Pseudo-Minkowskian coordinates in asymptotically flat space-times

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For a rich class of asymptotically flat vacuum space-times, we show that it is possible to introduce a global coordinate system in a canonical fashion that is analogous to the standard Minkowskian coordinate systems used in flat space. This is accomplished by studying the intersection of the future light cone of interior space-time points with future null infinity. This intersection, referred to as a light cone cut of future null infinity, is piecewise a two-surface which can be described analytically by a function of the coordinates of null infinity. This function (the light cone cut function) can be given a special spherical-harmonic decomposition with the coefficients depending on the interior points. The canonical pseudo-Minkowskian coordinates are defined from the four coefficients of the $l=0,1$ spherical harmonics. In Minkowski space-time this prescription yields precisely the standard Cartesian flat coordinates. [S0556-2821(97)05304-6]

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I. INTRODUCTION

It is the purpose of this paper to argue that for suitably defined classes of asymptotically flat vacuum space-times, sufficiently close to Minkowski space, there exists a special set of *global* coordinates, chosen in a canonical fashion, that are, in a precise sense, the counterparts of the ordinary flat-space Minkowski coordinates. These coordinates, which we will denote by x^a and refer to as a canonical set of coordinates (or as pseudo-Minkowski coordinates), are such that when the radiation data (the Bondi asymptotic shear) goes to zero (i.e., to flat-space data) the x^a become the conventional Minkowski coordinates. There is a 10-parameter transformation freedom in their choice that is analogous to the Poincaré transformations; this freedom does become the Poincaré group in the limit of vanishing data. (The boost freedom can sometimes be eliminated by requiring that the Bondi momentum have only a nonvanishing ''time'' component at i^0, leaving only the Poincaré translations.) That we can obtain this canonical choice of global coordinates is rather surprising. For many years it was believed that the (asymptotic) coordinate freedom associated with asymptotically flat space-times was that of the Bondi-Metzner-Sachs (BMS) group [1,2]. It was then shown that with the data chosen in a special class [2], defined by certain asymptotic fall-off properties (but still very general), the BMS group could be reduced to the Poincaré group [2]. Nevertheless, it was believed that the coordinates associated with these Poincaré transformations could only be defined in the neighborhood of future null infinity, $\mathcal{I}^+$. It is our claim that these pseudo-Minkowski coordinates can be extended throughout the space-time.

We will be concerned with two different ''suitably defined'' classes of asymptotically flat vacuum space-times. The first class, which has been shown to exist [3], are the asymptotically flat spaces obtained from hyperboloidal initial data (i.e., data given on spacelike hypersurfaces that intersect future null infinity on a two-sphere) that are sufficiently close to the Minkowski space data. The solutions are ''global'' to the future of the initial data surface and possess a smooth conformal extension, ''above'' the data surface, through null infinity, $\mathcal{I}^+$. We will refer to such space-times as *HF* spaces. The second class, originally conjectured by Penrose, has not yet been shown to exist, though there appears to be some hope [4] that in the near future its existence can be demonstrated. This class, which we will refer to as *RP* spaces, is the special case of future asymptotically simple space-times [5] where the *vacuum* Einstein equations are imposed. It is characterized by the condition that the conformal structure can be extended, in all null directions, through null infinity. The *RP* spaces presumably contain the *HF* spaces.

In principle, our argument for the existence of the pseudo-Minkowskian coordinates applies to generic asymptotically simple spaces, since it makes no use of the Einstein equations. Its value resides, however, in the fact that it applies equally well to the *HF* spaces and the *RP* spaces, if the latter can be shown to exist.

There are subtle technicalities underlying the ideas used here, which we have not addressed, and thus we have not attempted to state our results in a mathematically formal manner. We can think of our proofs as being essentially heuristic.

II. THE ARGUMENT

We begin with a brief discussion of some background issues.

