Note on the propagation of the constraints in standard 3+1 general relativity

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The well posedness of the evolution of the constraints in *standard* 3+1 general relativity is established by means of the Bianchi identities. Other related nonstandard 3+1 cases which can be analyzed along similar lines are considered as well. The well posedness of the propagation of the constraints is relevant to the problem of unconstrained numerical evolution. [S0556-2821(97)01910-3]

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I. INTRODUCTION

One strength of the 3+1 splitting of the Einstein equations is that, by virtue of the Bianchi identities, four out of the ten Einstein equations can be seen to be satisfied if (1) they are satisfied on a single slice and (2) the remaining six are imposed. This statement (in essence, if not literally) can be found in textbooks [1] and in the papers that have become cornerstones of the 3+1 formulation [2,3], as well as in other works [4,5], yet its full extent is very much defined by the context, in a sense that is exploited here. This property of the 3+1 formulation has been invoked in the literature to justify constructions that might conceivably extend beyond its range of validity, oftentimes without proper verification. Such appears to be, quite generally, the case of the numerical evolution of constrained initial data into metrics that satisfy the constraints at subsequent times. In principle, there are two alternatives. One alternative consists in coding all ten of the Einstein equations (*constrained* evolution). This carries the numerical burden of solving elliptic equations at every time step. The other alternative is to code only the evolution equations and impose the elliptic constraints only on the initial data, relying on the argument for the conservation of the constraints (*unconstrained* or *free* evolution). Unconstrained evolution is more appealing from the numerical point of view; yet, although there exists a considerable amount of work developing unconstrained evolution [6], a large part of the numerical evolution schemes that can be found in the literature are constrained. Some insights into the nature of the difficulty of implementing unconstrained codes have been given by Choptuik [7].

Specifically, there is a distinction between the concepts of conservation and stable propagation of the constraints. The constraints are known to be conserved in the sense that if they are chosen to vanish exactly on the initial slice, then they are exactly vanishing at subsequent times. On the other hand, a quantity propagates in a stable manner if it depends continuously on its initial values, so that small variations of initial data do not give rise to significantly different values at subsequent times [8]. If the constraints are to be solved numerically and only on the initial slice, then their propagation must be guaranteed not to be unstable, because all numerical initial data are subject to error, albeit small. It is essential that the small (even round-off) initial error in the vanishing of the constraints is prevented from an incontrollable growth. However, there is no mention of stable propagation of the constraints in any of the references that I have cited so far, a fact that seems to have been overlooked.

Here the propagation of the constraints is studied in a class of cases of interest, and it is found that the propagation is stable in some cases and unstable in other cases. These observations are based on the concept of well posedness, which is equivalent to existence, uniqueness, and stability simultaneously [8]. For first-order systems of partial differential equations (PDE's), the well posedness can be established by means of three algebraic criteria, namely symmetric, strict or strong hyperbolicity [9], although, if the system has nonconstant coefficients, general existence theorems are known only for the symmetric case. The arguments presented here might have been known to individuals, yet they do not seem to have appeared in the literature.

Mainly to fix the context and notation, in Sec. II, I briefly introduce the procedure for the 3+1 splittings of the Einstein equations. From the point of view of the theory of PDE's, there is no unique 3+1 formulation; the 3+1 versions of general relativity in [2,3,10–12] differ in that the six equations that are chosen for evolution include combinations of the constraints themselves in different nonequivalent ways. In the spirit of these works, I do allow for combinations of the six evolution equations with the constraints (though not in full generality) since this is an essential part of the problem of defining the context for the constraint propagation. I emphasize the choice made by York in [3], which has become of wide use in numerical relativity (see, for instance, [7,13,14]) and canonical gravity [4], and which I refer to as ''standard.'' The stability of the propagation of the constraints is analyzed in Sec. III. In the last section, I summarize the main results and briefly elaborate on their significance with respect to the problem of unconstrained numerical evolution.

Throughout this work, the word stability is used in the analytical context, as opposed to numerical.

