PHYSICAL REVIEW D VOLUME 56, NUMBER 8 15 OCTOBER 1997

# Dynamics of light cone cuts of null infinity

Simonetta Frittelli*
*Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260*

Carlos Kozameh
*FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000, Córdoba, Argentina*

Ezra T. Newman
*Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260*

(Received 31 March 1997)

In this work we explore further consequences of a recently developed alternate formulation of general relativity, where the metric variable is replaced by families of surfaces as the primary geometric object of the theory—the (conformal) metric is derived from the surfaces—and a conformal factor that converts the conformal metric into an Einstein metric. The surfaces turn out to be characteristic surfaces of this metric. The earlier versions of the equations for these surfaces and conformal factor were local and included all vacuum metrics (with or without a cosmological constant). In this work, after first reviewing the basic theory, we specialize our study to spacetimes that are asymptotically flat. In this case our equations become considerably simpler to work with and the meaning of the variables becomes much more transparent. Several related insights into asymptotically flat spaces have resulted from this. (1) We have shown (both perturbatively and nonperturbatively for spacetimes close to Minkowski space) how a ''natural'' choice of canonical coordinates can be made that becomes the standard Cartesian coordinates of Minkowski space in the flat limit. (2) Using these canonical coordinates we show how a simple (completely gauge-fixed) perturbation theory off flat space can be formulated. (3) Using the rigid structure of the spacetime null cones (with their intersection with future null infinity) we show how the asymptotic symmetries (the BMS group or rather its Poincaré subgroup) can be extended to act on the interior of the spacetimes. This apparently allows us to define approximate Killing vectors and approximate symmetries. We also appear to be able to define a local energy-momentum vector field that is closely related to the asymptotic Bondi energy-momentum four-vector. [S0556-2821(97)04420-2]

PACS number(s): 04.20.Gz, 04.20.Ha

## I. INTRODUCTION

In a recent series of papers, Einstein's theory of general relativity (GR) was reformulated and presented as a theory of characteristic hypersurfaces [1–4] rather than as a theory of the metric field. From this point of view the spacetime metric and associated connection are all derived concepts: the basic variables of the reformulation are special families of three-surfaces in a four-manifold $M^4$—from which a conformal metric can be found—and a scalar function (a conformal factor) which converts the conformal metric into a metric. The surfaces, which are obtained from solutions to partial differential equations, are automatically the characteristic surfaces of the (derived) metric and the metric automatically satisfies the Einstein field equations. This reformulation of GR has been referred to as the null-surface formulation (NSF) of GR and can be applied to the Einstein equations with or without sources. In the present work we will confine ourselves solely to the vacuum case.

More specifically, the NSF describes GR in terms of two functions on $M^4\times S^2$; one of the functions $Z(y^a,\zeta,\bar{\zeta})$ describes an $S^2$'s worth of surfaces through each spacetime point, while the other function $\Omega(y^a,\zeta,\bar{\zeta})$ plays the role of a conformal factor for a family of associated conformal metrics. The $y^a$ are local coordinates on $M^4$, and $\zeta$ is a stereographic coordinate on $S^2$. It is from this sphere's worth of surfaces themselves that the conformal metric—conformal to an Einstein metric—is constructed; the conformal factor then converts it into an Einstein metric. All the surfaces, for arbitrary but fixed $\zeta$, given by $Z(y^a,\zeta,\bar{\zeta})=\text{constant}$, are null surfaces with respect to this metric.

The partial differential equations satisfied by the two functions $Z$ and $\Omega$ (which are discussed in detail in [1–4]) can be imposed, in general, in any local region of an Einstein manifold. Roughly speaking the equations split into two sets: two (complex) equations, which we refer to as *metricity conditions*, guarantee that a Lorentzian metric can be constructed from the functions $Z$ and $\Omega$; the third (real) equation, referred to as $(E)$, imposes the vacuum Einstein equations on that metric. The purpose of the present work is to study these equations in the very important special case of asymptotically flat vacuum spacetimes. The main result of this study is that the meaning of the variables becomes more concrete and hence clearer and, furthermore, the structure of the equations changes and becomes much more transparent and simpler to use. It formally allows a straightforward (fully gauge fixed) perturbation theory.

In Sec. II, without giving any details or proofs, we outline (see [1–4] for details) the main features and equations of the NSF of GR. The main body of this work, contained in Sec.

*Electronic address: simo@artemis.phyast.pitt.edu

 

III, is divided into five subsections. In Sec. III A we review certain features of null infinity and discuss an alternate geometric meaning of the function $Z(y^a,\zeta,\bar{\zeta})$ which has, up to now, defined the null surfaces. In addition, the derivatives of $Z$ with respect to $(\zeta,\bar{\zeta})$ take on a simple geometric meaning. In Sec. III B, we discuss our main result, a single real equation [the light cone cut (LCC) equation] which, in the case of asymptotically flat spacetimes, replaces the two complex metricity conditions. Its derivation, which is quite lengthy, is contained in Appendix A. The LCC equation, plus equation $(E)$, are a pair of coupled equations for $Z$ and $\Omega$, and constitute the vacuum Einstein equations for asymptotically flat spaces. Section III C is devoted to describing an integral version of the LCC equation and $(E)$ and to a related perturbation expansion. In connection with this one can see how the gauge becomes fixed. In Sec. III D, self-dual (or anti-self-dual) vacuum metrics (via the good cut equation) are shown to satisfy the complexified version of the LCC equation, while in Sec. III E an alternate differential version of the LCC equation is given which displays interesting properties. In Sec. IV several isues that arise naturally from the asymptotically flat NSF are raised and discussed. Specifically, in Sec. IV A we show how from knowledge of $Z(y^a,\zeta,\bar{\zeta})$ one can obtain Bondi interior coordinates, $(u_B,r_B,\zeta_B,\bar{\zeta}_B)$. Understanding the insertion of Bondi coordinates in the context of the NSF is fundamental to the description of asymptotically flat solutions in terms of the NSF. In Sec. IV B, we give a preliminary analysis of how the asymptotic symmetries, the Bondi-Metzner-Sachs (BMS) group (or more accurately the Poincaré subgroup), yields several natural structures in the interior of the spacetime that reflect the group action at infinity $\mathcal{I}^+$. In particular, we discuss the introduction of a global pseudo Minkowskian coordinate system (equivalent to the previously mentioned gauge fixing) and the related global pseudo Poincaré transformations generated by the Poincaré transformations at infinity $\mathcal{I}^+$.

## II. THE NULL-SURFACE FORMULATION OF GR

In this section we will review the new formulation (the NSF) of classical general relativity [1–4]. In this formulation, the emphasis has been shifted away from the more standard type of field variable (metric, connection, holonomy, curvature, etc.) to, instead, families of three-dimensional surfaces on a four-manifold, $M^4$. On the manifold $M^4\times S^2$ (the sphere bundle over $M^4$ with no further structure), there are given differential equations for the determination of these surfaces. From the surfaces themselves, by differentiation and algebraic manipulation, a two-index symmetric tensor on $M^4$ can be defined. We will refer to this tensor as a ‘‘conformal metric,’’ although it actually represents a special member of the conformal class. The surfaces, which are our basic geometric quantities, are then, with no further conditions, the characteristic surfaces of this conformal metric (and of the whole conformal class). In addition, the equations allow for a choice of conformal factor that turns the conformal metric into a metric which satisfies the vacuum Einstein equations. Thus, the vacuum Einstein equations have been reformulated as equations for families of surfaces and a single (scalar) conformal factor. All geometric quantities, such as the metric, the connection, spin coefficients, Weyl and Ricci tensors, etc., can be expressed in terms of the surfaces and the conformal factor. The entire conformal information of the spacetime is coded into the surfaces, which will be described by

$$u=Z(y^a,\zeta,\bar{\zeta}). \tag{1}$$

For fixed value of $(\zeta,\bar{\zeta})$, Eq. (1) describes a local foliation of $M^4$ by the level surfaces of $Z(y^a,\zeta,\bar{\zeta})$. For changing values of $(\zeta,\bar{\zeta})$, $u=Z(y^a,\zeta,\bar{\zeta})=\text{const}$ describes a local two-parameter family of foliations.

With no proofs or derivations, as they have been given in extensive detail elsewhere, we will write out the Einstein equations for the families of surfaces, $u=Z(y^a,\zeta,\bar{\zeta})$, and the conformal factor, $\Omega(y^a,\zeta,\bar{\zeta})$. Though the vacuum metric can be written explicitly in terms of $Z$ and $\Omega$, we will not need it here but refer the reader to [1–4].

We begin with some preliminary definitions: from the assumed knowledge of $Z(y^a,\zeta,\bar{\zeta})$ we construct three additional functions by differentiating $Z$: two functions as the first derivatives with respect to $\zeta$ and $\bar{\zeta}$, and the third as the second mixed derivative with respect to both $\zeta$ and $\bar{\zeta}$. Using the ð and $\bar{ð}$ notation [5], we construct the four functions of $(y^a,\zeta,\bar{\zeta})$:

$$\theta^i(y^a,\zeta,\bar{\zeta})\equiv(\theta^0,\theta^+,\theta^-,\theta^1)\equiv(u,\omega,\bar{\omega},R)$$
$$\equiv(Z,ðZ,\bar{ð}Z,ð\bar{ð}Z). \tag{2}$$

Their gradients $\partial_a\theta^i$ [for any fixed value of $(\zeta,\bar{\zeta})$] form a covector basis. Using the set of dual vectors, $\theta_i^a(y^a,\zeta,\bar{\zeta})$ [satisfying $\theta_i^a\theta^j{}_{,a}=\delta_i^j$, for the same fixed value of $(\zeta,\bar{\zeta})$], we can define the directional derivatives

$$\partial_i\Phi=\theta_i^a\partial_a\Phi=\Phi_{,i}\quad (i=0,+,-,1). \tag{3}$$

Frequently we use $D\equiv\partial_1=\partial/\partial R$. Also for shorthand we define

$$\Lambda(y^a,\zeta,\bar{\zeta})\equiv ð^2Z\quad\text{and}\quad \bar{\Lambda}(y^a,\zeta,\bar{\zeta})\equiv\bar{ð}^2Z. \tag{4}$$

Using these variables and notation, the vacuum Einstein equations are

$$D^2\Omega=Q\Omega, \tag{E}$$

$$ð\Omega=\frac{1}{2}W\Omega, \tag{$m_{\rm I}$}$$

$$ð\Lambda_{,1}-2\Lambda_{,-}=[W+ð(\ln q)]\Lambda_{,1}, \tag{$m_{\rm II}$}$$

$$Q\equiv\frac{1}{4q}D\Lambda_{,1}D\bar{\Lambda}_{,1}+\frac{3}{8q^2}(Dq)^2-\frac{1}{4q}D^2q,$$

$$q\equiv 1-\Lambda_{,1}\bar{\Lambda}_{,1} \tag{5}$$

and

$$W\left(1-\frac{1}{4}\Lambda,_1\overline{\Lambda},_1\right)=\Lambda,_+ +\frac{1}{2}\bar{\eth}\Lambda,_1+\frac{1}{2}\Lambda,_1\overline{\Lambda},_- +\frac{1}{4}\Lambda,_1\eth\overline{\Lambda},_1 -\frac{1}{2}\eth\ \ln q-\frac{1}{4}\Lambda,_1\bar{\eth}\ \ln q. \tag{6}$$

[Equation $(E)$ is a direct translation into our variables of the trace-free part of the Einstein equations $R_{ab}-\frac{1}{4}g_{ab}R=0$ while Eqs. $(m_{\rm I})$ and $(m_{\rm II})$ guarantee that a metric exists.]

It is often useful to include among our ''Einstein equations'' $(E)$, $(m_{\rm I})$, and $(m_{\rm II})$, a fourth equation that follows immediately from the definitions (4): namely,

$$\bar{\eth}^2\Lambda=\eth^2\overline{\Lambda}. \tag{$I$}$$

Equations $(E)$, $(m_{\rm I})$, $(m_{\rm II})$, and $(I)$ are local coupled differential equations for the dependent variables $Z$ and $\Omega$. These four equations, though they appear not to have any obvious relationship to the Einstein equations, actually have the identical content as the vacuum equations with the possibility of a nonvanishing cosmological constant appearing as a constant of integration in the solutions.

As we emphasized earlier, from knowledge of $Z(y^a,\zeta,\overline{\zeta})$ and $\Omega(y^a,\zeta,\overline{\zeta})$, satisfying Eqs. $(E)$, $(m_{\rm I})$, $(m_{\rm II})$, and $(I)$, a vacuum metric can be easily constructed [1–4]. More specifically, our construction produces a sphere's worth of conformal metrics and a sphere's worth of conformal factors all of which are equivalent [from Eqs. $(m_{\rm I})$ and $(m_{\rm II})$], to a single unique metric obtained after a sphere's worth of coordinate transformations. In other words, we have a unique metric that has been given in a sphere's worth of coordinate systems parametrized by $(\zeta,\overline{\zeta})$.