Since, in either case, *RP* or *HF*, we are dealing with asymptotically flat space-times with smooth conformal extensions to null infinity, we can begin with the existence of $\mathcal{I}^+$, with its usual properties, e.g., it is $S^2\times R$, and that we can introduce standard Bondi coordinates of $\mathcal{I}^+$, namely, $(u,\zeta,\overline{\zeta})$ with $u\in R$ and $(\zeta,\overline{\zeta})$ the complex stereographic coordinates on S^2. Two-dimensional subsurfaces of $\mathcal{I}^+$ will be described by functions of the form

$$u=F(\zeta,\overline{\zeta}). \tag{1}$$

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They need not be differentiable or single valued, though they will be piecewise differentiable and piecewise single valued.

From the fact that the space-times we are dealing with are R^4 with a boundary (the boundary being either the initial data surface or $\mathcal{I}$), there will exist global coordinate systems (on R^4), any one of which we will refer to as y^a. In addition to the global y^a, there will also be local space-time Bondi coordinates in the neighborhood of $\mathcal{I}^+$ obtained by extending the Bondi coordinates of $\mathcal{I}^+$, $(u,\zeta,\bar{\zeta})$, into the interior via an affine parameter, r, along the null geodesics normal to the spheres, u = const, on $\mathcal{I}^+$. These local Bondi coordinates,

$$y_B^a=(u,\zeta,\bar{\zeta},r), \tag{2}$$

must be connected to the global y^a, in a neighborhood of $\mathcal{I}^+$, by some transformation,

$$y_B^a=Y^a(y^b). \tag{3}$$

It is this transformation that allows us to connect or ''tie'' the interior of the space-time to $\mathcal{I}^+$.

We now assume that, in the class of ''suitable space-times,'' the vacuum metrics are known in the global y^a coordinate system, $g^{ab}=g^{ab}(y^c)$, and that, in principle, the null geodesic equations can be integrated in the form

$$y^a=y^a(y_0^b,\zeta_0,\bar{\zeta}_0,s), \tag{4}$$

where y_0^b is an initial point, $(\zeta_0,\bar{\zeta}_0)$ are a choice of stereographic coordinates on the sphere of null directions at y_0^b, labeling the initial null directions, and s is an affine parameter along the null geodesics. [That the null directions $(\zeta_0,\bar{\zeta}_0)$ at the various points y_0^b can be related to each other smoothly follows from the existence of a global orthonormal tetrad.]

Remark. The theorem on the differentiability of solutions to ordinary differential equations [6] guarantees that the null geodesics $y^a(y_0^b,\zeta_0,\bar{\zeta}_0,s)$ are sufficiently differentiable functions of the initial conditions $y_0^b,\zeta_0,\bar{\zeta}_0$ in an open interval of the parameter s (later, our argument requires them to be twice differentiable). The degree of differentiability of Eq. (4) actually extends to the boundary $\mathcal{I}^+$. This can be seen clearly by conformally compactifying the space-time (the null geodesics are conformally invariant). Since the conformally compactified space-time can be extended past $\mathcal{I}^+$, then $\mathcal{I}^+$ becomes a finite surface contained in an open range of the parameter along the null geodesics.

Using Eq. (4) in Eq. (3), we have the expression for the null geodesics in the neighborhood of $\mathcal{I}^+$ in the Bondi coordinates; i.e.,

$$y_B^a=Y^a(y^c(y_0^b,\zeta_0,\bar{\zeta}_0,s)). \tag{5}$$

or

$$u=u(y_0^b,\zeta_0,\bar{\zeta}_0,s), \tag{6a}$$

$$\zeta=\zeta(y_0^b,\zeta_0,\bar{\zeta}_0,s) \quad \text{and complex conjugate,} \tag{6b}$$

$$r=r(y_0^b,\zeta_0,\bar{\zeta}_0,s). \tag{6c}$$

As our interest is in the behavior of the null geodesics at $\mathcal{I}^+$, we let s (and r) $\to\infty$, thus we have

$$u=u(y_0^b,\zeta_0,\bar{\zeta}_0,\infty)\equiv U(y_0^b,\zeta_0,\bar{\zeta}_0), \tag{7a}$$

$$\zeta=\zeta(y_0^b,\zeta_0,\bar{\zeta}_0,\infty)\equiv Y(y_0^b,\zeta_0,\bar{\zeta}_0) \quad \text{and complex conjugate.} \tag{7b}$$

