beta,W,U^A)$ on $r=r_0>2m$. The geometry of our problem is depicted in Fig. 1.

To prove (i), we only need to notice that the surface $\phi(x^a)\equiv u-u_0=0$ has gradient $\phi_{,a}=(1,0,0,0)$ and the characteristic matrix $\boldsymbol{A}^a\phi_{,a}$ is the coefficient of $\partial/\partial u$ in Eq. (24). Thus, the characteristic matrix has rank $2=16-14$, which means that $u=u_0$ is characteristic of multiplicity $\mu=14$, as long as $r>r_0$.

To prove (ii), we notice that the surface $\psi(x^a)\equiv r-r_0=0$ has gradient $\psi_{,a}=(0,1,0,0)$ and the characteristic matrix $\boldsymbol{A}^a\psi_{,a}$ is the coefficient of $\partial/\partial r$ in Eq. (24). This matrix has determinant $-(r-2m)r^5\det(q_{AB})$, which is nonsingular as long as $r_0>2m$. Thus $r=r_0$ is noncharacteristic as long as $r_0\neq 2m$.

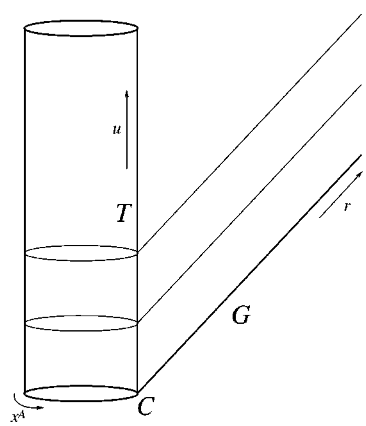

FIG. 1. The geometry of our characteristic problem. The surface $u=u_0$ is characteristic and null, whereas the surface $r=r_0$ is noncharacteristic and timelike.

To prove (iii), we only need to show that the 14×14 matrix $\boldsymbol{y}^{(\alpha)}\boldsymbol{A}^a\psi_{,a}\boldsymbol{z}^{(\beta)}$ has rank 14, so that the hypotheses of Duff's theorem are satisfied. This is straightforward, since the left null vectors of the coefficient of $\partial/\partial u$ in Eq. (24) are simply the 14 basis vectors $\boldsymbol{y}_i^{(\alpha)}=\delta_i^{(\alpha)}$ (with $i=2,\ldots,16$), while the right null vectors are $\boldsymbol{z}_j^{(\beta)}=(\delta_j^{(\beta)}-r\,\delta_j^J\delta_P^{(\beta)})$ (with $j=2,\ldots,16$, and where δ_j^J and $\delta_P^{(\beta)}$ represent Kronecker symbols which are nonvanishing only for values of j corresponding to the label for the two variables J_{AB} and for values of β corresponding to the label for the two variables P_{AB}); hence,

$$
\boldsymbol{y}^{(\alpha)}\boldsymbol{A}^a\psi_{,a}\boldsymbol{z}^{(\beta)}
=\begin{pmatrix}
-r(r-2m)/2 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & r^4 q_{AB} & & & & \\
-r & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 2 & \\
 & & & & & & & 1
\end{pmatrix}, \tag{25}
$$

which has rank 14 on the surface $u=u_0$, $r=r_0>2m$, wherever the angular coordinates do not degenerate, an assumption that is understood throughout this work.

The first-order system (24) by itself has more solutions than the second-order linearized characteristic equations (2). However, the solutions to the original second-order equations are singled out by prescribing data that satisfy Eqs. (14) and (15) on $r=r_0$. Equations (14) and (15) constitute constraints on the boundary data on the surface $r=r_0$, and are trivially propagated by the system (24). At this point we should remind the reader that, in order to obtain a solution to the full set of linearized Einstein equations, the data on $r=r_0$ must be further restricted by the four constraints, mentioned in Sec. I, which we ignore in our present considerations.