It is important to emphasize that our equations involve six independent variables and six derivative operators which in general do not commute among themselves. Specifically, we have

$$(\partial_i\partial_j-\partial_j\partial_i)\Phi=0, \tag{7a}$$

$$(\eth\partial_i-\partial_i\eth)\Phi=-T_i^j\partial_j\Phi, \tag{7b}$$

$$(\bar{\eth}\partial_i-\partial_i\bar{\eth})\Phi=-\overline{T_i^j}\partial_j\Phi, \tag{7c}$$

$$(\bar{\eth}\eth-\eth\bar{\eth})\Phi=2s\Phi, \tag{7d}$$

where $s$ is the spin weight of $\Phi$ and

$$T_j^0=\delta_j^+,\quad \overline{T_j^0}=\delta_j^-, \tag{8a}$$

$$T_j^+=\Lambda,_j,\quad \overline{T_j^+}=\delta_j^1, \tag{8b}$$

$$T_j^-=\delta_j^1,\quad \overline{T_j^-}=\overline{\Lambda},_j, \tag{8c}$$

$$qT_i^1=\{\Lambda,_i\overline{\Lambda},_+ +\eth\overline{\Lambda},_i+\overline{\Lambda},_-\delta_i^1+\overline{\Lambda},_0\delta_i^+-2\delta_i^-\}\Lambda,_1+\bar{\eth}\Lambda,_i +\Lambda,_-\overline{\Lambda},_i+\Lambda,_0\delta_i^-+\Lambda,_+\delta_i^1-2\delta_i^+, \tag{8d}$$

$$q\overline{T_i^1}=\{\overline{\Lambda},_i\Lambda,_- +\bar{\eth}\Lambda,_i+\Lambda,_+\delta_i^1+\Lambda,_0\delta_i^--2\delta_i^+\}\overline{\Lambda},_1+\eth\overline{\Lambda},_i +\overline{\Lambda},_+\Lambda,_i+\overline{\Lambda},_0\delta_i^++\overline{\Lambda},_-\delta_i^1-2\delta_i^-. \tag{8e}$$

*Remark:* If one considers Eqs. $(E)$ and $(m_{\rm I})$ as equations for $\Omega$ for given $\Lambda$, then the study of their integrability conditions should yield equations involving only $Z$ so that they, with $(m_{\rm II})$, would be equivalent to the ''conformal Einstein equations.''[1] The conformal Einstein equations (the vanishing of the Bach tensor being one of them) yield metrics that are conformal to vacuum metrics; i.e., such that a conformal factor would exist that would convert the conformal metric into a vacuum metric. This procedure has been partially successful. It was shown [8] that the conformal metric constructed from $Z$ must satisfy the vanishing of the Bach tensor as a necessary condition for the existence of $\Omega$. This condition is also sufficient [9] if we restrict ourselves to asymptotically flat metrics.

It is the purpose of the remainder of this work to study our version of the Einstein equations, namely Eqs. $(E)$, $(m_{\rm I})$, $(m_{\rm II})$, and $(I)$, for the case of asymptotically flat spacetimes. The meaning and structure of the equations change considerably, becoming simpler and much more transparent. Perturbatively, they become gauge fixed and the solution can be written (in principle) as a series of explicit quadratures over the sphere—a nonlinear version of D'Adhémar integrals.

## III. THE NSF EQUATIONS FOR ASYMPTOTICALLY FLAT SPACETIMES

### A. Light cone cuts of $\mathcal{I}^+$

We now make the specialization from a description of any (local) Einstein spacetime to the study of asymptotically flat vacuum spacetimes. In this case the geometrical descriptions of various quantities becomes cleaner and more precise. We begin with the fact that null infinity, $\mathcal{I}^+$, exists. $\mathcal{I}^+$ can be thought of as the future null boundary of the spacetime, the collection of the ''end points'' of all future directed null geodesics, where the asymptotically flat spacetime ''becomes flat.'' These ''end points'' form a three-surface, referred to as $\mathcal{I}^+$, that can be visualized as a light cone and be coordinatized by a Bondi coordinate system,

$$(u,\zeta,\overline{\zeta}), \tag{9}$$

where $u$ is the Bondi retarded time, and $(\zeta,\overline{\zeta})$ (in $S^2$) label the null generators of $\mathcal{I}^+$. (We note that since $\mathcal{I}^+$ is, in some sense, flat, it turns out that it possesses an invariance group. The invariance group and some of its ramifications are discussed later in this work.) Using $\mathcal{I}^+$ and its properties, we can introduce a special class of null surfaces in the interior of the spacetime described in the following fashion; our basic variable, the function which describes our family of null surfaces

$$u=Z(y^a,\zeta,\overline{\zeta}), \tag{10}$$

is chosen as the past null cones of the points $(u,\zeta,\overline{\zeta})$ of $\mathcal{I}^+$.

[1] We refer here to the equations that are derived in [6], which are integrability conditions for the equations used by Friedrich in [7], coincidentally referred to as ''conformal Einstein equations.''

In this description, the values of $(u,\zeta,\bar{\zeta})$ in Eq. (10) are considered fixed, while $y^a$ varies over the past light cone of $(u,\zeta,\bar{\zeta})$.

From this meaning to $Z$, there exists a *dual interpretation* of $u=Z(y^a,\zeta,\bar{\zeta})$; namely, if the spacetime point $y^a$ is held fixed in Eq. (10) but the $(\zeta,\bar{\zeta})$ is varied over $S^2$, we obtain a (piecewise differentiable) two-surface on $\mathcal{I}^+$, the so-called light cone cut of $\mathcal{I}^+$, defined as the intersection of the future light cone of the point $y^a$ with $\mathcal{I}^+$. It consists of all points of $\mathcal{I}^+$ reached by null geodesics from $y^a$. With this dual interpretation, now considered as primary, $Z$ is then referred to as the light cone cut function. (The light cone cuts of $\mathcal{I}^+$ for Minkowski space are smooth and topologically $S^2$, though in the general case they will have self-intersections and cusps).

From this dual point of view, we now have a geometric interpretation, not only of $Z(y^a,\zeta,\bar{\zeta})$ but also of $\omega\equiv\eth Z$ and $R\equiv\eth\bar{\eth}Z$. $\omega$ is the ''stereographic angle'' that the light cone cuts make with the Bondi $u=\text{const}$ cuts; i.e., it labels the backward direction of the null geodesics from the point $(u,\zeta,\bar{\zeta})$, on $\mathcal{I}^+$, to $y^a$. $R$ is a measure of the curvature of the cut and thus a measure of the ''focusing distance'' from $\mathcal{I}^+$ to $y^a$ along the null geodesic. The four functions

$$(Z,\eth Z,\bar{\eth}Z,\eth\bar{\eth}Z)\equiv\theta^i(y^a,\zeta,\bar{\zeta}), \tag{11}$$

which are defined geometrically on $\mathcal{I}^+$, describe the interior of the spacetime. They can (in principle) be inverted [see Eq. (2)], leading to

$$y^a=y^a(\theta^i,\zeta,\bar{\zeta}), \tag{12}$$

which gives the location of spacetime points in terms of (geometrical) information on $\mathcal{I}^+$, namely the set $\theta^i(y^a,\zeta,\bar{\zeta})$, all obtained from $Z(y^a,\zeta,\bar{\zeta})$. Since the complete conformal information of the spacetime is coded into $Z(y^a,\zeta,\bar{\zeta})$, it is coded into $\theta^i(y^a,\zeta,\bar{\zeta})$ as well. That local interior spacetime structure can be obtained from $\mathcal{I}^+$ is due to the fact that the lightcones have a rigid structure. This plays an important role in our later discussion.

### B. The light cone cut equation

Although the null-surface equations $(E)$, $(m_{\mathrm{I}})$, $(m_{\mathrm{II}})$, and $(I)$ are completely general in the sense that they contain, locally, all possible spacetimes (including singular and regular asymptotically flat spacetimes), they appear quite intractable as they stand; we have not been able to implement any systematic method of constructing solutions or of approximating solutions from them. It is thus highly desirable to achieve a simpler or clearer formulation of these equations. We will now show that a simple reformulation does exist for the regular asymptotically flat solutions of Einstein equations.

Our main result is that the complex equations $(m_{\mathrm{I}})$, $(m_{\mathrm{II}})$, and $(I)$ imply (via a lengthy derivation) a real equation referred to as the light cone cut (LCC) equation, displayed below, which has many of the properties that we seek: it is a single equation with a unique solution for given radiative data. Conversely, though not trivially, the solutions to the LCC equation coupled to Eqs. $(E)$ satisfy $(m_{\mathrm{I}})$, $(m_{\mathrm{II}})$, and $(I)$, which allows us to claim that the NSF of asymptotically flat spacetimes consists of just the LCC equation and Eqs. $(E)$, two coupled real equations for the two real unknowns $Z$ and $\Omega$.

The derivation of the LCC equation, being lengthy and largely technical, is given in Appendix A. It mainly consists of taking appropriate derivatives of Eqs. $(m_{\mathrm{I}})$, $(m_{\mathrm{II}})$, and $(I)$, and combining them in a suitable manner to obtain an equation which can be integrated up in the variable $R$. At this point an integration constant is introduced, in the form of a complex spin weight-2 function of three variables $(u,\zeta,\bar{\zeta})$ which[2] is denoted

$$\dot{\sigma}=\dot{\sigma}(u,\zeta,\bar{\zeta}), \tag{13}$$

where $\dot{}\equiv\partial/\partial u$. Further manipulations and one more integration in the variable $u$ are necessary. The asymptotic flatness is imposed by setting the new integration constant to zero, so that there remains only

$$\sigma(u,\zeta,\bar{\zeta}) \tag{14}$$

as the free complex datum. $\sigma$ turns out to be the free characteristic datum for asymptotically flat spacetimes and is referred to as the asymptotic Bondi shear [10]. The final equation is the LCC equation

$$\eth^2\bar{\eth}^2 Z=\bar{\eth}^2\sigma_R+\eth^2\bar{\sigma}_R+N[Z,\Omega], \tag{15}$$

where $\sigma_R=\sigma(Z,\zeta,\bar{\zeta})$ is the freely chosen Bondi shear ($\bar{\sigma}$ being the complex conjugate) with the variable ''$u$'' replaced by $Z$. We choose data such that $\dot{\sigma}(u,\zeta,\bar{\zeta})$ vanishes as $u\rightarrow\pm\infty$. Physically, this is a natural condition limiting the gravitational radiation to finite amounts. On this data we impose the gauge condition that $\sigma(u,\zeta,\bar{\zeta})\rightarrow 0$ as $u\rightarrow+\infty$. The preservation of this form of the data restricts the full BMS group to its Poincaré subgroup. (See Sec. IV B)

The quantity $N$ stands for

$$N=\frac{1}{2}\int_{-\infty}^{u}\mathcal{N}du', \tag{16}$$

where $\mathcal{N}$ is explicitly given by

[2]Though the integration function is introduced with the only condition of being independent of $R$, the NSF equations require this function to be also independent of $\omega$ and $\bar{\omega}$. This assertion is easily proved in the linearized approximation; the linearized result presumably holds in the exact case as well.