In principle, Eqs. (7) are known functions for any asymptotically flat space-time. Moreover, our earlier remark establishes that both $U(y_0^b,\zeta_0,\bar{\zeta}_0)$ and $Y(y_0^b,\zeta_0,\bar{\zeta}_0)$ are sufficiently differentiable functions of $y_0^b,\zeta_0,\bar{\zeta}_0$.

Future null-geodesic completeness implies that Eqs. (7) determine a unique point $(u,\zeta,\bar{\zeta})$ on $\mathcal{I}^+$ for every initial point y_0^b and every initial direction $(\zeta_0,\bar{\zeta}_0)$. As $(\zeta_0,\bar{\zeta}_0)$ range over the sphere of all initial directions, Eqs. (7) parametrically describe a mapping from the two-sphere into $\mathcal{I}^+$; namely, a two-surface on $\mathcal{I}^+$. It is the intersection of all null geodesics from y_0^b with $\mathcal{I}^+$. This surface can be described in the form (1) by inverting Eq. (7b) and using the inversion in Eq. (7a). We refer to the surface, when expressed in this form, as the *light cone cut* of $\mathcal{I}^+$ from the interior point y_0^a.

Because of future null-geodesic completeness, the light cone cuts stay away from i^+ (i.e., they do not blow up), but generically they will not be single valued or differentiable. This can be seen from the fact that, in general, due to the development of caustics in the space-time, Eq. (7b) will *not* have a unique inverse [i.e., there will be several values of $(\zeta_0,\bar{\zeta}_0)$ for each $(\zeta,\bar{\zeta})$]. If, however, Eq. (7b) is *inverted locally*, so that, locally, $\zeta_0=Y_0(y_0^a,\zeta,\bar{\zeta})$ and the ζ_0 is eliminated from Eq. (7a), we obtain the (piecewise) light cone cut function in the form of Eq. (1); i.e., as

$$u=Z(y_0^a,\zeta,\bar{\zeta}). \tag{8}$$

Emphasizing an essential point, we stress that though Eq. (8) is given only piecewise and is not globally smooth or single valued, however, Eq. (7a) is single valued and smooth. We will return to this point later.

As an important digression we look at both Eqs. (7) and (8) in Minkowski space. In a given Minkowski frame with standard coordinates x^a, the null geodesics (4) can be written as

$$x^a=x_0^a+s\ell^a(\zeta_0,\bar{\zeta}_0), \tag{9}$$

where $\ell^a(\zeta_0,\bar{\zeta}_0)$ is a null vector for all values of $(\zeta_0,\bar{\zeta}_0)$ that can be given in the form

$$\ell^a(\zeta,\bar{\zeta})=\frac{1}{\sqrt{2}(1+\zeta\bar{\zeta})}(1+\zeta\bar{\zeta},\zeta+\bar{\zeta},i(\bar{\zeta}-\zeta),-1+\zeta\bar{\zeta}). \tag{10}$$

For later use, we note that the four components of ℓ^a are linear combinations of the first four spherical harmonics $Y_{00}(\zeta,\bar\zeta)$ and $Y_{1m}(\zeta,\bar\zeta)$.

