The characteristic treatment of black holes

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The characteristic initial value problem has been successfully implemented as a robust computational algorithm (the PITT NULL CODE) to evolve 4-dimensional vacuum space-times. It has been applied to the calculation of gravitational waveforms emitted by black holes and to the event horizon structure in the merger of black holes. The characteristic code also has potential application to the binary black hole problem via Cauchy-characteristic matching.

Because the event horizon is itself a characteristic hypersurface, it can be analyzed by characteristic techniques as a stand-alone object. We have developed an analytic conformal model of null hypersurfaces which gives new insight into the intrinsic geometry of the pair-of-pants horizon found in the numerical simulation of the head-on collision of black holes and into the initially toroidal horizon found in the simulation of a collapsing, rotating cluster.

Most studies of black hole formation and merger have been restricted to axisymmetry. However, axisymmetric horizons, like the Schwarzschild horizon, are non-generic. When applied to a *non*-axisymmetric horizon, the characteristic approach reveals substantially new features. In particular, coalescing black holes generically go through a toroidal phase before they become spherical. The conformal structure of the event horizon supplies part of the data for a simulation of the exterior space-time. This provides a new way to calculate the post-merger waveforms from a binary black hole inspiral.

§1. Null Cone Evolution

A longterm project [1] to develop the pioneering work of Bondi [2] and Penrose [3] into a computational algorithm for the characteristic initial value problem has recently culminated in a highly accurate, efficient and robust code - the PITT NULL CODE [4]. Because the evolution proceeds on a space-time foliation by null cones which are are generated by the characteristic rays of the theory, this approach offers several advantages for numerical work. I will describe here the fruits of these investigations relevant to black hole physics.

In null cone coordinates, Einstein's equations reduce to propagation equations along the radial light rays, which can be integrated in hierarchical order for one variable at a time. This leads to a highly efficient characteristic marching algorithm. There is one complex evolution variable which describes the free degrees of freedom of the gravitational field and four auxiliary variables. A compactified grid, based upon Penrose's conformal description of null infinity, removes the necessity of an artificial outgoing radiation condition and makes possible a rigorous description of geometrical quantities such as the Bondi mass and news function. The news function supplies both the true waveform and polarization of the gravitational radiation incident on a distant antenna. Furthermore the grid domain is exactly the region in which waves propagate, which is ideally efficient for the purpose of radiation studies. Since each null cone extends from the source to null infinity, the radiation appears immediately with no need for numerical evolution to propagate it across the grid. In addition, the growth of a large redshift offers the bonus of forecasting event horizon formation.

The computational technique of shooting along characteristics is standard in one spatial dimension but the use of characteristic hypersurfaces as the underlying foliation for numerical evolution in higher dimensions is exclusive to relativity [5]. The basic approach is applicable to any of the hyperbolic systems occurring in physics, e.g. the wave equation, electromagnetic theory and hydrodynamics.

For general relativity, the computational grid is based on coordinates constructed from a family of outgoing null hypersurfaces emanating from a worldtube of topology $S^2 \times R$. Let u label these hypersurfaces, x^A $(A = 2, 3)$ be angular coordinates for the null rays and r be a surface area distance. Then, in the resulting $x^\alpha = (u, r, x^A)$ coordinates, the metric takes the Bondi-Sachs form [2),6)]

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right) du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B, \tag{1·1}$$

where $det(h_{AB}) = det(q_{AB}) = q(x^A)$, with q_{AB} a unit sphere metric. For purposes of including null infinity as finite grid points, the code uses a compactified radial coordinate.

The traditional $3+1$ decomposition of space-time used in the Cauchy formalism is not applicable here because the foliation by null hypersurfaces of constant u has a degenerate 3-metric and a null normal. However, an analogous $2+1$ decomposition can be made on the timelike worldtube of constant r, which has intrinsic metric

$${}^{(3)}ds^2 = -e^{2\beta}\frac{V}{r}du^2 + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du). \tag{1·2}$$

In this form, we can identify $r^2 h_{AB}$ as the metric of the surfaces of constant u which foliate the worldtube, $e^{2\beta}V/r$ as the square of the lapse function and $(-U^A)$ as the shift vector.