$$
\begin{aligned}
\mathcal{N}= & \overline{\Lambda,_0}(\Lambda,_1-\Lambda,_0-(\bar{\eth}\Lambda),_- -\frac{1}{4}J+\frac{3}{2}\int_\infty^R(K,_- -L,_+)dR')+\Lambda,_0\left(\overline{\Lambda,_1}-\overline{\Lambda,_0}-(\eth\overline{\Lambda}),_+ -\frac{1}{4}\overline{J}+\frac{3}{2}\int_\infty^R(\overline{K,_+}-\overline{L,_-})dR'\right) \\
& -\frac{1}{4}\bar{\eth}^2J+\frac{3}{2}\bar{\eth}^2\int_\infty^R(K,_- -L,_+)dR'-\frac{1}{4}\eth^2\overline{J}+\frac{3}{2}\eth^2\int_\infty^R(\overline{K,_+}-\overline{L,_-})dR'+\frac{1}{2}\eth\bar{\eth}^2(\overline{\Lambda,_1}\Lambda,_- +\Lambda,_1(\eth\overline{\Lambda}),_1) \\
& -\bar{\eth}^3\left(\frac{\Lambda,_1}{4}[3(\overline{\Lambda,_1}\Lambda,_- +\Lambda,_1(\eth\overline{\Lambda}),_1-2\Lambda,_1\bar{\eth}\ \ln\Omega)-K]\right)-\bar{\eth}^2(\Lambda,_-\overline{\Lambda,_-}+\Lambda,_1(\eth\overline{\Lambda}),_-)-\frac{1}{2}\eth\bar{\eth}^2K \\
& -\eth\bar{\eth}(\Lambda,_-\overline{\Lambda,_+}+\Lambda,_1(\eth\overline{\Lambda}),_+)-\eth[-\overline{\Lambda,_+}(\Lambda,_0-\Lambda,_1-(\bar{\eth}\Lambda),_-)+(\eth\overline{\Lambda}),_+((\bar{\eth}\Lambda),_1-\Lambda,_+) \\
& -\overline{\Lambda,_1}((\bar{\eth}\Lambda),_0-(\bar{\eth}\Lambda),_1)+\overline{K}-(\bar{\eth}\Lambda),_+(\eth\overline{\Lambda}),_1]+2\overline{\Lambda,_0}(\bar{\eth}\Lambda),_- +2(\eth\overline{\Lambda}),_0(\bar{\eth}\Lambda),_1+2\bar{\eth}(\overline{\Lambda,_0}\Lambda,_- +\Lambda,_1(\eth\overline{\Lambda}),_0)
\end{aligned}
\tag{17}
$$

with

$$
\begin{aligned}
J\equiv & 3\bar{\eth}L+\eth K+\Lambda,_+^2+2\Lambda,_1(\bar{\eth}\Lambda),_+ +3\Lambda,_-\overline{\Lambda,_-} \\
& +3\Lambda,_1(\eth\overline{\Lambda}),_- -(\eth\Lambda),_-\overline{\Lambda,_1}+(\bar{\eth}\Lambda),_1^2-(\eth\Lambda),_1(\eth\overline{\Lambda}),_1,
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
K\equiv & 4\left(1-\frac{1}{4}\Lambda,_1\overline{\Lambda,_1}\right)\eth\ \ln\Omega+\frac{1}{2}\Lambda,_1\overline{\Lambda,_1}(\bar{\eth}\Lambda),_1-\Lambda,_-\overline{\Lambda,_1} \\
& +\frac{1}{2}\Lambda,_1\overline{\Lambda,_-}+\frac{1}{2}\Lambda,_1^2\overline{\Lambda,_+}+\frac{1}{2}\Lambda,_1(\eth\overline{\Lambda}),_1,
\end{aligned}
\tag{19}
$$

and

$$
L\equiv-\Lambda,_1\Lambda,_+-\frac{1}{2}\Lambda,_1\bar{\eth}\Lambda,_1-\Lambda,_1^2\bar{\eth}\ \ln\Omega. \tag{20}
$$

$N$ is thus seen to be a rather complicated functional of both $Z$ and $\Omega$. It involves an integration over $u$ and over $R$ of polynomials which are second and third order in the derivatives of $\eth^2Z$ and linear in the derivatives of $\ln\Omega$. It is disturbing (and potentially troublesome) that $N$ actually contains the highest-order derivatives in the LCC equation. Nevertheless, the explicit expression for $N$ that is presented here is just one of many equivalent forms, since $N$ can be changed via the Eqs. $(I)$, $(m_{\mathrm{I}})$, and $(m_{\mathrm{II}})$; at this time we are not yet certain of which form of $N$ would be most advantageous.[3] We emphasize that in this form, the LCC equation is suitable for perturbations around flat space, since $N$ is nonlinear and vanishing in the limit of small $\Lambda$.

### C. Integral form of the asymptotically flat NSF

Equations (15) and $(E)$ are a pair of coupled equations for both $Z$ and $\Omega$ that will be shown to constitute the full set of Einstein equations for asymptotically flat spacetimes with the asymptotic free data already included as $\sigma(u,\zeta,\overline{\zeta})$. They can be written as the pair of integral equations

[3]It can be seen that, up to second order in $\Lambda$ and $\Omega$, $\mathcal{N}$ can actually be rewritten into manifestly real form [11].

$$
\Omega=1+\int_R^\infty\int_{R'}^\infty Q\Omega\, dR''dR' \tag{21}
$$

and

$$
\begin{aligned}
Z(y^a,\zeta,\overline{\zeta})= & Z_0(y^a,\zeta,\overline{\zeta}) \\
& +\int_{S^2}G(\zeta,\zeta')(\bar{\eth}^2\sigma_R+\eth^2\overline{\sigma_R}+N[Z,\Omega])dS'^2,
\end{aligned}
\tag{22}
$$

where $G(\zeta,\zeta')$ is the Green's function for the "double Laplacian" $\bar{\eth}^2\eth^2$, discussed in Appendix C.

Note that $\Omega$ is required to approach the value 1 as $R\rightarrow\infty$. This is a geometrical requirement arising from the fact that the null surfaces $Z=$const have been chosen so that they are asymptotically null planes. This requirement fixes the otherwise arbitrary "constants of integration" in Eq. $(E)$.

Similarly, when integrating Eq. (15) the kernel $Z_0$ of $\bar{\eth}^2\eth^2$ is introduced, which consists of a combination of the $l=0$ and $l=1$ spherical harmonics with the four coefficients being arbitrary functions of $y^a$: i.e.,

$$
Z_0=\sum_{l=0,1}f_{lm}(y^a)Y_{lm}(\zeta,\overline{\zeta}).
$$

The kernel can be simplified by introducing new (*canonical*) coordinates $x^a\Leftrightarrow x_{lm}$ from these four functions via $x_{lm}=f_{lm}(y^a)$ so that

$$
Z_0=\sum_{l=0,1}x_{lm}Y_{lm}(\zeta,\overline{\zeta}). \tag{23}
$$

Though it is far from obvious, it has been shown [12] that, when the spacetime is sufficiently close to flat space, the transformation $x_{lm}=f_{lm}(y^a)$, which defines the canonical coordinates $x^a$ from a global coordinate system $y^a$, is sufficiently well behaved for the canonical coordinates $x^a$ to be also global. The coordinates $x^a$ defined in this way transform, in a nontrivial manner, via the asymptotic Poincaré transformation group, and constitute actual cartesian coordinates if the spacetime is flat. The properties of these canonical (pseudo Minkowskian) coordinates driven by the asymptotic Poincaré transformations are discussed in Sec. IV B.

We can obtain insights into the structure of the two equations (21) and (22), by means of a perturbation scheme around flat space. The characteristic datum for flat space is $\sigma=0$. The solution is $Z=Z_0$ and $\Omega=1$. Perturbed solutions around flat space correspond to a choice of data $\sigma=\epsilon\sigma_1$ where $\epsilon$ is a small real parameter, and in general can be expressed as power series of the form $Z=Z_0+\Sigma_{n=1}^{M}\epsilon^n Z_n$ and $\Omega=1+\Sigma_{n=1}^{M}\epsilon^n\Omega_n$ where the index $n$ labels the $n$th-order correction to the flat solutions. It is important to note that, even though the free data are taken to be only first order, the restricted data have contributions to higher orders as well in the following sense:

$$\sigma_1(Z,\zeta)=\sigma_1(Z_0,\zeta)+\epsilon Z_1\dot{\sigma}_1(Z_0,\zeta)+\epsilon^2\left(Z_2\dot{\sigma}_1(Z_0,\zeta)+\frac{1}{2}Z_1\ddot{\sigma}_1(Z_0,\zeta)\right)+\cdots \equiv \sigma_{R_1}+\epsilon\sigma_{R_2}+\epsilon^2\sigma_{R_3}+\cdots . \tag{24}$$

Since $Q=1+O(\epsilon^2)$, the linearization is explicitly

$$\Omega=1+O(\epsilon^2) \tag{25}$$

and

$$Z(x^a,\zeta,\overline{\zeta})=Z_0(x^a,\zeta,\overline{\zeta})+\epsilon\int_{S^2}G(\zeta,\zeta')(\bar{\eth}^2\sigma_{R_1}+\eth^2\overline{\sigma_{R_1}})dS'^2+O(\epsilon^2). \tag{26}$$

For $n>1$, the $n$th corrections to flat solutions are found by direct integration from lower order corrections:

$$Z_n=\int_{S^2}G(\zeta,\zeta')[\eth^2\overline{\sigma_{R_n}}+\bar{\eth}^2\sigma_{R_n}+N_n(Z,\Omega)]dS'^2, \tag{27}$$

$$\Omega_n=\int_R^\infty\int_{R'}^\infty[Q(Z)\Omega]_n dR'dR''. \tag{28}$$

The process of integrating Eqs. (21) and (22) is very much simplified by the fact that $Q$ depends on $Z$ only through $\Lambda$ and has no linear term in $\Lambda$. From this it follows that the right-hand side in Eq. (28) depends only on $Z_{n-1},\ldots,Z_1$ and is thus known at order $n-1$. Thus, the right-hand side of Eq. (28) constitutes a source in terms of known lower orders in $Z$ and can be integrated, yielding $\Omega_n$. The $\Omega_n$ in Eq. (27) can now be thought of as a source for $Z_n$, and since $N$ is also second order in $\Lambda$, the entire right-hand side of Eq. (27) involves only known lower order $Z$ terms. The two equations thus decouple at every stage of the approximation allowing one to toggle back and forth between them. At every stage, the solutions are unique, due to our fixed choices of the kernel $Z_0$ and the boundary condition for $\Omega$. Thus the coupled equations (15) and ($E$) can be *uniquely* solved, given radiative data $\sigma$, by means of a perturbative expansion. We have not studied the difficult problem of the convergence of this expansion.

We claimed earlier that we have (perturbatively) a fixed gauge. This arises from our fixed choice of $Z_0=\Sigma_{l=0,1}x_{lm}Y_{lm}$, the fixed choice of null foliations and the boundary condition that $\Omega$ goes to 1 at $\mathcal{I}^+$. The $x^a$ play the dual role of constants of integration of the LCC equation and as the choice of local canonical (pseudo Minkowskian) coordinates. A general gauge transformation (coordinate transformation) away from these canonical coordinates would consist in choosing the $x^a$ as four arbitrary functions of $y^a$, namely $x^a=x^a(y^a)$. The property of uniqueness of (perturbative) solutions to Eqs. ($E$) and (15) in this gauge can be invoked to show that the coupled system ($E$)-(15) is equivalent to the full set of equations ($E$), ($m_{\rm I}$), ($m_{\rm II}$), and ($I$). For, if the solution is unique, it cannot be further restricted by imposing on it Eqs. ($m_{\rm I}$), ($m_{\rm II}$), and ($I$), and must therefore satisfy them identically. This argument is restricted to the perturbative solutions.

We point out, but do not further explore, that the integral equations (21) and (22) appear to be well suited for the use of fixed-point theorems on function spaces to approach the problem of existence of solutions. Unfortunately, finding a measure and studying properties of the map appears to be virtually insurmountable due to the complexity of $N$. The use of Newton's approximation is also suggested but carries the same difficulty.

### D. A digression: Self-dual spacetimes

As a mild digression we discuss a special case of Eqs. ($E$) and (15), namely the case of asymptotically flat vacuum *self-dual* spacetimes. These spacetimes are complex, and arise by allowing $u$ to become complex and treating $\zeta$ and $\overline{\zeta}$ as two independent complex coordinates. The two functions $\sigma(u,\zeta,\overline{\zeta})$ and $\overline{\sigma}(u,\zeta,\overline{\zeta})$ are no longer complex conjugates of each other, they become independent data. The choice of Bondi shear that corresponds to self-dual spacetimes is

$$\overline{\sigma}(u,\zeta,\overline{\zeta})=0 \tag{29}$$

with $\sigma=\sigma(u,\zeta,\overline{\zeta})$ as an arbitrary spin-weight-2 function of the three arguments. The self-dual spacetimes can be described in terms of complex light cone cut functions satisfying the so-called *good cut equation*, i.e.,

$$\eth^2 Z=\sigma(Z,\zeta,\overline{\zeta}). \tag{30}$$

We point out that, if the LCC equation is complexified by allowing all complex conjugate quantities to become independent, Eq. (15) is consistent with the good cut equation (30). By this we mean that every solution to the good cut equation is also a solution of the complexified LCC equation. Indeed, from Eq. (30) we have that $\Lambda=\sigma(Z,\zeta,\overline{\zeta})$ and hence that $\Lambda_{,1}=0$. This implies that $Q=0$ and hence that $\Omega=1$. Furthermore, this also implies $\Lambda_{,+}=\Lambda_{,-}=0$ as well as a further set of relations (see Appendix B) among the different derivatives of $\Lambda$ and $\overline{\Lambda}$ that are relevant to Eq. (15): namely,

$$0=2\overline{\Lambda}_{,-}+\eth\overline{\Lambda}_{,1}, \tag{31a}$$

$$0=2\overline{\Lambda}_{,+}-\bar{\eth}\overline{\Lambda}_{,1}, \tag{31b}$$

$$0=\frac{1}{2}\,\eth^3\overline{\Lambda}_{,1}+2\bar{\eth}\dot{\sigma}-2\eth\overline{\Lambda}_{,1}\dot{\sigma}-\overline{\Lambda}_{,1}\eth\dot{\sigma}, \tag{31c}$$

with the result that $N$ in Eq. (15) vanishes identically. In other words, the null-surface description of self-dual spacetimes is

$$\Omega=1 \quad \text{and} \quad \eth^2 Z=\sigma(Z,\zeta,\bar{\zeta}). \tag{32}$$

The space of the (complex) solutions to Eq. (32), known as $\mathcal{H}$ space, has been extensively studied. Reference [13] reviews the theory and describes the significance of Eqs. (31).