The transformation (3), between the x^a and the Bondi $y_B^a=(u,\zeta,\bar\zeta,r)$, is given by

$$x^a=ut^a+r\ell^a(\zeta,\overline{\zeta}),\quad t^a=\frac{1}{\sqrt{2}}(1,0,0,0). \tag{11}$$

Finally, for Minkowski space, after a brief calculation by following null geodesics and passing to the limit $s\to\infty$, Eqs. (7) become [7–9]

$$u=x_0^a\ell_a(\zeta_0,\overline{\zeta_0}) \tag{12a}$$

and

$$(\zeta,\overline{\zeta})=(\zeta_0,\overline{\zeta_0}), \tag{12b}$$

hence, for the light cone cut function (8) we have

$$u=Z_0(x_0^a,\zeta,\overline{\zeta})\equiv x_0^a\ell_a(\zeta,\overline{\zeta}), \tag{13}$$

which describes a smooth embedding of a sphere on $\mathcal{I}^+$. Using our observation that the ℓ^a are combinations of the first four spherical harmonics, and dropping the label 0, Eq. (13) can be rewritten as

$$u=\sum_{\substack{l=0,1\\ m=-l,l}} x_{l,m}Y_{lm}(\zeta,\overline{\zeta}) \tag{14}$$

with

$$x_{0,0}\equiv\sqrt{2\pi}t,$$

$$x_{1,0}\equiv-\sqrt{\frac{2\pi}{3}}z,$$

$$x_{1,1}\equiv\sqrt{\frac{\pi}{3}}(x-iy),$$

$$x_{1,-1}\equiv-\sqrt{\frac{\pi}{3}}(x+iy), \tag{15}$$

or

$$x_{l,m}\Leftrightarrow x^a.$$

In other words, in Minkowski space, the Minkowski coordinates x^a are the coefficients of the first four spherical harmonics of the light cone cut function. If the Minkowski space had been described by some other global coordinates y^a, then the light cone cuts would still have had the form

$$u=Z(y^a,\zeta,\overline{\zeta})\equiv\sum_{\substack{l=0,1\\ m=-l,l}} f_{l,m}(y^b)Y_{lm}(\zeta,\overline{\zeta})\equiv f^a(y^b)\ell_a(\zeta,\overline{\zeta}),$$

where $f_{l,m}(y^b)$ are the appropriate coefficients in the expansion. By comparison with Eq. (14), the x^a coordinates could then be obtained from

$$x_{l,m}=\int_{S^2}Z_0(y^a,\zeta,\overline{\zeta)Y_{l,m}}(\zeta,\overline{\zeta})dS^2\equiv f_{l,m}(y^a), \tag{16}$$

for $l=0,1$, where dS^2 is the area element on the unit sphere, $dS^2=(2/i)[d\zeta\wedge d\bar\zeta/(1+\zeta\bar\zeta)^2]$. This relationship, which can be expressed equivalently by $x^a=f^a(y^b)$, gives the coordinate transformation from y^a to the canonical x^a.

It is precisely this observation that we will use to obtain the pseudo-Minkowskian coordinates x^a in the *HF* and *RP* spaces.

The basic idea is to consider the curved space light cone cut function (8),

$$u=Z(y^a,\zeta_0,\overline{\zeta_0})$$

(in the following, we are dropping the sublabel 0 in y_0^a), find the coefficients of its first four spherical harmonics, and identify them as the pseudo-Minkowskian coordinates. More explicitly, in the HF and RP spaces, we intend to define the four pseudo-Minkowskian coordinates by the transformation

$$x_{l,m}=\int_{S^2}Z(y^a,\zeta,\overline{\zeta)Y_{l,m}}(\zeta,\overline{\zeta})dS^2\equiv f_{l,m}(y^a), \tag{17}$$

for $l=0,1$. There is a difficulty with this definition; namely, can we give meaning to the integrals (17), since the cut function $Z(y^a,\zeta,\bar\zeta)$ is multivalued (or is given piecewise)? The solution is actually quite simple; the integral is pulled back to the sphere of null directions at the point y^a. The integral in Eq. (17) becomes [from Eqs. (7a) and (7b)]

$$\begin{aligned}x_{l,m}=&\int_{S_0^2}U(y^a,\zeta_0,\overline{\zeta_0})\overline{Y_{l,m}}(\zeta(y^a,\zeta_0,\overline{\zeta_0}),\overline{\zeta}(y^a,\zeta_0,\overline{\zeta_0}))\\ &\times J(y^a,\zeta_0,\overline{\zeta_0})dS_0^2\\ \equiv& f_{l,m}(y^a),\end{aligned} \tag{18}$$