The application of Duff's theorem to the Einstein equations in the form (24) is, thus, extremely simple. This is due to our having at hand the characteristic initial-value problem in its *canonical form relative to the characteristic surface* (see [11], p. 130), namely, in coordinates adapted to the initial characteristic surface. Any characteristic initial-value problem reduced to canonical form splits its variables into a set of *normal variables*, which evolve according to the evolution equations, and a set of *null variables*, for which no evolution equation is available. The normal variables are linearly independent combinations of

$$\boldsymbol{v}\equiv \boldsymbol{A}^a\phi_a\boldsymbol{u}, \tag{26}$$

which are $R-\mu$ in number since the rank of the characteristic matrix is $R-\mu$. In our case, there are two normal variables, P_{AB}.

The null variables are linearly independent combinations of

$$\boldsymbol{w}^{\alpha}\equiv \boldsymbol{y}^{(\alpha)}\boldsymbol{A}^a\psi_a\boldsymbol{u}, \tag{27}$$

which are μ in number. In our case, there are 14 null variables $(M_{ABC},B_A,Q^A,J_{AB},\beta,W,U^A)$. The splitting is a consequence of the existence of the 14 left null vectors of the characteristic matrix, since contracting Eq. (4) on the left with $\boldsymbol{y}^{(\alpha)}$ produces 14 ''hypersurface equations'' due to the vanishing of the coefficient of $\partial/\partial u$ on the initial characteristic surface.

IV. INCLUSION OF THE NULL BOUNDARY

The arguments in the previous section guarantee the existence and uniqueness of solutions constructed from data given on an initial null slice $u=u_0$ and a boundary world tube $r=r_0$, the solutions extending up to any finite values of u and r greater than the starting ones. This is important, since r can reach a value as large as desired. However, a rigourous study of the gravitational radiation can only be attained at *null infinity* (corresponding to $r\to\infty$, namely, the boundary of the spacetime). Ideally, one would like to be able to reach *infinity* in a practical manner and in a way that would let us infer the properties of the obtained solution.

Our analysis of the previous section is not well suited for this purpose, because the boundary is located at $r=\infty$, which is outside the domain of our coordinate r. In particular, our previous argument does not rule out solutions that diverge as $r\to\infty$, but only those that diverge at finite radius r. (The null variables are controlled by the proof of existence, whereas the normal variables are controlled by the assumption of regular—analytic—data on the entire initial surface.)

We need to introduce a coordinate that would ''bring'' infinity into a finite radius, such as

$$x\equiv \frac{r}{R+r}. \tag{28}$$

The points at the boundary of the spacetime are thus reached when the compactified radial coordinate x takes the value 1. The introduction of this compactified coordinate is motivated by [18], which in turn is inspired from Penrose's compactification [20].

In asymptotically flat spacetimes, the variable W diverges at infinity at the rate of r^2 (see [13,16]). Therefore it is convenient to introduce a slight modification and define a normalized variable that behaves regularly at infinity:

$$\tilde{W}\equiv \frac{W}{r^2}=\frac{(1-x)^2}{x^2}\frac{W}{R^2}. \tag{29}$$

The remaining variables do not diverge at infinity. The change of coordinate $r\to x$ transforms our linearized equations (16)–(23) into the following system:

$$x^2R^2P_{AB,u}-\frac{1}{2}x^2R^2\left(D_BQ_A-\frac{1}{2}q_{AB}D_CQ^C\right)-\frac{1}{2}x(1-x)^2P_{AB,x}[xR-2m(1-x)]$$
$$+x(1-x)RJ_{AB,u}+x(1-x)R\left(D_BU_A-P_{AB}-\frac{1}{2}q_{AB}D_CU^C\right)-(1-x)^2\left(D_AB_B-\frac{1}{2}q_{AB}D_CB^C\right)=0, \tag{30a}$$

$$(1-x)^2M_{ABC,x}-RP_{AB,C}=0, \tag{30b}$$

$$(1-x)^2B_{A,x}=0, \tag{30c}$$

$$x^3(1-x)R^2Q_{A,x}+4x^2R^2Q_A+x(1-x)RD^BP_{AB}+4(1-x)^2B_A=0, \tag{30d}$$

$$(1-x)^2J_{AB,x}-RP_{AB}=0, \tag{30e}$$

$$(1-x)^2\beta,_x=0, \tag{30f}$$

$$4x^2(1-x)^2R\tilde{W},_x+4xR\tilde{W}-4xRD_AU^A+(1-x)(2D_AB^A-D_AQ^A-2q^{AB}q^{CD}D_CM_{ABD})=0, \tag{30g}$$