A stronger statement can be made in the linear regime. It is interesting to notice that in the linearization of the complexified LCC equation, with $\bar{\sigma}(u,\zeta,\bar{\zeta})=0$, it is *completely equivalent* to the good cut equation with regularity imposed on $Z$; i.e., every solution of the complexified linearized LCC equation with vanishing $\bar{\sigma}$, satisfies the good cut equation as well. In order to see this we write the complexified linearized LCC equation with vanishing $\bar{\sigma}$: i.e.,

$$\bar{\eth}^2\eth^2 Z=\bar{\eth}^2\sigma(Z_0,\zeta,\bar{\zeta}).$$

If $Z$ is regular in the sense that it admits an expansion in terms of spin-0 spherical harmonics, this equation implies that $\eth^2 Z$ is equal to $\sigma(Z_0,\zeta,\bar{\zeta})$ only up to the addition of the kernel of $\bar{\eth}^2$ acting on spin-2 functions. Since this kernel is vanishing (there are no nonzero spin-2 functions that are annihilated by $\bar{\eth}^2$), then $\eth^2 Z=\sigma(Z_0,\zeta,\bar{\zeta})$.

Presumably, the exact complexified LCC equation with $\bar{\sigma}=0$ with some appropriate regularity conditions on $Z$ should be equivalent to the exact good cut equation.

### E. Alternative version of the null-surface equations

In this section we will show that there is a version of the asymptotically flat null-surface equations that have a lack of symmetry between the $\eth$ and $\bar{\eth}$ derivatives; i.e., they will have a chirality or handedness; technically, they will depend on a choice of the complex structure on the $(\zeta,\bar{\zeta})$ sphere.

As a model, we first examine the case of self-dual spacetimes, as in the previous section. The light cone cut functions $Z$ for self-dual spacetimes satisfy the good cut equation, which can equivalently be written in the form

$$\eth^2 Z=\Lambda(Z,\eth Z,\bar{\eth}Z,\eth\bar{\eth}Z,\zeta,\bar{\zeta}) \tag{33}$$

with $\Lambda$ representing a source term to be specified by the additional equation

$$\Lambda(\theta^i,\zeta,\bar{\zeta})=\sigma(u,\zeta,\bar{\zeta}). \tag{34}$$

In this case, Eq. (33) contains only $\eth$ derivatives of $Z$, since the solution $\Lambda$ to Eq. (34) does not depend on $\theta^+$ or $\theta^1$.

As a next step in complexity, we discuss the analogous procedure for anti-self-dual spacetimes. In this case, the cut function must satisfy

$$\eth^2 Z=\Lambda(Z,\eth Z,\bar{\eth}Z,\eth\bar{\eth}Z,\zeta,\bar{\zeta}) \tag{35a}$$

but where now $\Lambda$ is first determined by a solution of

$$0=\frac{1}{2}\,\bar{\eth}^3\Lambda_{,1}+2\eth\dot{\bar{\sigma}}-2\bar{\eth}\Lambda_{,1}\dot{\bar{\sigma}}-\Lambda_{,1}\bar{\eth}\dot{\bar{\sigma}}, \tag{35b}$$

where asymptotic flatness and appropriate regularity conditions are imposed on $\Lambda$. [Note that Eq. (35b) is the complex conjugate of Eq. (31c).] Equations (35) are understood in the following manner. Equation (35b) constitutes a fourth-order nonlinear equation for the function $\Lambda$ in the independent variables $(\theta^i,\zeta,\bar{\zeta})$. The solution $\Lambda$ is then expressed in terms of $Z$ by $\Lambda=\Lambda(Z,\eth Z,\bar{\eth}Z,\eth\bar{\eth}Z,\zeta,\bar{\zeta})$ and placed on the right-hand side of Eq. (35a), which, in turn, becomes a nonlinear second-order equation for $Z$ in the independent variables $(x^a,\zeta,\bar{\zeta})$. This alternative version of the anti-self-dual equations, however, does not completely favor the $\eth$ derivatives over the $\bar{\eth}$ derivatives because of the presence of $\bar{\eth}Z$ and $\eth\bar{\eth}Z$ in the right-hand side of Eq. (35a).

Analogously, we can treat the linearized approximation of real general relativity in the following manner; the real cut function $Z$ would be found by

$$\eth^2 Z=\Lambda(Z,\eth Z,\bar{\eth}Z,\eth\bar{\eth}Z,\zeta,\bar{\zeta}) \tag{36a}$$

and $\Lambda$ is determined by

$$0=\frac{1}{2}\,\bar{\eth}^3\Lambda_{,1}+2\eth\dot{\bar{\sigma}}, \tag{36b}$$

with appropriate regularity conditions and asymptotic flatness imposed on $\Lambda$. Here $\sigma(u,\zeta,\bar{\zeta})$ enters in the solution to Eq. (36b) as a constant of integration. It is simple to show that, by integrating up Eq. (36b), taking $\bar{\eth}^2$ of Eq. (36a), and entering the solution $\Lambda$ into the right-hand side of Eq. (36a), we recover the linearized LCC equation.

This same procedure can be applied in the exact real case. The counterparts of Eq. (36) are

$$\eth^2 Z=\Lambda(Z,\eth Z,\bar{\eth}Z,\eth\bar{\eth}Z,\zeta,\bar{\zeta}) \tag{37a}$$

and an equation for the determination of $\Lambda(\theta^1,\zeta,\bar{\zeta})$: namely,

$$2\eth\dot{\bar{\sigma}}=\bar{\eth}^2\Lambda_{,+}+\bar{\eth}[\Lambda_{,1}(\eth\overline{\Lambda})_{,+}+\Lambda_{,-}\overline{\Lambda}_{,+}]-\overline{\Lambda}_{,+}[\Lambda_{,0}-\Lambda_{,1}-(\bar{\eth}\Lambda)_{,-}]+(\eth\overline{\Lambda})_{,+}[(\bar{\eth}\Lambda)_{,1}-\Lambda_{,+}]$$
$$-\eth\left(\frac{3}{2}\int_R^\infty(\overline{K}_{,+}-\overline{L}_{,-})\,dR'-\frac{1}{4}\,\overline{J}\right)-\overline{\Lambda}_{,1}[(\bar{\eth}\Lambda)_{,0}-(\bar{\eth}\Lambda)_{,1}]+\overline{K}-(\bar{\eth}\Lambda)_{,+}(\eth\overline{\Lambda})_{,1}. \tag{37b}$$

[Equation (37b) is derived in Appendix A, where it appears labeled as Eq. (A14).] The approach to the light cone cuts via Eq. (37) does not seem to have immediate practical uses. It has a handedness built in, which, however, does not completely favor the $\eth$ derivative over the $\bar{\eth}$ derivatives for the same reason as above, namely, the presence of $\bar{\eth}Z$ and $\eth\bar{\eth}Z$ in the right-hand side of Eq. (37b). It does not appear that a completely chiral description (such as sought by Penrose as part of the twistor program) would be feasible even in a perturbative fashion, since at the linearized level it is clear, from Eq. (36b), that $\Lambda,_1$ is not vanishing.

## IV. FURTHER ISSUES AND DISCUSSION

We have presented here an unconventional description of the vacuum Einstein equations applied to asymptotically flat spacetimes in terms of either the light cone cuts of $\mathcal{I}^+$ or, equivalently, the past null cones of the points of $\mathcal{I}^+$; i.e., in terms of $Z$ and $\Omega$. This formulation has certain advantages (and of course certain disadvantages) over the conventional treatment. We are interested in studying what new insights it can give us into Einstein manifolds or into solution-generating techniques.

As a possible application we plan in the future to study (to second order) the problem of the classical scattering of data from $\mathcal{I}^-$ to $\mathcal{I}^+$, i.e., if past data are given and a solution is evolved from it, what will the future data look like? This appears to us to be technically difficult but conceptually straightforward with the use of the fixed pseudo Minkowski gauge.

Another problem is to study asymptotically flat metrics in the asymptotic region in terms of $Z$ and $\Omega$. In order to accomplish this (since the metrics in the asymptotic region are expressed in interior Bondi coordinates) we must first find the relationship between our pseudo Minkowskian coordinates and the interior Bondi coordinates. Finding this relationship is a pretty geometric excercise using the light cone cut function. This is done in Sec. IV A.

Of considerably more interest to us is the question of what, if any, effect can be seen in the interior of the asymptotically flat space that might be induced by the asymptotic symmetries. That this is a distinct possibility could be conjectured from the rigidity of the light cone structure that we are dealing with. This, in fact, is what happens; there are several different but related objects that can be found (or defined) in the interior, that transform under representations of the Poincaré group, some of them via finite-dimensional representations, others via infinite-dimensional representations. One of the more intriguing results is the following: As we mentioned earlier, given a particular Bondi coordinate system at $\mathcal{I}^+$, there is a canonical choice of coordinates $x'^a$ throughout the spacetime (the pseudo Minkowskian coordinates). If the asymptotic Bondi coordinates $(u,\zeta,\bar{\zeta})$ are changed via an asymptotic Poincaré transformation, this induces a transformation of the pseudo Minkowskian coordinates to new pseudo Minkowskian coordinates, $x'^a$. The relationship of the new $x$'s to the old ones is a nonlinear realization of the Poincaré group. We have an equation of the form

$$x'^a = x'^a(x^b,\Lambda^b_c,d^b). \tag{38}$$

The $(\Lambda^a_b,d^a)$ are the ten parameters of the asymptotic Poincaré transformation. If Eq. (38) is treated as a ten-parameter set of motions, we can differentiate Eq. (38) with respect to any of the parameters to obtain ten vector fields that can be ''defined'' as four translations, three rotations, and three boosts. Obviously they are not symmetries of the spacetime, though they do become the spacetime symmetries in the case of vanishing radiation data—the case of flat spacetime. It is suggestive that these vector fields might be thought of as defining global approximate symmetries, thus, perhaps, allowing the discussion of approximate conservation laws via Noether-type theorems. There are other, more dynamical, objects that can also be obtained which transform nicely; e.g., local energy-momentum four-vector fields can be obtained from the Bondi mass aspect mapped down to the points $x^a$. We will not go into these issues in great detail here, since they will be treated elsewhere. On the other hand, we do want to give the gist of the ideas here. In order to do this, however, we find it appropriate to first give an outline of the theory of infinite dimensional representations of the Lorentz group [14,15]. This material is presented in Sec. IV B.

In Sec. IV C we will very briefly discuss how the asymptotic form of the NSF might be of use in one attempt to ''quantize'' GR.

### A. Introduction of interior Bondi coordinates

The Bondi coordinates $(u,\zeta,\bar{\zeta})$ of $\mathcal{I}^+$ can be extended into the interior of the spacetime, in a neighborhood of $\mathcal{I}^+$, in the following manner. From a given cut $u=\text{const}$ at $\mathcal{I}^+$, coordinates $\zeta_B$ and $\bar{\zeta}_B$ can be assigned as labels for the null geodesics that meet the cut orthogonally. The coordinate system is completed by defining a parameter $r_B$ that varies along these null geodesics. In other words the coordinate system is defined by choosing null geodesics labeled by where they interesect $\mathcal{I}^+$ [i.e., $(u,\zeta,\bar{\zeta})$] and are othogonal to the $u=\text{const}$ cuts; $r_B$ is chosen as a geodesic parameter. We refer to $(u_B,r_B,\zeta_B,\bar{\zeta}_B)$ as *interior Bondi coordinates.*

Here we study how to transform the coordinates $y^a$ into interior Bondi coordinates $(u_B,r_B,\zeta_B,\bar{\zeta}_B)$ in a neighborhood of $\mathcal{I}^+$. We show that the cut function $Z(y^a,\zeta,\bar{\zeta})$ actually encodes the coordinate transformation.