II. 3+1 SPLITTING

On the manifold M with generic coordinates x^a, a foliation by spacelike level surfaces of a function $t(x^a)$ is assumed. The unit normal $n^a \equiv -g^{ab}N\nabla_a t=(t^a-N^a)/N$ (where N and N^a are the lapse function and shift vector,

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respectively) defines a projector on the spatial surfaces $h^a{}_b$, such that $h^{ab}\equiv g^{ab}+n^a n^b$ is the induced metric on the surfaces. The notation and conventions throughout this work belong to Chap. 10 and Appendix E of [1].

The Einstein equations $G_{ab}-T_{ab}=0$ can be projected by means of the normal n^a and the projector $h^a{}_b$, splitting into three distinct sets of equations: the double normal projection $(G_{cd}-T_{cd})n^c n^d=0$, the mixed spatial-normal projection $(G_{cd}-T_{cd})n^c h^d{}_a=0$, the double spatial projection $(G_{cd}-T_{cd})h^c{}_a h^d{}_b=0$. The projections of the stress-energy tensor T_{ab} are denoted $\rho\equiv T_{cd}n^c n^d$, $J_a\equiv -T_{cd}n^c h^d{}_a$, and $S_{ab}\equiv T_{cd}h^c{}_a h^d{}_b$. The left-hand sides occurring in the preceding equations are hereby denoted

$$\mathcal{C}\equiv G_{cd}n^c n^d-\rho, \tag{1}$$

$$\mathcal{C}_a\equiv -G_{cd}n^c h^d{}_a-J_a, \tag{2}$$

$$\mathcal{E}_{ab}\equiv G_{cd}h^c{}_a h^d{}_b-S_{ab}. \tag{3}$$

Equations (1) and (2) define the scalar and vector constraints (also referred to as Hamiltonian and momentum constraints) respectively, whereas Eq. (3) defines the spatial Einstein equation. In this language, general relativity is obtained by setting $\mathcal{C}=0$, $\mathcal{C}_a=0$, and $\mathcal{E}_{ab}=0$ or by imposing $\mathcal{C}=0$, $\mathcal{C}_a=0$, and $\mathcal{E}_{ab}+m_{ab}\mathcal{C}+l^c_{ab}\mathcal{C}_c=0$, for arbitrary m_{ab} and l^c_{ab} symmetric in a,b. Clearly, as long as the constraints are vanishing, the evolution admits many alternative expressions. I restrict to the special case $l^c_{ab}=0$ and $m_{ab}=-\mu h_{ab}$ for some constant μ. This defines a one-parameter freedom of choice $\mathcal{F}_{ab}(\mu)$ for the evolution equation

$$\mathcal{F}_{ab}(\mu)\equiv\mathcal{E}_{ab}-\mu h_{ab}\mathcal{C}=0. \tag{4}$$

The "standard" and most widely adopted choice of the parameter μ over many years has been provided by York in [3]. Essentially, the Einstein equations $G_{ab}-T_{ab}=0$ are rewritten in the form $R_{ab}=T_{ab}-1/2g_{ab}T$ and then projected onto the spatial slice. This procedure corresponds to the choice $\mu=1$. To show this, we first define

$$\begin{aligned}\mathcal{Y}_{ab}&\equiv h^c{}_a h^d{}_b\left(R_{cd}-T_{cd}+\frac{1}{2}g_{cd}T\right)\\&=h^c_a h^d{}_b\left(R_{cd}-T_{cd}-\frac{1}{2}g_{cd}R+\frac{1}{2}g_{cd}(T+R)\right),\end{aligned} \tag{5}$$

from which it is clear that

$$\mathcal{Y}_{ab}=\mathcal{E}_{ab}+\frac{1}{2}h_{ab}(R+T). \tag{6}$$

Since $-(R+T)=g^{ab}(G_{ab}-T_{ab})=(h^{ab}-n^a n^b)(G_{ab}-T_{ab})=h^{ab}\mathcal{E}_{ab}-\mathcal{C}$ then

$$\mathcal{Y}_{ab}=\mathcal{E}_{ab}-\frac{1}{2}h_{ab}h^{cd}\mathcal{E}_{cd}+\frac{1}{2}h_{ab}\mathcal{C}. \tag{7}$$