for $l=0,1$, where $J(y^a,\zeta_0,\overline{\zeta_0})$ is the Jacobian of the transformation (7b) and the integral is taken over the sphere of initial null directions. That the Jacobian exists follows from the smoothness of $Y(y_0^b,\zeta_0,\overline{\zeta_0})$ [Eq. (7b)]. As all the functions in the integrand are now well defined, the integral is well defined. We thus have Eq. (18), or

$$x^a=f^a(y^b) \tag{19}$$

as our proposed transformation from the global y^a to the canonical x^a.

The only remaining issue is whether Eq. (19) is well behaved globally; i.e., does the Jacobian of Eq. (19) exist and is it different from zero for all y^a? This can be answered affirmatively by first remembering that $U(y_0^b,\zeta_0,\overline{\zeta_0})$ [from Eq. (7a)] is a smooth function of y_0^b and then observing that for sufficiently small values of the data the HF and (presumably) the *RP* spaces are smoothly connected to Minkowski space and, second, that the transformation (19) applied to Minkowski space [namely, Eq. (16)] has a nonvanishing Jacobian. It now follows, by continuity, that for sufficiently small data the Jacobian of Eq. (19) is nonvanishing. We have thus shown that the transformation (18) and (19) yields a special or canonical set of coordinates that in the flat-space case are the standard Cartesian flat coordinates.

III. DISCUSSION

There are several issues that need further discussion.

(1) We began with a given asymptotically flat space-time and from a *specific* choice of Bondi coordinate system on $\mathcal{I}^+$ we could find the pseudo-Minkowskian coordinates x^a. As there is considerable freedom in the choice of Bondi coordinates (the freedom of the BMS group), it appears that our ''canonical'' coordinates are not very canonical. However, the freedom of the BMS group can be reduced. We first make the physically reasonable restriction of the data to that for which the magnetic part of the Bondi shear vanishes at both i^+ and i^0. One can then restrict the BMS group to the Poincaré group by requiring that the BMS frame be chosen such that the (full) Bondi shear (the characteristic datum) vanishes at i^+; i.e., the shear vanishes in the limit as $u\to\infty$. The remaining freedom is the Poincaré group. [In the case of RP spaces one can restrict the freedom even more; the shear can be made to vanish at i^0 (as $u\to-\infty$) and the boost frame can be chosen uniquely so that the initial Bondi four-momentum (at $u\to-\infty$) has only a nonvanishing time component. In this case one is left with only the Poincaré translations.]

(2) In the case where we cannot (or do not wish to) restrict ourselves to just the translations, we can study how the pseudo-Minkowskian coordinates transform under boost transformations. They transform, in fact, as part of an infinite dimensional representation of the Lorentz group. In the language of Gelfand *et al.* [10,11], the function $Z(y^a,\zeta,\bar{\zeta})$ transforms under the (reducible, but not completely reducible, infinite-dimensional) $D_{(1,1)}$ representation which possesses an invariant subspace $E_{(1,1)}$ spanned by the $l=0,1$ spherical harmonics. ($D_{(1,1)}/E_{(1,1)}$ is an irreducible representation and $E_{(1,1)}$ by itself is the vector representation.) The effect of this is that if Z is expanded in spherical harmonics, the coefficients of the $l=0,1$ harmonics do not enter into the $l>1$ harmonics under the transformation, but the higher-harmonic coefficients do enter and affect the $l=0,1$ coefficients. The x^a ''sit'' in the invariant subspace but do not transform just among themselves; the higher-harmonic coefficients get mixed in.

We point out that by taking different inverse powers of $Z(y^a,\zeta,\bar{\zeta})$ it is possible to construct objects that do transform as finite irreducible representations (scalars, vectors, symmetric tensors, etc.) under the Lorentz transformations [11]. Some of these objects are under study, though at the moment we are not able to advance any interpretation associated with them.