A Schwarzschild geometry in outgoing Eddington-Finklestein coordinates is given by the choice $\beta = 0$, $V = r - 2m$, $U^A = 0$ and $h_{AB} = q_{AB}$. In the nonspherically symmetric case, in order to computationally treat derivatives of tensor fields on the sphere, we introduce two stereographic coordinate patches with a complex unit sphere dyad vector satisfying $q_{AB} = q_{(A}\bar{q}_{B)}$. This allows use of the spin weight *eth* operator [7)] to express the covariant derivatives of tensor fields on the sphere as spin-weighted fields. Our computational *eth* formalism [8)] accurately calculates covariant derivatives in spherical coordinates.

The conformal 2-metric h_{AB} can be represented by its dyad component $J = h_{AB}q^A q^B/2$. The role of h_{AB} as the null hypersurface data for the characteristic initial value problem can thus be transferred to J. The Einstein equations impose no constraints so that the complex function J encodes the two degrees of freedom of the gravitational field of the null hypersurface.

The evolution algorithm is a computational version of the mixed initial value problem based upon a worldtube and a null hypersurface [9)]. Consider a convex worldtube whose interior contains the sources and whose exterior is an asymptotically flat region of spacetime. If such a worldtube is sufficiently large it admits a slicing whose outgoing normal null hypersurfaces extend to infinity without developing caustics. Given J on the initial null hypersurface, Einstein's equations propagate data from the world tube outward along the outgoing characteristics to determine the exterior space-time. The required worldtube data consists of the conformal 2-geometry of a foliation of the the worldtube and the mass and angular momentum aspects of the initial slice. There are constraint equations on the worldtube which are hyperbolic versions of the elliptic constraints of the Cauchy problem. These constraints are generalized mass and angular momentum conservation laws. In addition, the lapse and shift associated with the foliation of the world tube represent gauge freedom which must be specified.

The worldtube can be shrunk to a nonsingular worldline, in which case the conservation equations reduce to regularity conditions. The code was first implemented this way in axisymmetric form [10]. However, this has high computational cost because of the Courant-Friedrich-Lewy condition which forces a small time step near the vertex of the null cone. In the 3-dimensional case, this would make evolution computationally unfeasible on a uniform grid. This perhaps can be circumvented by an adaptive grid but at present, 3-dimensional null cone evolution is only feasible in th exterior of a worldtube. The world tube can be null, as well as timelike, which allows important application to black holes.

Given worldtube data and initial data, the evolution provides the waveform at future null infinity $\mathcal{I}^+$ - first in a super-accelerated frame determined by the gauge conditions adopted on the world tube and then converted to an asymptotic inertial frame to give the Bondi news function, whose real and imaginary parts are the standard plus and cross polarization modes. This has been tested to be second order accurate in grid size in a wide number of test beds including linearized waves and nonlinear waves propagating outside a black hole (which are independently constructed by solving the Robinson-Trautman equation) [4].

§2. Black holes

The Pitt Null Code is a new tool for the accurate simulation of highly curved space-times. As is historically asked with the discovery of new solutions to Einstein's equations: "What can you learn from them?" The crucial issue in this question is the physical relevance of the boundary data determining the space-time, in particular the worldtube data. We would be in great shape if we knew the right data on a worldtube surrounding the inspiral of a binary black hole. But the determination of such data is essentially the guts of the binary black hole problem.

2.1. *Nonlinear scattering off a black hole*

One simple choice of worldtube is supplied by the ingoing $r = 2m$ hypersurface in a Schwarzschild spacetime (the white hole horizon). The induced worldtube data automatically satisfies the conservation conditions. With this worldtube data, we pose the nonlinear version of the classic perturbative problem [11] of gravitational wave scattering off a Schwarzschild black hole by setting initial data on an outgoing null hypersurface extending to $\mathcal{I}^+$ consisting of an ingoing pulse of compact radial support with various angular modes.

We have calculated the news function radiated from this system [4]. In the perturbative regime, the news results from the backscattering of the incoming pulse off the effective potential of the interior Schwarzschild black hole [11] and, as expected, scales linearly with the amplitude of the incoming pulse. However, in the nonlinear regime, the news scales stronger than linearly with amplitude and the waveform reveals nonlinear generation of additional modes. In this regime, the mass of the system is dominated by the incoming pulse, which essentially backscatters off itself in a nonlinear way. In the extreme nonlinear regime the total back scattered energy radiated to $\mathcal{I}^+$ is itself much larger than the mass of the interior black hole. In dimensionless units the Bondi news $= 400$, corresponding to a radiation power of 10^{13} solar masses/sec.

