FINITE DIFFERENCING ON THE SPHERE

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ABSTRACT

We describe a finite difference version of the eth formalism, which allows use of spherical coordinates in 3-dimensional systems with global second order accuracy. We briefly present the application of the formalism to the evolution of linear scalar waves and to the calculation of the curvature scalar of a curved geometry on a topologically spherical manifold.

1. Introduction

Spherical coordinates and spherical harmonics are standard analytic tools in the description of radiation. The eth formalism [1,2] and the associated spin-weighted spherical harmonics [1,3] allow a simple and unified extension of these analytic techniques to vector and tensor fields. In computational work, spherical coordinates have mainly been used in axisymmetric systems, where the polar singularities may be regularized by standard tricks. In the absence of symmetry, these techniques do not easily generalize and they would be especially prohibitive to develop for tensor fields. Here we present a finite difference version of the eth formalism, which allows use of spherical coordinates in computational relativity with global second order accuracy.

2. The Eth Formalism

We introduce two stereographic coordinate patches (North and South) covering the sphere, $\zeta_N = \tan(\theta/2)\, e^{i\phi}$ and $\zeta_S = 1/\zeta_N$. Let (q_S, p_S) and (q_N, p_N) be the real and imaginary parts of ζ_S and ζ_N, respectively. Both patches extend between $-1 \le q \le 1$ and $-1 \le p \le 1$. A uniform, square, numerical grid is introduced in each patch. Functions and their derivatives are represented with the usual second order difference method. At the boundaries, the derivatives are obtained using functional values supplied by interpolation from the opposite patch. For second order accurate numerical differentiations a fourth order interpolation scheme is necessary and has been developed.

We next introduce a complex vector basis in each patch.In the S patch we make the choice $e^a_S = (1+\zeta_S\bar{\zeta}_S)(\delta^a_1 + i\delta^a_2)/2$, so that its real and imaginary parts line up with the S axes. Similarly, in the N patch, $e^a_N = (1+\zeta_N\bar{\zeta}_N)(\delta^a_1 + i\delta^a_2)/2$. Tensor objects are now contracted with various combinations of basis vectors and reduce to spin weighted scalars [1]. A spin weighted scalar on the sphere is represented by a set of grid values on the two patches and an integer spin value.

A covariant (unit sphere metric) derivative of a tensor field is reduced to derivatives of scalar fields via the introduction of the eth and eth-bar operators. Their action on spin weighted scalars is given by

$$\begin{aligned} \eth v &= q^a\partial_a v + n\zeta v \\ \bar{\eth} v &= \bar{q}^a\partial_a v - n\bar{\zeta} v, \end{aligned} \tag{1}$$

where n is the spin weight of the scalar. With the above prescription, a tensor equation on the sphere is reduced to scalar equations involving fields of different spin weights. All derivatives are reduced to eth and eth-bar operators which have a simple, everywhere regular, finite difference representation.

3. Applications and Tests

A crucial first implementation of the scheme is the discretization of the Laplace operator on the sphere. In terms of the complex coordinate $\zeta = (q,p)$ we have

$$D^2\Psi = (1+\zeta\bar{\zeta})^2\partial_\zeta\partial_{\bar{\zeta}}\Psi = (1+q^2+p^2)^2[\partial_{qq}+\partial_{pp}]\Psi. \tag{2}$$

The centered finite difference approximation of Eq. (2) is now standard since the operator is conformal to the cartesian form. At the boundary, a virtual grid point is implied and acquires a value through interpolation from the opposite patch. The discretization is confirmed to be globally second order accurate.

A complete implementation of the formalism, involving repeated evaluations of the Laplace operator on numerical data, is the numerical solution of the 3-D wave equation in spherical coordinates. An algorithm for computing the solution of the wave equation in the characteristic initial value formulation is known. [4] In retarded time coordinates the wave equation takes the form

$$2g_{,ur} - g_{,rr} + \frac{D^2 g}{r^2} = 0. \tag{3}$$

The main accuracy problem in Eq. (3) is now treated , since the angular momentum term is regular throughout each patch. The linearity of the problem allows a thorough accuracy check since sufficiently general exact solutions are easily identified. Using exact multipole solutions we verified second order global convergence for large harmonic values.

As a tensorial illustration of the forgoing methods, we consider a problem which arises in many different contexts in general relativity: Given the metric h_{ab} of a topological sphere, calculate the scalar curvature. The metric is uniquely determined by its unit sphere dyad components $K = h_{ab}\, e^a \bar{e}^b/2$ and $J = h_{ab}\, e^a e^b/2$. The scalar curvature corresponding to h_{ab} is given by

$$R = 2K - \eth\bar{\eth}K + \frac{1}{2}[\bar{\eth}^2 J + \eth^2 \bar{J}] + \frac{1}{4K}[\bar{\eth}\bar{J}\eth J - \bar{\eth}J\eth\bar{J}]. \tag{4}$$

The comparison of the numerical evaluation of R with exact calculations provides a first test of accuracy. A strong global test is suggested by the Gauss-Bonnet theorem for spherical topologies, namely

$$\oint R dS = 8\pi. \tag{5}$$

Starting from an arbitrary metric we compute the numerical curvature and then integrate over the sphere using a second order integration scheme. The integration must take into account the overlap between the coordinate patches. A simple and natural choice is to use the equator $\zeta\bar{\zeta} = 1$ as the smooth and symmetric boundary of the integration within each patch. We checked the convergence of the Gauss-Bonnet integral over a wide range of curvature radii and verified the remarkable robustness of the method.

In summary, the finite difference implementation of the eth formalism we presented offers a robust and accurate method for developing numerical schemes in 3-D. The traditional reliance on cartesian coordinates can be relaxed and spherical coordinate topologies can be used whenever the nature of the problem renders them suitable.

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5. References

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