The standard formulation consists then in setting $\mathcal{Y}_{ab}=0$ as well as the constraints. By taking the trace on Eq. (7) and setting it equal to zero we obtain $h^{cd}\mathcal{E}_{cd}=3\,\mathcal{C}$ and finally

$$\mathcal{Y}_{ab}=\mathcal{E}_{ab}-h_{ab}\mathcal{C}=0. \tag{8}$$

This shows that $\mathcal{Y}_{ab}=\mathcal{F}_{ab}(\mu)$ for $\mu=1$. In terms of the intrinsic metric h_{ab}, the explicit form of Eq. (8) and of the constraints $\mathcal{C}$ and $\mathcal{C}_a$ is that of Eqs. (35), (39), (23), and (24) of [3].

III. THE PROPAGATION OF THE CONSTRAINTS

The issue of the propagation of the constraints is stated here as follows. The constraints are so called because they do not involve second-order time derivatives of the metric. It is desirable, however, to achieve a formulation of the evolution of the induced metric by means of propagation equations, with no constraints. This could be done if the constraints were propagated by virtue of the evolution of the metric, because, in such an instance, the constraints would only affect the choice of the data on the initial surface, being otherwise irrelevant to the evolution. The essential issue is to show that $\mathcal{C}$ and $\mathcal{C}_a$ are propagated if the metric satisfies proper evolution equations $\mathcal{F}_{ab}(\mu)=0$. While it holds true that vanishing initial constraints evolve into vanishing constraints at all times, the extended assumption that the evolved constraints depend continuously on their initial values has less often been studied in the literature. If this were actually the case, then initial constraints that are almost vanishing (but for a small error) should keep almost vanishing at subsequent times, and it should be possible to numerically evolve the Einstein equations without imposing the constraints other than on the choice of initial data, a procedure that is currently known as unconstrained evolution.

In this section, it is shown that, within the class defined in the previous section, the Bianchi equations and the evolution of the metric determine a linear homogeneous evolution system of PDE's for the constraints, and that the *choice* of the proper evolution equations for the metric determines whether the evolution of the constraints is well posed. Clearly, homogeneity and well posedness ensure the vanishing of the constraints at subsequent times in the domain of determinacy [15] of the initial surface. The stability of the propagation of the constraints is obtained as a byproduct, and, though it may not have played a fundamental role in analytic developments, it is essential to the problem of unconstrained evolution, as argued before.

The Bianchi equations, $\nabla^a(G_{ab}-T_{ab})=0$, which in the 3+1 picture take the form $\nabla^a(\mathcal{E}_{ab}+n_a\mathcal{C}_b+n_b\mathcal{C}_a+n_a n_b\mathcal{C})=0$, hold as a consequence of the Bianchi identities $\nabla^a G_{ab}=0$ and the conservation of the stress-energy tensor $\nabla^a T_{ab}=0$. These equations can be projected onto the surface and onto the normal. Consider first the normal projection

$$\begin{aligned}0&=n^c\nabla^d(G_{cd}-T_{cd})=n^c\nabla^d(\mathcal{E}_{cd}+n_c\mathcal{C}_d+n_d\mathcal{C}_c+n_c n_d\mathcal{C})\\&=-\mathcal{E}_{cd}D^c n^d-2\mathcal{C}_d n^c\nabla_c n^d-D^d\mathcal{C}_d-\mathcal{C}D_c n^c-n^d\nabla_d\mathcal{C}.\end{aligned} \tag{9}$$

Here, the derivative operator D_a is the covariant derivative of the spatial metric h_{ab}, given by [1]

$$D_c V^{a_1\cdots a_k}{}_{b_1\cdots b_l}=h^{a_1}{}_{d_1}\cdots h^{e_l}{}_{b_l}h^f{}_c\nabla_f V^{d_1\cdots d_k}{}_{e_1\cdots e_l}. \tag{10}$$

The identities $n^c n_c = -1$, $n^c h^b{}_c = 0$, $n^c \mathcal{E}_{cd} = 0$, $n^d \mathcal{C}_d = 0$, $n^d \nabla_c n_d = 0$, and $\nabla^c n_c = D^c n_c$ have been used to obtain the result (9).