(3) One might ask what coordinate conditions on the metric are implied by the use of the canonical x^a. Though in general we do not know the answer to this question, in linear theory the metric comes out [12] in the analogue of the Coulomb gauge; i.e., for the linearized metric $g^{ab}=\eta^{ab}+h^{ab}$, we have $h^{0a}=0$.

(4) The cut function $u=Z(y^a,\zeta,\bar{\zeta})$ contains all conformal information of the space-time. This includes an explicit means of calculating the conformal metric from knowledge of Z. If, in addition, an appropriate conformal factor (to make the conformal metric into a vacuum metric) is given [say $\omega(y^a)$] then the space-time metric can be obtained. In the situation discussed here, Z was obtained by integrating the null geodesic equations from the knowledge of an asymptotically flat vacuum metric. A new point of view is that the cut function Z and the conformal factor ω are to be the fundamental variables of the theory and the metric is to be a derived concept. One can reformulate general relativity (GR) completely in terms of the $Z(y^a,\zeta,\bar{\zeta})$ and $\omega(y^a)$. We refer to this formulation as the null surface formulation (NSF) of GR. The vacuum Einstein equations can be replaced by (coupled) differential equations for Z and ω [13,14]. There are actually two versions of the NSF, a general version that applies, locally, to any vacuum space-time, and a more special version that is applied to asymptotically flat space-times. In the latter case, one begins with the choice of a Bondi frame on $\mathcal{I}^+$, and the choice of the Bondi shear as free characteristic data. (With no loss of generality this data can be chosen to vanish at either i^+ or i^0 to obtain either the *HF* or *RP* spaces.) A very attractive result obtained from this approach is that *the space-time points themselves arise as constants of integration* and are, in fact, the coefficients of the first four spherical harmonics; i.e., our canonical coordinates, x^a, arise naturally as four constants of integration. We actually first observed the appearance of the canonical coordinates in this manner. A detailed paper is being prepared expounding this point of view.

(5) We note that for asymptotically flat space-times (not necessarily vacuum Einstein) with a definite choice of a Bondi frame on $\mathcal{I}^+$, there is, in addition to the canonical x^a, a canonically defined flat metric, η, that exists on the same manifold. Given the cut function $Z(x^a,\zeta,\bar{\zeta})$ for the space-time, we also have the natural flat-space cut function (13), namely $u=Z_0(x^a,\zeta,\bar{\zeta})\equiv x^a\ell_a(\zeta,\bar{\zeta})$, which leads (with $\omega=1$) to the standard flat metric in the x^a coordinates (see the Appendix).

(6) There is an important subtle issue that should be raised and discussed. If not clarified, it could lead to confusion. Consider the situation that we have two metrics in our ''suitably defined classes'' that are given on the same manifold using the same global coordinates y^a. Using the same Bondi coordinates $(u,\zeta,\bar{\zeta})$ on $\mathcal{I}^+$, we can calculate and obtain the light cone cut functions for each metric, say, $Z_1(y^a,\zeta,\bar{\zeta})$ and $Z_2(y^a,\zeta,\bar{\zeta})$. We can then introduce our canonical pseudo-Minkowski coordinates x^a [via Eq. (19)] for each cut function and obtain

$$x_1^a=f_1^a(y^b) \quad \text{and} \quad x_2^a=f_2^a(y^b). \tag{20}$$

The point that we want to make is that the two sets x_1^a and x_2^a cannot be identified; i.e., the two functions $f_1^a(y^b)$ and $f_2^a(y^b)$ are always different when the two metrics are different (see the Appendix).

(7) Though it is not at all clear at this time what uses these canonical cordinates might have, their very existence is surprising. They, however, do appear to be needed in the attempt [15] to apply Ashtekar's asymptotic quantization [16] procedure to the quantization of the NSF of GR. Here a unique gauge (in this case analogous to the Coulomb gauge of electrodynamics) is needed.