We have at our disposal a sphere's worth of coordinate transformations, from the $y^a$ to our family of null coordinates, $\theta^i=\theta^i(y^a,\zeta)$, defined by Eq. (2). However, the interior Bondi coordinates do not correspond with one of these coordinate systems for any given value of $\zeta$. At every value of $(\zeta,\bar{\zeta})$, the coordinate $\theta^0$ defines the past null cone from $(u,\zeta)$ at scri. The coordinates $(\theta^+,\theta^-)$ label all null geodesics within this past light cone. The value of $\theta^\pm$ that corresponds to the null geodesic that meets scri orthogonally is $\theta^+=\theta^-=0$, for the reasons described in the discussion below. Thus we can transform from $y^a$ into interior Bondi coordinates $(u_B,r_B,\zeta_B,\bar{\zeta}_B)$ by the set of four implicit functions

$$u_B=Z(y^a,\zeta_B(y^a),\bar{\zeta}_B(y^a)),\quad 0=\eth Z(y^a,\zeta_B,\bar{\zeta}_B),$$
$$0=\bar{\eth}Z(y^a,\zeta_B,\bar{\zeta}_B),\quad r_B=\eth\bar{\eth}Z(y^a,\zeta_B(y^a),\bar{\zeta}_B(y^a)). \tag{39}$$

They encode $y^a=y^a(u_B,r_B,\zeta_B,\bar{\zeta}_B)$ and are, as can be clearly seen, intrinsically connected to the cut function $Z$.

To show that $\theta^+=\theta^-=0$ corresponds to the orthogonal null geodesic, first we point out that $Z$ assigns a sphere's worth of values of $u$ at scri for every interior point $y^a$ via $u=Z(y^a,\zeta,\bar{\zeta})$, by means the null geodesics from $y^a$, i.e., via the light cone cuts. The issue is how to pick the correct value of $\zeta$ on the light cone cut, which corresponds with the null geodesic that leaves scri orthogonally, thus assigning to $y^a$ a unique value of $u$. The light cone cut is a two-surface at scri which is tangent to a Bondi cut ($u=$const) at only a discrete set of points, at which the tangent vanishes. By solving $\eth Z(y^a,\zeta,\bar{\zeta})=0$ and $\bar{\eth} Z(y^a,\zeta,\bar{\zeta})=0$ for $\zeta$ and $\bar{\zeta}$ as functions of $y^a$ (with one choice of these "tangent points," such as the one that minimizes the value of $\theta^0$), we obtain $(\zeta_B,\bar{\zeta}_B)$ as functions of $y^a$. Then by substitution we obtain $u_B=Z[y^a,\zeta_B(y^a),\bar{\zeta}_B(y^a)]$, i.e., $u_B$ as a function $y^a$. The remaining coordinate, the geodesic parameter $r_B$), can be obtained from the NSF picture, simply by the substitution into $r=\eth\bar{\eth}Z(y^a,\zeta,\bar{\zeta})$, yielding $r_B=\eth\bar{\eth}Z(y^a,\zeta_B(y^a),\bar{\zeta}_B(y^a))$.

As an illustration, here we show how this procedure can be applied in a flat spacetime to obtain the interior Bondi coordinates from standard Cartesian coordinates. Since the Cartesian coordinates $x^a=(t,x,y,z)$ can always be obtained [12] from an arbitrary coordinate system $y^a$ by knowledge of $Z(y^a,\zeta,\bar{\zeta})$, this example covers the most general flat case.

The light cone cut function for flat space in Cartesian coordinates has the form (23), or equivalently $Z_0=x^a\ell_a$, where

$$\ell_a=\frac{1}{\sqrt{2}(1+\zeta\bar{\zeta})}((1+\zeta\bar{\zeta}),-(\zeta+\bar{\zeta}),i(\zeta-\bar{\zeta}),(1-\zeta\bar{\zeta})). \tag{40}$$

Applying $\eth$ to $Z_0$ and setting it equal to zero we have $x^a m_a=0$, where $m_a\equiv\eth\ell_a=[1/\sqrt{2}(1+\zeta\bar{\zeta})](0,(\bar{\zeta}^2-1),i(\bar{\zeta}^2+1),(-2\bar{\zeta}))$. This is a quadratic equation for $\zeta$ as a function of $x^a$, and the solutions are

$$\bar{\zeta}_B(x^a)=\frac{z\pm\sqrt{x^2+y^2+z^2}}{x+iy}, \tag{41}$$

and the complex conjugate

$$\zeta_B(x^a)=\frac{z\pm\sqrt{x^2+y^2+z^2}}{x-iy}. \tag{42}$$

Choosing the positive sign (which gives the smallest value of $u$ upon substitution), and substituting this value of $(\zeta_B(x^a),\bar{\zeta}_B(x^a))$ into $u=x^a\ell_a(\zeta_B(x^a))$ and $r=x^a\eth\bar{\eth}\ell_a(\zeta_B(x^a))$, we obtain

$$u_B(x^a)=\frac{1}{\sqrt{2}}(t-\sqrt{x^2+y^2+z^2}), \tag{43}$$

$$r_B(x^a)=\sqrt{2(x^2+y^2+z^2)}, \tag{44}$$

which completes the transformation. The inverse transformation giving $x^a$ in terms of the interior Bondi coordinates is

$$x^a=\sqrt{2}u_Bt^a+r_B\ell^a(\zeta_B,\bar{\zeta}_B), \tag{45}$$

where $t^a\equiv(1,0,0,0)$.

### B. Aspects of Lorentz covariance in asymptotically flat spacetimes

It has been known, since shortly after the seminal work of Bondi [16,5] on gravitational radiation and the discoveries of the related asymptotic symmetries—the Bondi-Metzner-Sachs (BMS) group—that, with an appropriate choice of the gauge applied to the characteristic data, one could obtain the Poincaré subgroup as the invariance group of the asymptotic region of the spacetime. We have recently realized that this asymptotic invariance has consequences that go well beyond the asymptotic regions, and, in fact, has local influences throughout the spacetime. By this we do not mean that we can find spacetime symmetries resembling the Poincaré group; however, we are finding a large number of interesting local structures that transform under either the finite- or infinite-dimensional (reducible, nonunitary) representations of the Lorentz group. These structures arise because of the existence of the asymptotic symmetry. Though, at the present, we do not yet understand the physical significance of most of these quantities, they nevertheless are intriguingly suggestive. Our purpose here is to give a simple preliminary discussion of them. We will concentrate on the homogeneous Lorentz group in detail and simply mention how the Poincaré translations enter the discussion.

We begin with a brief review of some of the ideas associated with the finite- and infinite-dimensional (reducible, nonunitary) representations of the homogeneous Lorentz group [14,15].

We first recall that the representations are labeled by two numbers ($k_0$ and $c$) where $k_0$ is integer or half-integer and $c$ is any complex number [14] or alternatively by $s$ and $w$ with $s$ (the spin weight) being either integer or half-integer and $w$ (the conformal weight) being complex. However for the sake of simplicity we will confine ourselves to a special subset of these representations, the so-called $s=0$ and $w=\ldots,-4,-3,-2,0,1,2,3,\ldots$ representations. The linear vector space associated with each of these representations has a dual space which also lies in this same class. The representations can thus be organized in dual pairs, the pairs being

$$(w,w')=(-2,0),(-3,1),(-4,2),\ldots,(-n,n-2), \quad n\geqslant 2. \tag{46}$$

A vector, in any one of these $s=0$ representations, can be expressed as a regular function on the sphere, i.e., by

$$\eta_{(w)}(\zeta,\bar{\zeta})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\eta_{(w)}^{lm}Y_{lm}(\zeta,\bar{\zeta}) \tag{47}$$

with the constants $\eta_{(w)}^{lm}$ being the components of the vector in the $Y_{lm}(\zeta,\bar{\zeta})$ basis. The $(w)$ labels the representation and also describes how the vectors of the representation transform under the Lorentz transformation. The Lorentz transformation is given in the form of the fractional linear transformation (Mobius transformation), or (almost) equivalently by an $SL(2,C)$ transformation. Specifically, we have

$$\eta'_{(w)}(\zeta',\overline{\zeta'})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\eta'^{lm}_{(w)}Y_{lm}(\zeta',\overline{\zeta'}) \tag{48}$$

with

$$\eta'_{(w)}(\zeta',\overline{\zeta'})=K^{w}\eta_{(w)}(\zeta,\overline{\zeta}) \tag{49}$$

and

$$\zeta'=\frac{a\zeta+b}{c\zeta+d},\quad ad-bc=1, \tag{50}$$

$(a,b,c,d)$ complex and

$$K=v^{-1},$$

$$v=v^{I}\ell_{I}(\zeta,\overline{\zeta})=\sum_{l=0}^{1}\sum_{m=-l}^{l}v^{lm}Y_{lm}(\zeta,\overline{\zeta})$$
$$=(1+\zeta\overline{\zeta})^{-1}((a\zeta+b)(\overline{a\zeta}+\overline{b})+(c\zeta+d)(\overline{c\zeta}+\overline{d})). \tag{51}$$

The four components $v^I$ represent a unit Lorentz vector, namely the velocity of the ''boost.'' For a rotation, $v^I=(1,0,0,0)$ and we have that $K=1$. The Lorentz vector $\ell_I$ is defined to have the components as the spacetime vector $\ell_a$ in Eq. (40). We are using capital Latin indices $I,J$ to denote Lorentz objects.

Equations (47)–(51) contain the full description of the $s=0$ and integer $w\neq-1$ representations. Though these representations are all infinite dimensional, they are not totally reducible; they do contain invariant subspaces. For the cases of $w\geqslant0$, these invariant subspaces are finite dimensional and yield the finite-dimensional representations; for $w<0$ the invariant subspaces are all infinite dimensional. Specifically for fixed $w\geqslant0$, the invariant subspace is defined by vectors of the form

$$\eta_{(w)}=\sum_{l=0}^{w}\sum_{m=-l}^{l}\eta^{lm}_{(w)}Y_{lm}(\zeta,\overline{\zeta}). \tag{52}$$

In particular the scalar representation is given by $w=0$, yielding as the invariant subspace

$$\eta_{(0)}=\eta^{00}_{(0)}Y_{00}(\zeta,\overline{\zeta}), \tag{53}$$

which are simply constants. The ordinary vector representation is given by by $w=1$, yielding

$$\eta_{(1)}=\sum_{l=0}^{1}\sum_{m=-l}^{l}\eta^{lm}_{(1)}Y_{lm}(\zeta,\overline{\zeta}), \tag{54}$$

and the symmetric trace-free representation by $w=2$, with

$$\eta_{(2)}=\sum_{l=0}^{2}\sum_{m=-l}^{l}\eta^{lm}_{(2)}Y_{lm}(\zeta,\overline{\zeta}). \tag{55}$$

There are ''intertwining'' operators [14,15] that map vectors from one representation to another. We will have an interest in the special case where the map is from an infinite-dimensional negative-$w$ representation to the invariant subspace of its dual; i.e., from $w=-n<-1$ to the finite-dimensional representation $w'=n-2$; e.g., from $w=-2$ to $w'=0$ or $w=-3$ to $w'=1$. Explicitly, the mappings are given by

$$\eta'_{(w')}(\zeta,\overline{\zeta})=\oint G_{(w',w)}(\zeta,\lambda)\,\eta_{(w)}(\lambda,\overline{\lambda})\,\frac{d\lambda d\overline{\lambda}}{(1+\lambda\overline{\lambda})^{2}}, \tag{56}$$

where $G_{(w',w)}(\zeta,\lambda)$ is a Green's function for every $w$. For instance, for $w=-2$, the Green's function is $G_{(0,-2)}=1$ and it allows us to obtain Lorentz scalars from the $w=-2$ representation. For $w=-3$, the Green's function is $G_{(1,-3)}=\ell^{I}(\lambda)\ell_{I}(\zeta)$ and it yields Lorentz four-vectors from the $w=-3$ representation.

Returning to the theory of asymptotically flat spacetimes, we briefly review some of the ideas [5] concerning the BMS group, the symmetry group of $\mathcal{I}^+$.

Given the Bondi coordinates of $\mathcal{I}^+$, $(u,\zeta,\overline{\zeta})$, the BMS transformation is given by

$$u'=K(u+\alpha(\zeta,\overline{\zeta})),$$
$$\zeta'=\frac{a\zeta+b}{c\zeta+d},\quad ad-bc=1, \tag{57}$$

where $K$ is given by Eq. (51) and $\alpha(\zeta,\overline{\zeta})$ is an arbitrary function of conformal weight 1 on $S^2$, the so-called supertranslation. By demanding that the Bondi shear $\sigma(u,\zeta,\overline{\zeta})$ (the free gravitational characteristic data), vanishes at future infinity and remains zero after a BMS transformation, one can show that $\alpha(\zeta,\overline{\zeta})$ must be restricted to contain only $l=0,1$ harmonics (the four translations), and Eq. (57) becomes the Poincaré group. In this manner the Poincaré group becomes the symmetry group of $\mathcal{I}^+$.