Consider now the spatial projection

$$\begin{aligned} 0 &= h^c{}_a \nabla^d (G_{cd} - T_{cd}) = h^c{}_a \nabla^d (\mathcal{E}_{cd} + n_c \mathcal{C}_d + n_d \mathcal{C}_c + n_c n_d \mathcal{C}) \\ &= D^c \mathcal{E}_{ac} + \mathcal{E}_{ad} n^c \nabla_c n^d + \mathcal{C}_a D_c n^c + \mathcal{C}_d n_a n^c \nabla_c n^d + \mathcal{C}_d D^d n_a \\ &\quad + 2\mathcal{C} n^d \nabla_d n_a + n^d \nabla_d \mathcal{C}_a . \end{aligned} \tag{11}$$

Equations (9) and (11) constitute the evolution of the constraints $\mathcal{C}$ and $\mathcal{C}_a$, since the time derivative of the constraints is contained in the terms $n^d \nabla_d \mathcal{C}$ and $n^d \nabla_d \mathcal{C}_a$. This is emphasized by rearranging terms in the following way

$$n^d \nabla_d \mathcal{C} = -D^d \mathcal{C}_d - \mathcal{E}_{cd} D^c n^d + \mathcal{L}(\mathcal{C}, \mathcal{C}_d), \tag{12}$$

$$n^d \nabla_d \mathcal{C}_a = -D^c \mathcal{E}_{ac} - \mathcal{E}_{ad} n^c \nabla_c n^d + \mathcal{L}_a(\mathcal{C}, \mathcal{C}_d), \tag{13}$$

where $\mathcal{L}$ and $\mathcal{L}_a$ collect the homogeneous terms in Eqs. (9) and (11). (Assuming the evolution equations $\mathcal{E}_{ab} = 0$ are satisfied, since the right-hand side of Eqs. (12) and (13) vanishes identically if $\mathcal{C} = \mathcal{C}_a = 0$, then the derivative of the constraints out of the initial surface vanishes as well. This is the argument for the conservation of the constraints [2,5]. This argument applies just as well to any combination of the spatial equations with the constraints.)

In the following, I raise the index of the vector constraint by $\mathcal{C}^a \equiv h^{ab} \mathcal{C}_b$, which makes the equations more transparent without loss of generality. Consider the case where the evolution equations $\mathcal{F}_{ab}(\mu) = 0$ are satisfied. This is equivalent to $\mathcal{E}_{ab} = \mu h_{ab} \mathcal{C}$. Equations (12) and (13) take the form

$$n^d \nabla_d \mathcal{C} = -D_d \mathcal{C}^d + \mathcal{L}(\mathcal{C}, \mathcal{C}^d), \tag{14}$$

$$h_{ab} n^d \nabla_d \mathcal{C}^b = -\mu D_a \mathcal{C} + \mathcal{L}_a(\mathcal{C}, \mathcal{C}^d). \tag{15}$$

In the particular case $\mu = 1$ the system (14), (15) is manifestly well posed because it is symmetric hyperbolic, and the stable propagation of the constraints holds. This is the case of the standard evolution equations given by York, as shown in the previous section. This is equivalent to the statement that the unconstrained evolution of $\mathcal{Y}_{ab} = 0$ is not hampered from the analytical point of view.

The system is not symmetric for a generic μ, but the well posedness can still be established by means of either strict or strong hyperbolicity. In a coordinate system $x^a = (t, x^i)$ adapted to the foliation, the principal part of the system (14), (15) has the following 3+1 form:

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathcal{C} \\ \mathcal{C}^i \end{pmatrix} = \mathbf{A}^j \frac{\partial}{\partial x^j}\begin{pmatrix} \mathcal{C} \\ \mathcal{C}^i \end{pmatrix}, \tag{16}$$

where the matrices $\mathbf{A}^j$ are readily identified from (14), (15). The principal symbol is defined as the matrix $\mathbf{A}(\xi) \equiv \mathbf{A}^j \xi_j$ (where ξ_i is an arbitrary spatial covector of unit length) and has the form