(8) Asymptotically flat self-dual space-times can be obtained from $\omega=1$ and $Z(x^a,\zeta,\bar\zeta)$'s that are regular and satisfy the ''good cut'' equation [17], namely $\bar\eth^2 Z=\bar\sigma(Z,\zeta,\bar\zeta)$ for an arbitrary choice of the function $\bar\sigma(Z,\zeta,\bar\zeta)$. The space-time points enter the solution $Z(x^a,\zeta,\bar\zeta)$ as the constants of integration, and appear as the coefficients of the first four spherical harmonics; i.e., the $x_{l,m}$ or x^a appear here again in the same canonical fashion as earlier.

(9) We wish to point out a shortcoming in the use of the canonical coordinates for the *HF* spaces; there will certainly be ''ghost''coordinates for any *HF* space; i.e., there will be numerical values of the coordinates x^a such that no point in the *HF* space exists that corresponds to them. Intuitively, these missing points lie to the past of the past Cauchy horizon. It seems likely to us, however, that this problem will disappear for the *RP* spaces.

(10) It might be conjectured that canonical coordinates of this kind could exist in the Christodoulou-Klainerman spaces [18]. However, the $\mathcal{I}^+$ of the Christodoulou-Klainerman spaces is not smooth; thus it is not clear that our construction applies to this case.

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APPENDIX

We will describe here an illustrative example of what has been discussed in the main text. We will compare the canonical coordinates of two different flat space-times, namely, two different flat metrics on the same manifold, R^4 with future null boundary $\mathcal{I}^+$. For computational ease, however, we begin with a fixed Bondi frame $(u,\zeta,\bar\zeta)$ at $\mathcal{I}^+$ and give two *known* cut functions, Z_1 and Z_0, in terms of the same Bondi frame, rather than obtaining Z_1 and Z_0 by integrating the null geodesics of the two metrics. From knowledge of the cut function, a prescription is available for the transformation between the Bondi coordinates and an arbitrary set of interior coordinates in the neighborhood of $\mathcal{I}^+$. We give this prescription explicitly in the case of the two canonical sets of x^a's obtained from the two cut functions. We make use of these two transformations to find the relationship between the two sets of canonical coordinates. It eventually becomes clear that Z_1 and Z_0 are indeed the cut functions of two different flat metrics.

We begin with the cut function

$$u=Z_1(y^a,\zeta,\bar\zeta)\equiv f^a(y^b)\ell_a(\zeta,\bar\zeta)+\alpha(\zeta,\bar\zeta), \tag{A1}$$

where the four $f^a(y^b)$ are chosen arbitrarily (with a nonvanishing Jacobian), and $\alpha(\zeta,\bar\zeta)$ is an arbitrary regular function on the sphere *that has no l = 0,1 spherical harmonics*. Introducing the canonical coordinates, in this case, is straightforward; i.e., $x_1^a=f^a(y^b)$, and Eq. (A1) becomes

$$u=Z_1(x^a,\zeta,\bar\zeta)\equiv x_1^a\ell_a(\zeta,\bar\zeta)+\alpha(\zeta,\bar\zeta). \tag{A2}$$

The coordinate transformation [12] between x_1^a and the interior Bondi coordinates $y_B^a=(u,\zeta,\bar\zeta,r)$ is given implicitly by

$$\begin{aligned} u&=Z_1(x_1^a,\zeta,\bar\zeta),=x_1^a\ell_a(\zeta,\bar\zeta)+\alpha(\zeta,\bar\zeta),\\ 0&=\eth Z_1(x_1^a,\zeta,\bar\zeta)\\ &=x_1^a m_a(\zeta,\bar\zeta)+\eth\alpha(\zeta,\bar\zeta)\quad\text{and complex conjugate},\\ r&=\bar\eth\eth Z_1(x_1^a,\zeta,\bar\zeta)=x_1^a(n_a(\zeta,\bar\zeta)-\ell_a(\zeta,\bar\zeta))+\bar\eth\eth\alpha(\zeta,\bar\zeta). \end{aligned} \tag{A3}$$