We are now in position to show how, from the representation theory, the asymptotic symmetries (Lorentz or Poincaré group) can be felt in the interior of the spacetime from several different but related points of view.

First we point out and emphasize that the cut function $Z$ transforms as a $w=1$ function under the BMS transformation $u'=Ku$; i.e.,

$$Z'(x^a,\zeta',\overline{\zeta'})=K(\zeta,\overline{\zeta})Z(x^a,\zeta,\overline{\zeta}). \tag{58}$$

We note, but do not explore in detail here, that for Minkowski space, the cut function $Z_0(x^a,\zeta,\overline{\zeta})$ contains only the first four harmonics, $l=0,1$, and the coefficients are the standard flat spacetime coordinates $(t,x,y,z)$ and hence the application of Eq. (58) is just the ordinary (coordinate) Lorentz transformation; i.e., we have simply the finite-dimensional $w=1$ representation. However, in the general asymptotically flat case (with all harmonics in $Z$), the first four components can again be taken as the spacetime coordinates $x^a$ (this constitutes a canonical choice of global pseudo Lorentzian coordinates that exist when the spacetime is sufficiently close to Minkowski space [12]), but now the coordinate transformation generated by Eq. (58) is much more complicated (this follows from the fact that $Z$ is in the infinite dimensional $w=1$ representation); the coefficients of the higher harmonics map down to the $l=0,1$ harmonics (the finite dimensional invariant subspace) yielding a transformation that is in general nonlinear but dependent on the six parameters of the Lorentz transformation. If the $Z$ is thought of as a natural decomposition into the finite dimensional invariant subspace and the infinite dimensional remainder, i.e.,

$$Z=x^a\ell_a(\zeta,\overline{\zeta})+\sum_{l=2}^{\infty} z^{lm}(x^a)Y_{lm}(\zeta,\overline{\zeta}) \tag{59}$$

and

$$Z'=x'^a\ell_a(\zeta',\overline{\zeta'})+\sum_{l=2}^{\infty} z'^{lm}(x'^a)Y_{lm}(\zeta',\overline{\zeta'}), \tag{60}$$

then from Eq. (53) the transformation has the form

$$x'^a=\Lambda^a_b x^b+\Lambda_{lm}(\Lambda^a_b)z^{lm}(x^a). \tag{61}$$

Note that the components of $\ell_a$ [given as in Eq. (40)] are just a linear combination of the $l=0,1$ spherical harmonics. The decomposition of Eq. (59) is into the four components of invariant subspace of the $w=1$ representation plus the remaining infinite number of components. In this representation the infinite number of components from the $l\geqslant 2$ terms map down to the $l=0,1$ terms yielding Eq. (61). (Note that from the invariant subspace property of the first four components, they do not map ''up'' to the higher $l$ components, which transform among themselves.)

Equation (61) can now be thought of as the generalization of the Lorentz transformation (or the generalization of Killing symmetries) to approximate (or pseudo Killing) symmetries for the spacetimes. A more detailed paper is being prepared on this issue.

The Bondi mass aspect [10] (an asymptotic component of the Weyl tensor whose $l=0,1$ harmonics have been interpreted as the Bondi mass and momentum) is given as a function on $\mathcal{I}^+$ of the form $\Psi_2^0(u,\zeta,\overline{\zeta})$. It is known to transform under the (asymptotic) Lorentz group as a $w=-3$ quantity. If we now restrict the value of $\Psi_2^0(u,\zeta,\overline{\zeta})$ to a light cone cut by substituting $u=Z(x^a,\zeta,\overline{\zeta})$ we obtain a $w=-3$ function of the form $\Psi_2^0(x^a,\zeta,\overline{\zeta})$. By applying the $w=-3$ intertwining operator (56), it is mapped into a $w'=1$ finite dimensional vector that has the form

$$p=\sum_{l=0}^{1}\sum_{m=-l}^{l} p^{lm}Y_{lm}(\zeta,\overline{\zeta}), \quad p^{lm}\Leftrightarrow p^I(x^a), \tag{62}$$

i.e., a Lorentz four-vector field on the spacetime. To obtain a *spacetime* covector field one needs a soldering form, i.e., an object of the form $e^a_I$, so that

$$p^a(x^b)=p^I(x^b)e^a_I. \tag{63}$$

This would then yield a spacetime energy-momentum vector field that presumably could be interpreted as the total energy-momentum passing through the future null cone of the point $x^a$. One would have, via the integral curves of this vector field, preferred curves through spacetime. Unfortunately, at the moment we do not yet have a good means of obtaining the soldering form, though there are several suggestions that must be explored.

A fourth place that the asymptotic Lorentz symmetries enter into the interior local geometry is when we consider a curve in the spacetime, $x^a=x^a(\tau)$, with $v^a=dx^a/d\tau$, and define $dZ/d\tau=V(x^a,v^a,\zeta,\overline{\zeta})=Z_{,a}v^a$. From the properties of $Z$, $V$ is a $w=1$ function. From this it follows that $V^{-2}$ is a $w=-2$ function. Via the intertwining operator for $w=-2$ functions (56), $V^{-2}$ maps into the finite dimensional $w=0$ representation. This yields a scalar function $\Phi(x^a,v^a)$ on the tangent bundle that is homogeneous of degree $-2$ in the $v^a$. In the case where the cut function $Z$ is that of either Minkowski space or of a self-dual space, there is the result [17,13] that $\Phi^{-1}=g_{ab}v^a v^b$; i.e., it is the spacetime norm of $v^a$. In the general case, we do not yet know the meaning of $\Phi(x,v)$ but it is difficult to believe that it is not significant. For example, it has the structure of a scalar function on the tangent bundle and hence could be used as a Lagrangian, with

$$p_a=\frac{\partial\Phi(x^a,v^a)}{\partial v^a}. \tag{64}$$

If $\Phi(x^a,v^a)$ were used as a Lagrangian for the curves $x^a=x^a(\tau)$, we can only guess, at the present moment, of its meaning; in flat spaces and self-dual spaces, it yields geodesic motion—perhaps in the general case, since $\Phi(x,v)$ depends only on the conformal structure, it yields the equations of the conformal geodesics. In any case we feel it is very worthwhile investigating the possible meanings of $\Phi$.

In this discussion we have tried to point out that there is a very rich Lorentzian structure in the interior of any (sufficiently weak) asymptotically flat spacetime, that is inherited from the asymptotic symmetries and propagated rigidly throughout the spacetime, via the light cone structure of the spacetime itself. To our knowledge this structure has not been previously observed; what significance it may have or what use it can be put to are both, at the moment, open questions.

### C. Quantum comments

As was mentioned in earlier sections, for a asymptotically flat spacetimes our variable $Z$ represents the past null cones from points at $\mathcal{I}^+$ or alternatively the intersection of the future directed null cones from interior spacetime points. From the first point of view they are the null surfaces that most resemble the null planes of Minkowski spacetime and, in fact, are the null planes in the flat space case. From the second point of view, in the flat case, they are strictly the spheres on $\mathcal{I}^+$ representing the intersection of the flat space light cones with $\mathcal{I}^+$; the position, on $\mathcal{I}^+$, being determined by the coefficients of the four $l=0,1$ harmonics—namely the Minkowski flat coordinates. In linear theory or in full theory, we keep the same type of coordinates (our pseudo Minkowski coordinates)—the first four harmonic coefficients—but now the cones or planes become deformed. In our basic equation, the LCC equation, it is the $\sigma_B$ that plays the role of a source term driving the equation and causing the deformations of the surfaces.

We discuss now a few (rather unexpected) results that arise when we treat the LCC equation as an operator equation, representing an attempt at developing a quantized version of our NSF of GR.

Since (complex) $\sigma_B$ represents the free data for the gravitational radiation field, it and its complex conjugate can be considered as the basic (canonical) fields of the classical simplectic manifold satisfying Poisson bracket relations among themselves. The idea is to promote $\sigma_B$ to a quantum operator $\hat{\sigma}_B$ obeying commutation relations (obtained via the Poisson brackets) on $\mathcal{I}^+$. This procedure constitutes the Ashtekar asymptotic quantization program [18]. We then insert this operator $\hat{\sigma}_B$ into our LCC equation, thus also promoting $Z$ to $\hat{Z}$.

We remark on two aspects of this procedure.

The quantization of $\sigma_B$ by no means implies we are considering linear gravity. It represents an attempt to extend the asymptotic quantization procedures of Ashtekar [18] to the interior of the spacetime. It is greatly aided by the fact that we have restricted our diffeomorphism freedom to just the Lorentz group by our canonical choice of coordinates. In principle we should be able to extend the operator solution of the LCC equation into the interior. At present we do not know how to handle the nonlinear terms in the field equations for the quantum $\hat{Z}$. We have thus far [19,20] only analyzed the linear coupling between $\hat{Z}$ and $\hat{\sigma}_B$ given by Eq. (15). It is certainly possible that the nonlinear terms will lead to the same complications that are produced in other field theories when quadratic and higher order products of fields arise in the field equations. As the context here is different from other field theories—we are dealing here with *spacetime surfaces* that are to be made into operators rather than with conventional fields—the final status is not clear.

Nevertheless, even at a linearized level the field equations for $\hat{Z}$ produce a rather surprising result, namely, the spacetime points themselves become quantum operators with nonvanishing commutation relations. The classical spacetime manifold disappears and looses its status as a background stage, becoming a quantum object. We emphasize that our NSF formalism is quite unconventional and we do not have (at least from the fundamental starting point) a metric field on a manifold, but instead our basic starting variables are families of surfaces; it is thus not surprising that the ''quantization'' should lead to a nonconventional result. This idea is being further investigated.

## ACKNOWLEDGMENTS

The authors are thankful for stimulating conversations with many friends and colleagues, in particular Roger Penrose, Abhay Ashtekar, Helmut Friedrich, and Bernd Schmidt. We are grateful to Jaideep Ghosh, Tom Kling, and Gilberto Silva-Ortigoza for a thorough check and proofreading of the equations and to Mirta Iriondo for sharing with us her second-order results. This work was supported by CONICET and by the NSF on Grant No. PHY 92-05109.

## APPENDIX A: DERIVATION OF THE LIGHT CONE CUT EQUATION

Before we proceed with the calculations, it is useful to recall that the following expressions for $T_i^1$ and $\overline{T_i^1}$ are equivalent to Eqs. (8d) and (8e):

$$T_i^1=(\bar{\eth}\Lambda),_i-2\,\delta_i^+ , \tag{A1}$$

$$\overline{T_i^1}=(\eth\overline{\Lambda}),_i-2\,\delta_i^- . \tag{A2}$$

The equivalence can be verified by using the commutators (7) to pull $\bar{\eth}$ through $\partial_i$ in the term $(\bar{\eth}\Lambda),_i$, and using Eqs. (8a)–(8c).

Similarly, Eqs. $(m_{\rm I})$ and $(m_{\rm II})$, respectively, take on the following equivalent expressions:

$$\Lambda,_+=-(\bar{\eth}\Lambda),_1+K \tag{A3}$$

and

$$\Lambda,_-=\frac{1}{3}(\eth\Lambda),_1+L, \tag{A4}$$

with $K$ defined by Eq. (19) and $L$ by Eq. (20).

First, we take $\partial_+$ of Eq. (A4) and $\partial_-$ of Eq. (A3); then we obtain two equations by subtraction and addition:

$$4\Lambda,_{+-}=((\eth\Lambda),_+-(\bar{\eth}\Lambda),_-),_1+3L,_++K,_- \tag{A5}$$

and

$$0=\left(\frac{1}{3}(\eth\Lambda),_++(\bar{\eth}\Lambda),_-\right),_1+L,_+-K,_- . \tag{A6}$$

In Eq. (A6), we commute $\partial_+$ through $\eth$ and $\partial_-$ through $\bar{\eth}$, and subsequently use Eqs. (A4) and (A3) to eliminate $\Lambda,_+$ and $\Lambda,_-$, obtaining

$$\begin{aligned}0=\Bigg(&\frac{4}{3}(\Lambda,_0-2\Lambda,_1)+\frac{1}{3}(\bar{\eth}(\eth\Lambda),_1-\eth(\bar{\eth}\Lambda),_1)+\bar{\eth}L+\frac{1}{3}\,\eth K\\&+\frac{1}{3}(\Lambda,_+^2+\Lambda,_1(\bar{\eth}\Lambda),_+)+\overline{\Lambda},_-\Lambda,_-+\Lambda,_1(\eth\overline{\Lambda}),_-\Bigg),_1\\&+L,_+-K,_- .\end{aligned} \tag{A7}$$

We commute $\partial_1$ out from both terms in $(\bar{\eth}(\eth\Lambda),_1-\eth(\bar{\eth}\Lambda),_1)$ and subsequently use Eq. (A5) to eliminate the combination $(\bar{\eth}\Lambda),_--(\eth\Lambda),_+$ which appears on commuting. In this way, Eq. (A7) becomes

$$2(\Lambda,_{01}-\Lambda,_{11}-\Lambda,_{+-})=-3L,_++K,_--\frac{1}{2}J,_1 \tag{A8}$$

with $J$ given by Eq. (18). Since $\Lambda,_{+-}=-(\bar{\eth}\Lambda),_{1-}+K,_-$, from Eq. (A3), then we can integrate Eq. (A8) up in the variable $R$ to obtain

$$\Lambda,_0-\Lambda,_1+(\bar{\eth}\Lambda),_-=2\dot{\sigma}-\frac{1}{4}J+\frac{3}{2}\int_R^\infty(K,_--L,_+)dR', \tag{A9}$$

where an integration ''constant'' $\dot{\sigma}(u,\zeta,\overline{\zeta})$ has been introduced, which can be seen to represent the $u$-derivative of the Bondi shear by studying the limit of (A9) as $R\to\infty$.