$$\mathbf{A}(\xi) = \left(\begin{array}{c|ccc} N^i \xi_i & -N\xi_1 & -N\xi_2 & -N\xi_3 \\ \hline -\mu N \xi^1 & N^i \xi_i & 0 & 0 \\ -\mu N \xi^2 & 0 & N^i \xi_i & 0 \\ -\mu N \xi^3 & 0 & 0 & N^i \xi_i \end{array}\right), \tag{17}$$

where $\xi^i \equiv h^{ij} \xi_j$. The characteristic polynomial of (14), (15) is the determinant of $\mathbf{A}(\xi) - v\mathbf{I}$ (where $\mathbf{I}$ is the identity matrix in four dimensions), and its roots v are interpreted as the characteristic speeds [16]. Defining $\xi_d = (v, \xi_1, \xi_2, \xi_3)$, the characteristic polynomial has the form $(n^d \xi_d)^2 [(n^c \xi_c)^2 - \mu]$. Therefore the system is ill posed if $\mu < 0$, since for $\mu < 0$ two roots of the characteristic polynomial are complex $(n^c \xi_c = \pm i \sqrt{|\mu|} \Rightarrow v = N^i \xi_i \pm i \sqrt{|\mu|} N)$. This means that the error in satisfying $\mathcal{C} = \mathcal{C}_a = 0$ grows exponentially. Consequently, unconstrained evolution does not give a solution to the Einstein equations in the case $\mu < 0$.

If $\mu > 0$ the system (14), (15) is not strictly hyperbolic, since there is a double root $(n^d \xi_d = 0)$. The system can still be strongly hyperbolic if it can be established that there is a complete set of eigenvectors of the principal symbol for $\mu > 0$. This matrix has exactly four independent eigenvectors: two with eigenvalue $v = N^i \xi_i$ which can be chosen as $(0,1,-\xi_1/\xi_3,0)$ and $(0,0,1,-\xi_2/\xi_3)$, or more generally as $(0,b^i)$ with b^i such that $\xi_i b^i = 0$; one more with eigenvalue $v = N^i \xi_i + \sqrt{\mu} N$, which can be chosen as $(1, -\sqrt{\mu}\xi^i)$; and the fourth one with eigenvalue $v = N^i \xi_i - \sqrt{\mu} N$, which can be chosen as $(1, \sqrt{\mu}\xi^i)$. This shows that the system (14), (15) is strongly hyperbolic for any positive value of μ. Strongly hyperbolic systems are well posed under other smoothness and symmetrization conditions, which must apply to the eigenvectors. Instead of pursuing this argument, it is worth noticing that, with $\mu > 0$, a redefinition of variables $\tilde{\mathcal{C}} = \mathcal{C}$ and $\tilde{\mathcal{C}}^i = \mathcal{C}^i / \sqrt{\mu}$ casts the system into manifest symmetric hyperbolic form, which ensures the well posedness without further manipulations. The systems of PDE's that can be symmetrized by a simple rescaling of the variables may be referred to as symmetrizable, although I have not made this distinction in past works [10].

For $\mu = 0$, however, two eigenvectors coincide, so that there remain only three linearly independent eigenvectors, namely $(1,0,0,0)$, $(0,1,0,-\xi^1/\xi^3,0)$, and $(0,0,1,-\xi^2/\xi^3)$. Therefore, the system (14), (15) cannot be seen to be well posed by any of the three standard criteria if $\mu = 0$, which implies that unconstrained evolution is not guaranteed to give a solution to the Einstein equations. In this case the system is weakly hyperbolic with nonconstant coefficients. Although linear weakly hyperbolic systems with constant coefficients are known to be ill posed [9], in our case, due to the variable coefficients, general statements are more difficult to make with full rigor.

The characteristics of the propagation of the constraints are surfaces with normal covectors $\xi_d = (v, \xi_1, \xi_2, \xi_3)$ which satisfy either (a) $v - N^i \xi_i = 0$, namely $n^d \xi_d = 0$, or (b) $(v - N^i \xi_i)^2 - \mu N^2 = 0$, equivalently, $\xi^d \xi_d = 1 - \mu$. The first set of characteristics are timelike with respect to g_{ab}, and tangent to n^d. The second set of characteristics are null with respect to g_{ab} if $\mu = 1$, timelike if $0 < \mu < 1$ and spacelike if $\mu > 1$.