Here $m^a\equiv\eth\ell^a$ and $n^a\equiv\bar\eth\eth\ell^a+\ell^a$. The four vectors $\ell^a,m^a,\bar m^a,n^a$ satisfy $\ell^a n_a=-m^a\bar m_a=1$ while all the other possible scalar products among them are zero. This fact can be used to find the transformation explicitly:

$$x_1^a=(u-\alpha)(n^a+\ell^a)+(r-\bar\eth\eth\alpha)\ell^a+\eth\alpha\bar m^a+\bar\eth\alpha m^a. \tag{A4}$$

Our formalism associates a flat metric $\eta^{(1)}$ with Z_1 by

$$ds_1^2=\eta_{ab}^{(1)}dx_1^a dx_1^b=(dt_1)^2-(dx_1)^2-(dy_1)^2-(dz_1)^2. \tag{A5}$$

The flat metric (A5) can be ''tied'' or connected to $\mathcal{I}^+$ via Eq. (A4), and its light cone cuts are described by Eq. (A2).

The ''natural'' flat-space cuts of $\mathcal{I}^+$ are obtained from the cut function [see Eq. (13)]

$$u=Z_0(x_0^a,\zeta,\bar\zeta)=x_0^a\ell_a(\zeta,\bar\zeta), \tag{A6}$$

which, in turn, leads to the flat metric

$$ds_0^2=(dt_0)^2-(dx_0)^2-(dy_0)^2-(dz_0)^2=\eta_{ab}^{(0)}dx_0^a dx_0^b. \tag{A7}$$

The transformation from the x_0^a to the interior Bondi coordinates is now

$$\begin{aligned} u&=Z_0(x_0^a,\zeta,\bar\zeta)=x_0^a\ell_a(\zeta,\bar\zeta),\\ 0&=\eth Z_1(x_0^a,\zeta,\bar\zeta)=x_0^a m_a(\zeta,\bar\zeta)\quad\text{and complex conjugate},\\ r&=\bar\eth\eth Z_0(x_0^a,\zeta,\bar\zeta)=x_0^a(n_a(\zeta,\bar\zeta)-\ell_a(\zeta,\bar\zeta)), \end{aligned} \tag{A8}$$

or, explicitly,

$$x_0^a=u(n^a+\ell^a)+r\ell^a. \tag{A9}$$

The light cone cuts of the metric (A7) are described by Eq. (A6).

If the Bondi coordinates are eliminated between Eqs. (A4) and (A9), we can obtain the relation between the two sets of global coordinates, x_0^a and x_1^a. To eliminate the Bondi coordinates, we first solve $x_0^a \overline{m_a} = 0$ for ζ, and obtain [19]

$$\zeta = \frac{z_0 + R_0}{x_0 - i y_0} \quad \text{and complex conjugate,}$$

where

$$R_0 \equiv \sqrt{x_0^2 + y_0^2 + z_0^2}. \tag{A10}$$

The transformation is then

$$x_1^a = x_0^a - \alpha(n^a + l^a) - \eth\overline{\eth\alpha} l^a + \eth\alpha \overline{m^a} + \overline{\eth\alpha} m^a, \tag{A11}$$

where ζ, on the right-hand side, is given by Eq. (A10). As this is not a Lorentz transformation (it is not even linear) we see that we have two distinctly different flat metrics. Notice, furthermore, that none of the two flat metrics is more ''natural'' than the other one. Had we chosen to describe the cuts in an alternative Bondi slicing $u' = u - \alpha$, supertranslated with respect to the original one, the cut function Z_1 would have had $l = 0,1$ spherical harmonics only, whereas Z_0 would have had higher-order harmonics as well. Thus, in this alternative Bondi frame, $\eta^{(1)}$ would look ''natural.''

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[19] As $x_0^a \overline{m_a} = 0$ is a quadratic equation for ζ, there are two solutions $\zeta = (z_0 \pm R_0)/(x_0 - i y_0)$. We chose the positive sign.