$$(1-x)^2U^A,_x-RQ^A=0, \tag{30h}$$

where the coefficients are analytic functions of the coordinates. (Analyticity of the coefficients is one condition required for Duff's arguments.) In matrix notation, the system (30) reads

$$\left[\begin{pmatrix} x^2R^2 & 0 & 0 & 0 & x(1-x)R & 0 & 0 & 0 \\ & 0 & & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}\frac{\partial}{\partial u}+(1-x)\right.$$
$$\times\begin{pmatrix} -x(1-x)(xR-2m(1-x))R/2 & & & & & & & \\ & (1-x) & & & & & & \\ & & (1-x) & & & & & \\ & & & x^3R^2q_{AB} & & & & \\ & & & & (1-x) & & & \\ & & & & & (1-x) & & \\ & & & & & & 4x^2R & \\ & & & & & & & (1-x) \end{pmatrix}$$
$$\left.\times\frac{\partial}{\partial x}+\tilde{\boldsymbol{L}}^B\frac{\partial}{\partial x^B}+\tilde{\boldsymbol{B}}\right]\begin{pmatrix} P_{AB} \\ M_{ABC} \\ B_A \\ Q^A \\ J_{AB} \\ beta \\ \tilde{W} \\ U^A \end{pmatrix}=0. \tag{31}$$

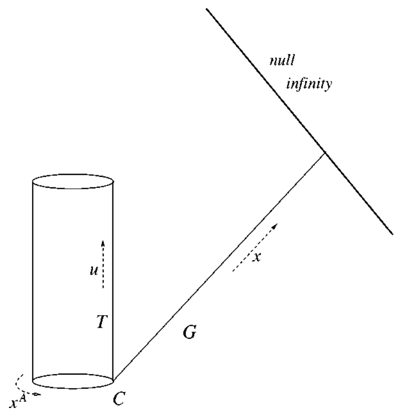

FIG. 2. The region where existence and uniqueness hold, for data specified on the initial null slice $u=u_0$ and world tube at $x=x_0$.

We can see that, by the same arguments as in the previous section, the prescription of P_{AB} on $u=u_0$ and the remaining variables on $x=x_0$ with $2m/(R+2m)<x_0<1$ yields a unique regular solution in the range $u\geqslant u_0$ and $x_0\leqslant x\leqslant 1$. The arguments follow because the only modifications with respect to the previous section have been the appearance of factors multiplying existent nonvanishing elements of the matrices, but the vanishing elements of the coefficient matrices remain the same. The region where the solution exists and is unique is shown in Fig. 2.

This means that we can construct a solution at null infinity by starting at an interior worldtube (and initial null slice). The solution at null infinity is regular and can be used to read the radiation given off by the isolated system. This scheme has been implemented successfully in numerical codes [13] and is currently being actively pursued.

On the other hand, it is also clear now that the surface $x=1$ is different from any other surface $x=x_0>0$. The surface $x=1$ is actually a characteristic surface of the system (30), since the matrix that is the coefficient of $\partial/\partial x$ is degenerate at $x=1$. Therefore, we cannot guarantee by this method the existence and uniqueness of solutions to the characteristic problem with data given at null infinity in addition to data given on a starting null surface. This is the inverse problem: knowing the radiation at infinity, can we reconstruct the gravitational fields in the interior in a unique regular fashion?

Apparently the answer is *yes*, certainly not from Duff's theorem, but from work by Sachs and Friedlander in the 1960s. Sachs showed that the characteristic initial value problem has a unique solution (if the solution exists) when additional data is prescribed at null infinity [2]. On the other hand, Friedlander obtained an argument, based on the scalar wave equation, to ensure the existence of regular solutions constructed from asymptotic data on null surfaces [4]. Both works used asymptotic expansions in terms of inverse powers of r. Friedrich [6] adapted Duff's theorem to the case when the boundary is a characteristic of the system of equations, to show existence and uniqueness in the case of the conformal Einstein equations. Possibly his argument could be used in our case of Bondi-Sachs type coordinates.