Equation (A9) is our starting point to eventually obtain the LCC equation. Since the procedure is lengthy, we outline it here and proceed with the exact calculations afterwards.

The LCC equation is of the form $\eth^2\bar\eth^2 Z = \eth^2\bar\sigma+\bar\eth^2\sigma+\cdots$ , where $\cdots$ represents nonlinear terms in $Z$ or linear terms in derivatives of $\Omega$. To obtain this equation, we take a number of manipulations on Eq. (A9) and on $\eth^2\bar\Lambda=\bar\eth^2\Lambda$, denoted Eq. ($I$) in the main text. The outline of the calculation is the following. From Eq. (A9), one could obtain $\bar\eth^2\Lambda_{,0}=\eth^2\dot{\bar\sigma}+\bar\eth^2\dot\sigma+\cdots$ by taking $\bar\eth^2$ on both sides, only if one could show that

$$\bar\eth^2(-\Lambda_{,0}-\Lambda_{,1}+(\bar\eth\Lambda)_{,-})=-2\eth^2\dot{\bar\sigma}+\cdots \tag{A10}$$

which relates $\Lambda$ to the complex conjugate $\bar\sigma$. To show that Eq. (A10) holds, we apply $\partial_-$ of Eq. ($I$):

$$(\bar\eth^2\Lambda)_{,-}=(\eth^2\bar\Lambda)_{,-}\,. \tag{A11}$$

We commute $\partial_-$ once through $\bar\eth$ in the left-hand side of Eq. (A11) to obtain $(\bar\eth^2\Lambda)_{,-}=\bar\eth((\bar\eth\Lambda)_{,-}+\Lambda_{,0}-\Lambda_{,1})+\cdots$ and then use Eq. (A9) again to substitute in the $2\dot\sigma$ so that the left-hand side of Eq. (A11) becomes

$$(\bar\eth^2\Lambda)_{,-}=2\bar\eth\dot\sigma+\cdots\,. \tag{A12}$$

We also commute $\partial_-$ through $\eth^2$ in the right-hand side of Eq. (A11), so that Eq. (A11) becomes

$$2\bar\eth\dot\sigma=\eth^2\bar\Lambda_{,-}+\cdots\,. \tag{A13}$$

The complex conjugate of Eq. (A13) is

$$2\eth\dot{\bar\sigma}=\bar\eth^2\Lambda_{,+}+\cdots\,. \tag{A14}$$

The final step is to take $\eth$ of both sides in Eq. (A14) and show that the right-hand side becomes

$$\eth\bar\eth^2\Lambda_{,+}\cdots=\bar\eth^2(\Lambda_{,0}-\Lambda_{,1}-(\bar\eth\Lambda)_{,-})+\cdots\,. \tag{A15}$$

This last step is quite involved. Once Eqs. (A14) and (A15) are shown to hold, then Eq. (A10) holds as well and one obtains the LCC equation simply by substitution in $\bar\eth^2$ of Eq. (A9) and subsequent integration in $u$.

In the remainder of this appendix we perform this procedure explicitly, namely, we show that the explicit forms of Eqs. (A12), (A14), (A15), and (A10) hold and use them to derive the LCC equation.

*Algebraic derivation 1. Equation (A12) holds by virtue of the commutation relations, the metricity condition* ($m_I$) *and Eq. (A9).*

The derivation begins with commuting $\partial_-$ through $\bar\eth$ in $(\bar\eth^2\Lambda)_{,-}$ to obtain

$$\begin{aligned}(\bar\eth^2\Lambda)_{,-}=&\,\bar\eth((\bar\eth\Lambda)_{,-}+\Lambda_{,0}-\Lambda_{,1})+\Lambda_{,-}(\bar\Lambda_{,0}-\bar\Lambda_{,1})\\&+\Lambda_{,1}((\eth\bar\Lambda)_{,0}-(\eth\bar\Lambda)_{,1})-K+\bar\Lambda_{,-}(\bar\eth\Lambda)_{,-}\\&+(\eth\bar\Lambda)_{,-}(\bar\eth\Lambda)_{,1},\end{aligned} \tag{A16}$$

where a further commutation of $\partial_0$ through $\bar\eth$ was made and the following version of the commutator $[\partial_{,1},\bar\eth]$ was used:

$$(\bar\eth\Lambda)_{,1}=\frac12\,\bar\eth\Lambda_{,1}+\frac12(\Lambda_{,-}\bar\Lambda_{,1}+\Lambda_{,1}(\eth\bar\Lambda)_{,1}+K). \tag{A17}$$

This version of the commutator is obtained by use of Eq. (A3) to eliminate $\Lambda_{,+}$ from the original version of the commutator. Substituting $(\bar\eth\Lambda)_{,-}+\Lambda_{,0}-\Lambda_{,1}$ from Eq. (A9) into Eq. (A16) we obtain

$$\begin{aligned}2\bar\eth\dot\sigma=&\,(\bar\eth^2\Lambda)_{,-}-\bar\eth\left(\frac32\int_\infty^R(K_{,-}-L_{,+})dR'-\frac14 J\right)\\&-\Lambda_{,-}(\bar\Lambda_{,0}-\bar\Lambda_{,1})-\Lambda_{,1}((\eth\bar\Lambda)_{,0}-(\eth\bar\Lambda)_{,1})+K\\&-\bar\Lambda_{,-}(\bar\eth\Lambda)_{,-}-(\eth\bar\Lambda)_{,-}(\bar\eth\Lambda)_{,1}\end{aligned}$$

which is the explicit form of Eq. (A12). □

*Algebraic derivation 2. Equation (A14) holds by virtue of the commutation relations and Eqs. (I) and (A12).*

The derivation begins with commuting $\partial_-$ through $\eth^2$ in $(\eth^2\bar\Lambda)_{,-}$ to obtain

$$\begin{aligned}(\eth^2\bar\Lambda)_{,-}=&\,\eth^2\bar\Lambda_{,-}+\eth((\bar\eth\Lambda)_{,-}\bar\Lambda_{,1}+\Lambda_{,-}\bar\Lambda_{,+})+\Lambda_{,-}(\eth\bar\Lambda)_{,+}\\&+(\bar\eth\Lambda)_{,-}(\eth\bar\Lambda)_{,1}.\end{aligned} \tag{A18}$$

On the other hand, by using Eq. ($I$) to change the first term on the right-hand side of Eq. (A12) into a term in $\bar\Lambda$, Eq. (A12) gives

$$\begin{aligned}2\bar\eth\dot\sigma=&\,(\eth^2\bar\Lambda)_{,-}-\bar\eth\left(\frac32\int_\infty^R(K_{,-}-L_{,+})dR'-\frac14 J\right)\\&-\Lambda_{,-}(\bar\Lambda_{,0}-\bar\Lambda_{,1})-\Lambda_{,1}((\eth\bar\Lambda)_{,0}-(\eth\bar\Lambda)_{,1})+K\\&-\bar\Lambda_{,-}(\bar\eth\Lambda)_{,-}-(\eth\bar\Lambda)_{,-}(\bar\eth\Lambda)_{,1}.\end{aligned} \tag{A19}$$

Using Eq. (A18) to eliminate $(\eth^2\bar\Lambda)_{,-}$ from the right-hand side of Eq. (A19) and taking the complex conjugate the following equation is obtained:

$$\begin{aligned}2\eth\dot{\bar\sigma}=&\,\bar\eth^2\Lambda_{,+}+\bar\eth(\Lambda_{,1}(\eth\bar\Lambda)_{,+}+\Lambda_{,-}\bar\Lambda_{,+})-\bar\Lambda_{,+}(\Lambda_{,0}-\Lambda_{,1}-(\bar\eth\Lambda)_{,-})+(\eth\bar\Lambda)_{,+}((\bar\eth\Lambda)_{,1}-\Lambda_{,+})\\&-\eth\left(\frac32\int_\infty^R(\overline{K_{,+}}-\overline{L_{,-}})dR'-\frac14\bar J\right)-\bar\Lambda_{,1}((\bar\eth\Lambda)_{,0}-(\bar\eth\Lambda)_{,1})+\bar K-(\bar\eth\Lambda)_{,+}(\eth\bar\Lambda)_{,1},\end{aligned}$$

which is the explicit form of Eq. (A14). □

*Algebraic derivation 3. Equation (A15) holds by virtue of Eqs. (A4) and (A3), and the commutation relations.*

The derivation requires the use of two auxiliary results. First, there is a relationship between $L$ and $K$. Using Eq. (A17) one can express $L$ as

$$L=-\frac{1}{2}\Lambda,_1(\Lambda,_+ +(\bar{\eth}\Lambda),_1-\Lambda,_-\overline{\Lambda},_1-\Lambda,_1(\eth\overline{\Lambda}),_1+2\Lambda,_1\bar{\eth}\ln\Omega), \tag{A20}$$

which becomes

$$L=-\frac{1}{2}\Lambda,_1(K-\Lambda,_-\overline{\Lambda},_1-\Lambda,_1(\eth\overline{\Lambda}),_1+2\Lambda,_1\bar{\eth}\ln\Omega), \tag{A21}$$

if Eq. (A3) is used to eliminate $\Lambda,_+$.

The second auxiliary relation is the following version of the commutator $[\partial_1,\eth]$ on $\Lambda$, which is obtained by use of Eqs. (A4), (A3), and (A21) into the original commutator:

$$\frac{1}{3}(\bar{\eth}\Lambda),_1=\frac{1}{2}\,\eth\Lambda,_1+\frac{1}{4}\Lambda,_1(K+\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1-2\Lambda,_1\bar{\eth}\ln\Omega). \tag{A22}$$

To derive Eq. (A15) we begin by noting that, after commuting $\partial_-\bar{\eth}$ in $(\bar{\eth}\Lambda),_-$, the combination $\Lambda,_0+\Lambda,_1-(\bar{\eth}\Lambda),_-$ can be rewritten as

$$\Lambda,_0+\Lambda,_1-(\bar{\eth}\Lambda),_-=3\Lambda,_1-\bar{\eth}\left(\frac{1}{2}\,\eth\Lambda,_1+\frac{1}{4}\Lambda,_1(-K+3(\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1-2\Lambda,_1\bar{\eth}\ln\Omega))\right)-\Lambda,_-\overline{\Lambda},_--\Lambda,_1(\eth\overline{\Lambda}),_- \tag{A23}$$

simply by using Eqs. (A4) and (A22). The first two terms on the right-hand side of Eq. (A23) can be changed into $\Lambda,_1-\frac{1}{2}\eth\bar{\eth}\Lambda,_1$ by commuting $\bar{\eth}\eth$. Applying $\bar{\eth}^2$ to the resulting expression and commuting $\bar{\eth}^2$ through $\eth\bar{\eth}\Lambda,_1$ we obtain

$$\begin{aligned}&\bar{\eth}^2(\Lambda,_0+\Lambda,_1-(\bar{\eth}\Lambda),_-)\\&\quad=-\frac{1}{2}\,\eth\bar{\eth}^3\Lambda,_1-\bar{\eth}^3\left(\frac{1}{4}\Lambda,_1(-K+3(\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1-2\Lambda,_1\bar{\eth}\ln\Omega))\right)-\bar{\eth}^2(\Lambda,_-\overline{\Lambda},_-+\Lambda,_1(\eth\overline{\Lambda}),_-).\end{aligned} \tag{A24}$$