IV. CONCLUDING REMARKS

An analysis of the well posedness of the propagation of the constraints has been presented in the case in which the six spatial evolution equations $\mathcal{E}_{ab}$ are not set equal to zero, but are set proportional to the scalar (Hamiltonian) constraint. This encompasses two distinct cases ($\mu>0$ and $\mu<0$). The implicances for the two cases are opposite. The case $\mu>0$ is well posed, whereas the case $\mu<0$ is not, being thus unsuited for unconstrained numerical evolution. In more general situations where the constraints are combined with the evolution equations in more involved ways, it is necessary to ensure that the well posedness of the propagation of the constraints will be preserved by the combinations.

For the case of the standard formulation of [3], which corresponds with $\mu=1$, it has been shown that the constraints propagate according to a symmetric hyperbolic system with either timelike or null characteristics. The outer sheet of the characteristic cone is null, and defines the domain of determinacy [15] of the initial manifold. In other words, if initial data for g_{ab} satisfy the constraints, and a solution g_{ab} of Eq. (8) is evolved from the initial data, then the constraints are satisfied by g_{ab} everywhere within the domain of determinacy of the initial manifold. This statement applies to the continuous (exact) versions of the 3+1 equations. This is relevant to the problem of unconstrained evolution of the standard 3+1 equations for the following reasons.

In order to evolve the metric numerically, a discretized set of equations is used in the place of the continuous equations. The solution of the discretized evolution equations satisfies the corresponding continuous versions only up to some finite order of accuracy, which means that the evolution of the constraints in this case is not exactly homogeneous, at the very least. In implementing an unconstrained numerical scheme, it is important that the discretized constraints be preserved by the (discretized) evolution *at the same order of accuracy*. Apparently, this is not a trivial problem [17]. This problem has been addressed by Choptuik in [7], who refers to it as the issue of *consistency* of the finite-difference scheme. Choptuik concludes that it is possible to implement an unconstrained finite-difference scheme to solve the evolution equations which preserves the discretized version of the constraints at the same order of accuracy [18]. The argument is strongly based, however, on the fundamental assumption of well posedness of the corresponding continuous equations, which here is shown to hold. Unconstrained evolution also requires the evolution equations $\mathcal{Y}_{ab}=0$ to be well posed, which does not appear to have been established yet. However, unconstrained numerical evolution has been achieved in the spherically symmetric [7] as well as in the full three-dimensional case [13].

If one does not combine the scalar constraint, the imposition of the spatial Einstein equations $\mathcal{E}_{ab}=0$, which corresponds to $\mu=0$, can not be seen to be well posed here [19]. This appears to be the case of Eqs. (7.315a) and (7.315b) of [2], since they can be seen to correspond to the addition of a term $h_{ab}\mathcal{C}$ to the standard $\mathcal{Y}_{ab}$. This case deserves deeper study.

Regarding the role of the lapse function and shift vector, it is clear, from the treatment in the previous sections, that the results hold as stated for any choice of lapse and shift independent of the constraint functions $\mathcal{C}$ and $\mathcal{C}_a$, so long as the evolution equations are well posed. In particular, any dynamical choice of the lapse and shift as functions of the metric or first-order derivatives of the metric is allowed, with the understanding that the evolution must be well posed. The results would no be guaranteed if the lapse and shift are chosen as functions of $\mathcal{C}$ and $\mathcal{C}_a$.

It is possible to implement further combinations with the constraints in a nontrivial manner, excluded in the present work. In [10,12], the evolution equations (which in terms of h_{ab} and $\mathcal{L}_n h_{ab}$ are first order in time and second order in space) are cast into fully first-order form by introducing (combinations of) the first space derivatives of h_{ab} as new variables. Subsequently, the evolution of these new variables is combined with the vector constraint. The analysis based on the Bianchi equations is not applicable to this case. The propagation of the constraints in this case will be studied elsewhere.

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[18] See Phys. Rev. D **44**, 3125 (1991).
[19] The treatment of the Cauchy problem for general relativity in Chap. 10 of [1], p. 260, appears to communicate the opposite idea.