V. CONCLUSION

Since the concept of asymptotic flatness was introduced [20], null infinity has become an ideal setting for the study of the gravitational radiation given off by an isolated system. From a numerical point of view, when modeling a particular problem, it is convenient to choose a set of field variables and foliation such that the variables involved in the simulation appear as smooth as possible. This configuration will usually imply fewer steep gradients which is always desirable for stability reasons. In particular, when dealing with gravitational waves that propagate along null directions, a description with respect to lightlike surfaces results in a smoother appearance of these disturbances than that obtained from a spacelike foliation of the spacetime. (This is analogous to describing the propagation of outgoing eletromagnetic waves; while in a spacelike foliation they depend on time and space, in a foliation adapted to the null cones they only depend on time; thus being constant on each null cone.)

Consequently, as more extensively argued in [3], the characteristic approach constitutes a valuable tool for numerical models aiming to study gravitational radiation and has been actively pursued [13,21,17,22]. Its effectiveness as a tool to model three-dimensional spacetimes has, in recent years, been demonstrated to an impressive degree of reliability [13,14], taking three-dimensional simulations in numerical relativity to the regime of arbitrarily-long-time evolution [23].

Numerical integration, as well as other approximation methods, share the properties of physical systems approached by observation and measurement, in the sense that there is an error inherent to the procedure which must be controlled. Courant's judgement on the necessary requirements that a system of partial differential equations must satisfy in order to represent a physical system (p. 227 of [19]) is as follows: (1) The solution must exist; (2) the solution should be uniquely determined; (3) the solution should depend continuously on the data (requirement of stability).

The first requirement is important because it eliminates the possibility that a solution might diverge at a finite time. Any divergent solutions would eventually spoil the integration by unavoidable mixing at the order of error. The second requirement is natural. But it is the third requirement that is crucial to any observable natural phenomenon, because of the associated finite precision in measurement, or to any approximation method, such as a numerical integration one, because of finite errors in the prescription of initial data and in the discretization.

A problem that satisfies all three requirements above is referred to as well posed, regardless of the possibility of additional properties possessed by the problem, such as hyperbolicity or parabolicity. A suitable method for determining whether a problem is well posed is by establishing estimates between the norm of the solution at a later time and the norm of the solution at the initial time, referred to as energy estimates (see, for instance, [19], p. 661, and [24]). Energy estimates, in turn, lead to algebraic criteria in the case of hyperbolic problems, namely, strict, strong, or symmetric hyperbolicity, which simplify in great measure the task of establishing well posedness. These standard criteria are applicable in cases of Cauchy initial-value problems, namely, when data are specified on a surface that is spacelike with respect to the equations. In recent years, the Cauchy approach to general relativity has yielded a variety of well-posed schemes (see, for instance, [8,9,25,10]).

Characteristic initial-value problems do not fit the framework of the standard algebraic criteria for well posedness. Furthermore, so far, there appear to exist no standard criteria for the well posedness of generic characteristic initial-value problems. Some results exist for special characteristic problems. A theorem due to Balean [26] establishes the well posedness of the scalar wave equation in three spatial dimensions, with data specified on a null-timelike boundary. To our knowledge, no such result is available yet for the null-timelike boundary-value problem in general relativity, although it now appears possible [27] to establish it by combining techniques appearing in [28] and [29]. In view of the results obtained here and, especially, from the fact that numerical evolution can be carried out to long times, it is reasonable to expect that the well posedness of CE could be established, although nonstandard methods to do this may be necessary.

Although uniqueness and existence are not sufficient conditions for well posedness, they are quite necessary, raising the characteristic scheme to the level of a generic Cauchy problem. It may be objected that the validity of the existent results is limited to the linearized regime of the Einstein equations. However, because of the quasilinear character of the Einstein equations (namely, the fact that the highest derivatives appear linearly), the results might be extended to the exact case as well. The existence and uniqueness of a quasilinear system may in principle be obtained from the existence and uniqueness of the corresponding linearized system via iterations (see p. 975 of [19]). We do not concern ourselves with this issue at this time.

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