We use Eq. (A17) to rewrite $-\frac{1}{2}\eth\bar{\eth}^3\Lambda,_1=\eth\bar{\eth}^2(-\frac{1}{2}\bar{\eth}\Lambda,_1)$, which is the first term on the right-hand side of Eq. (A24), in terms of $(\bar{\eth}\Lambda),_1$, and consequently eliminate $(\bar{\eth}\Lambda),_1$ in terms of $\Lambda,_+$ by means of Eq. (A3): namely,

$$-\frac{1}{2}\,\eth\bar{\eth}^3\Lambda,_1=\eth\bar{\eth}^2\left(\Lambda,_++\frac{1}{2}\Lambda,_-\overline{\Lambda},_1+\frac{1}{2}\Lambda,_1(\eth\overline{\Lambda}),_1-\frac{1}{2}K\right). \tag{A25}$$

Inserting Eq. (A25) into (A24) and rearranging terms we obtain

$$\begin{aligned}\eth\bar{\eth}^2\Lambda,_+=&\,\bar{\eth}^2(\Lambda,_0+\Lambda,_1-(\bar{\eth}\Lambda),_-)-\frac{1}{2}\,\eth\bar{\eth}^2(\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1)+\frac{1}{2}\,\eth\bar{\eth}^2K+\bar{\eth}^2(\Lambda,_-\overline{\Lambda},_-+\Lambda,_1(\eth\overline{\Lambda}),_-)\\&+\bar{\eth}^3\left(\frac{1}{4}\Lambda,_1(-K+3(\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1-2\Lambda,_1\bar{\eth}\ln\Omega))\right),\end{aligned}$$

which is the explicit form of Eq. (A15).□

*Algebraic derivation 4. Equation (A10) holds by virtue of Eqs. (A14) and (A15).*

The derivation consists of taking an ð to both sides of Eq. (A14) and substituting $\eth\bar{\eth}^2\Lambda,_+$ (which will appear on the right-hand side) by using Eq. (A15). In this way we obtain

$$\begin{aligned}2\eth^2\dot{\bar{\sigma}}=&\,\bar{\eth}^2(\Lambda,_0+\Lambda,_1-(\eth\Lambda),_-)-\frac{1}{2}\,\eth\bar{\eth}^2(\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1)+\frac{1}{2}\,\eth\bar{\eth}^2K+\bar{\eth}^2(\Lambda,_-\overline{\Lambda},_-+\Lambda,_1(\eth\overline{\Lambda}),_-)\\&+\bar{\eth}^3\left(\frac{1}{4}\Lambda,_1(-K+3(\Lambda,_-\overline{\Lambda},_1+\Lambda,_1(\eth\overline{\Lambda}),_1-2\Lambda,_1\bar{\eth}\ln\Omega))\right)\\&+\eth[-\overline{\Lambda},_+(\Lambda,_0-\Lambda,_1-(\bar{\eth}\Lambda),_-)+(\eth\overline{\Lambda}),_+((\bar{\eth}\Lambda),_1-\Lambda,_+)]+\eth[-\overline{\Lambda},_1((\bar{\eth}\Lambda),_0-(\eth\Lambda),_1)+\overline{K}-(\eth\Lambda),_+(\eth\overline{\Lambda}),_1]\\&-\eth^2\left(\frac{3}{2}\int_\infty^R(\overline{K},_+-\overline{L},_-)dR'-\frac{1}{4}\overline{J}\right)+\eth\bar{\eth}(\Lambda,_1(\eth\overline{\Lambda}),_++\Lambda,_-\overline{\Lambda},_+),\end{aligned}$$

which is the explicit form of Eq. (A10). □

*Algebraic derivation 5. The LCC equation follows from Eqs. (A9) and (A10).*

To derive the LCC equation, we first add and subtract $\Lambda,_0$ in the left-hand side of Eq. (A9), obtaining

$$2\Lambda,_0-(\Lambda,_0+\Lambda,_1-(\bar{\eth}\Lambda),_-)=2\dot{\sigma}+\frac{3}{2}\int_\infty^R(K,_- - L,_+)dR'-\frac{1}{4}J. \tag{A26}$$

We can now take $\bar{\eth}^2$ to both sides of Eq. (A26) and consequently use Eq. (A10) to eliminate $\bar{\eth}^2(\Lambda,_0+\Lambda,_1-(\bar{\eth}\Lambda),_-)$ in favor of $\eth^2\dot{\bar{\sigma}}$. Thus we obtain a $u$ derivative of the LCC equation:

$$\begin{aligned}2\bar{\eth}^2\Lambda,_0={}&2(\bar{\eth}^2\dot{\sigma}+\eth^2\dot{\bar{\sigma}})-\frac{1}{4}\bar{\eth}^2J+\frac{3}{2}\bar{\eth}^2\int_\infty^R(K,_- - L,_+)dR'-\frac{1}{4}\eth^2\overline{J}+\frac{3}{2}\eth^2\int_\infty^R(\overline{K,_+}-\overline{L,_-})dR'\\&+\frac{1}{2}\eth\bar{\eth}^2(\overline{\Lambda,_1}\Lambda,_- + \Lambda,_1(\overline{\eth\Lambda}),_1)-\bar{\eth}^3\left(\frac{\Lambda,_1}{4}[3(\overline{\Lambda,_1}\Lambda,_- + \Lambda,_1(\overline{\eth\Lambda}),_1-2\Lambda,_1\bar{\eth}\ln\Omega)-K]\right)\\&-\bar{\eth}^2(\Lambda,_-\overline{\Lambda,_-}+\Lambda,_1(\overline{\eth\Lambda}),_-)-\frac{1}{2}\eth\bar{\eth}^2K-\eth\bar{\eth}(\Lambda,_-\overline{\Lambda,_+}+\Lambda,_1(\overline{\eth\Lambda}),_+)-\eth[(\overline{\eth\Lambda}),_+((\bar{\eth}\Lambda),_1-\Lambda,_+)\\&-\overline{\Lambda,_+}(\Lambda,_0-\Lambda,_1-(\bar{\eth}\Lambda),_-)-\overline{\Lambda,_1}((\bar{\eth}\Lambda),_0-(\bar{\eth}\Lambda),_1)+\overline{K}]+\eth((\bar{\eth}\Lambda),_+(\overline{\eth\Lambda}),_1).\end{aligned} \tag{A27}$$

This equation can be integrated in $u$ (after commuting $\partial_0$ through $\bar{\eth}^2$ in $\bar{\eth}^2\Lambda,_0$ and $\bar{\eth}^2\dot{\sigma}$, and through $\eth^2$ in $\eth^2\dot{\bar{\sigma}}$), giving the LCC equation, labeled Eq. (15) in the main text. □

## APPENDIX B: SELF- AND ANTI-SELF-DUAL RELATIONS

The light cone cuts of self-dual spacetimes satisfy

$$\Lambda=\sigma \quad \text{and} \quad \Omega=1. \tag{B1}$$

The following is a list of relations obtained by using this information to evaluate equations derived in the previous appendix, the definitions of $K$, $L$, and $J$ given in the main text and the commutators acting on $\Lambda$ and $\overline{\Lambda}$.

From Eq. (B1) we immediately obtain

$$\Lambda,_0=\dot{\sigma}, \tag{B2a}$$

$$\Lambda,_1=0, \tag{B2b}$$

$$\Lambda,_+=0, \tag{B2c}$$

$$\Lambda,_-=0, \tag{B2d}$$

and

$$\eth\ \ln\Omega=0, \tag{B3a}$$

$$\bar{\eth}\ \ln\Omega=0, \tag{B3b}$$

which immediately lead to

$$L=0, \tag{B4}$$

$$K=0. \tag{B5}$$

From the commutation relations, (B1) and (B2), we obtain

$$(\eth\Lambda),_-=0, \tag{B6a}$$

$$(\bar{\eth}\Lambda),_1=0, \tag{B6b}$$

$$(\bar{\eth}\Lambda),_+=0, \tag{B6c}$$

$$(\bar{\eth}\Lambda),_-=\dot{\sigma}, \tag{B6d}$$

$$(\bar{\eth}\Lambda),_0=\bar{\eth}\dot{\sigma}. \tag{B6e}$$

From Eqs. (B6), (B2), (B5), and (B5) we obtain

$$J=0. \tag{B7}$$

Now we turn to the ‘‘complex conjugates’’ of the quantities we have evaluated so far. From Eqs. (B1), (B2), (B3), and (B6) we obtain

$$\overline{K}=0, \tag{B8}$$

which, if inserted into the complex conjugate of Eq. (A21), leads to

$$\overline{L}=0. \tag{B9}$$

From the definition of $\overline{L}$, namely the complex conjugate of Eq. (20), we obtain

$$2\overline{\Lambda,_-}+\eth\overline{\Lambda,_1}=0. \tag{B10}$$

From the commutation relations we see that

$$(\eth\overline{\Lambda}),_-=\eth\overline{\Lambda,_-}+\dot{\sigma}\overline{\Lambda,_1}, \tag{B11a}$$

$$(\eth\overline{\Lambda}),_1=\eth\overline{\Lambda,_1}+\overline{\Lambda,_-}, \tag{B11b}$$

which, if inserted into the complex conjugate of Eq. (A4) and using Eq. (B9) yields

$$2\overline{\Lambda,_+}-\bar{\eth}\overline{\Lambda,_1}=0. \tag{B12}$$

Using Eq. (B12), as well as all the relevant expressions obtained so far, into the complex conjugate of Eq. (18), we see that

$$\overline{J}=\frac{1}{2}(\eth\overline{\Lambda,_1})^2-\overline{\Lambda,_1}\eth^2\overline{\Lambda,_1}+2\overline{\Lambda,}_1^2\dot{\sigma}. \tag{B13}$$

Finally, with all the expressions that have been obtained so far, the complex conjugate of Eq. (A14) (shown explicitly in the algebraic derivation A) reduces to

$$2\bar{\eth}\dot{\sigma}+\frac{1}{2}\eth^3\overline{\Lambda,_1}-2\dot{\sigma}\eth\overline{\Lambda,_1}-\eth\dot{\sigma}\overline{\Lambda,_1}=0, \tag{B14}$$

which is the well-known equation for self-dual spacetimes displayed in [13] [Eqs. (2.28)].

The anti-self-dual results are the ‘‘complex conjugates’’ of these.

## APPENDIX C: THE GREEN's FUNCTION

Here we display the Green's function $G(\zeta,\eta)$ for the equation

$$\eth^2\bar{\eth}^2F=A \tag{C1}$$

for the spin-weight-zero function $F$ on the sphere, where $A$ is a regular spin-weight-zero source with only $l\geqslant 2$ spherical harmonics:

$$G(\zeta,\eta)=\frac{1}{4\pi}\ell(\zeta)\ell(\eta)\ln(\ell(\zeta)\cdot\ell(\eta)). \tag{C2}$$

Green's functions for higher order $\eth$ have been obtained in [21].


[1] S. Frittelli, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **36**, 4975 (1995).
[2] S. Frittelli, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **36**, 4984 (1995).
[3] S. Frittelli, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **36**, 5005 (1995).
[4] S. Frittelli, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **36**, 6397 (1995).
[5] E. T. Newman and R. Penrose, J. Math. Phys. **5**, 863 (1966).
[6] C. N. Kozameh, E. T. Newman, and K. P. Tod, Gen. Relativ. Gravit. **17**, 343 (1985).
[7] H. Friedrich, Commun. Math. Phys. **91**, 445 (1983).
[8] M. S. Iriondo, C. N. Kozameh, and A. T. Rojas, J. Math. Phys. **38**, 4714 (1997).
[9] L. J. Mason, J. Math. Phys. **36**, 3704 (1995).
[10] E. T. Newman and T. W. J. Unti, J. Math. Phys. **3**, 891 (1962).
[11] Mirta S. Iriondo (private communication).
[12] S. Frittelli and E. T. Newman, Phys. Rev. D **55**, 1971 (1997).
[13] M. Ko, M. Ludvigsen, E. T. Newman, and K. P. Tod, Phys. Rep. **71**, 53 (1981).
[14] I. M. Gelfand, M. I. Graev, and N. Velenkin, *Generalized Functions* (Academic Press, New York, 1966), Vol. V.
[15] E. T. Newman and R. Posadas, J. Math. Phys. **11**, 3145 (1970).
[16] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London **A269**, 21 (1962).
[17] R. O. Hansen, E. T. Newman, R. Penrose, and K. P. Tod, Proc. R. Soc. London **A363**, 445 (1978).
[18] A. Ashtekar, *Asymptotic Quantization* (Bibliopolis, Naples, 1987).
[19] S. Frittelli *et al.*, Class. Quantum Grav. **14**, A143 (1997).
[20] S. Frittelli *et al.*, Phys. Rev. D **56**, 889 (1997).
[21] J. Ivancovich, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **30**, 45 (1989).