2.2. *Radiation from the capture of matter by a black hole*

We have incorporated a crude hydrodynamic code for a perfect fluid into the null code. The combined 3-dimensional null code has been tested for stability and accuracy to verify that nothing breaks, at least in the regime that hydrodynamical shocks do not form. The results establish the feasibility of a characteristic *matter plus gravity* evolution [12]. We used the code to simulate a localized blob of matter falling into a black hole, verifying that the motion of the peak of the blob approximated a geodesic and monitoring the waveform of the emitted gravitational radiation at $\mathcal{I}^+$.

This simulation is a prototype of a neutron star orbiting a black hole. Thus a refined characteristic hydrodynamic code would open the way to explore an important astrophysical problem. Recently, such a high resolution shock-capturing code has been successfully implemented in the null formulation in a background geometry in the case of spherical symmetry [13] and this code is being generalized to 3-dimensions. We are planning a collaborative project to combine our characteristic gravitational code with this characteristic hydro code.

2.3. *Black hole in a box*

Characteristic evolution can also be based upon a family of *ingoing* null cones with data given on a worldtube at their *outer* boundary and on an initial *ingoing* null cone. We implemented such a code to evolve black holes in the region interior to the worldtube by locating a marginally trapped surface (MTS) on the ingoing cones and excising the singular region inside it [14]. This is a characteristic version of the conventional strategy for evolving black holes by excising the interior of an apparent horizon, as initially suggested by W. Unruh [15]. The ingoing null code locates the MTS, tracks it during the evolution and stably excises its interior from the numerical grid.

We used this code to simulate a distorted "black hole in a box" [16]. Data at the outer worldtube was induced from a Schwarzschild or Kerr spacetime but the worldtube was allowed to move relative to the stationary trajectories; i.e. with respect to the grid the worldtube is fixed but the black hole moves inside it. The initial null data consisted of a pulse of radiation which subsequently travels outward to the worldtube where it is reflected back toward the black hole. The approach of the system to equilibrium was monitored by the area of the MTS, which is equivalent to its Hawking mass. When the worldtube is stationary (static or rotating in place), the distorted black hole inside evolves to equilibrium with the boundary. A boost or other motion of the worldtube with respect to the black hole does not affect this result. The marginally trapped surface always reaches equilibrium with the outer boundary, confirming that the motion of the boundary is "pure gauge".

The code essentially runs "forever" even when the worldtube wobbles with respect to the black hole to produce artificial periodic time dependence. "Forever" cannot be rigorously attained in any finite simulation but we appeal to the characteristic time necessary to obtain accurate waveforms for the inspiral and merger of two black holes. The inspiral from a a post-Newtonian orbit at $r = 20M$ to the innermost stable orbit at $6M$ lasts $\approx 10,000M$. We have successfully evolved an initially distorted, wobbling black hole for a time of $60,000M$, longer than needed for a smooth transition from the post-Newtonian regime to merger, if this success could be duplicated in the ultimate binary black hole code [17)]. This capability is important because the coordinates used to simulate a binary may not become exactly stationary after merger and ring-down to final equilibrium.

This is the most demanding black hole simulation achieved by any code to date. Results can be viewed at http://artemis.phyast.pitt.edu/animations.

2.4. *Cauchy-Characteristic-Matching (CCM)*

The sole weakness of characteristic evolution is its limitation to regions admitting a nonsingular foliation by global null cones. This problem arises when focusing by gravity reverses the expansion of the null rays and produces caustics. Focusing can be magnified by locating the lens far from the source. In a spacetime with miniscule curvature containing, say, two peanuts, no global null cones exist if the peanuts are sufficiently far apart (approximately 10^{10} light years). In that case, no matter where the vertex of a cone is placed, the rays heading toward one of the peanuts would be refocused. It is not strong curvature by itself but the combination of curvature and large scale inhomogeneities that makes caustics unavoidable. In the spherically symmetric gravitational collapse of matter, the light cones from the center of symmetry extend smoothly to null infinity until the center enters the event horizon. Global null cones with point vertices exist even for a binary neutron star with orbital separation less than 5 neutron star radii.

Given the appropriate worldtube data for a binary system in its interior, characteristic evolution can supply the exterior spacetime and radiated waveform. But determination of the worldtube data for the complete evolution of a binary system requires the Cauchy evolution of the interior. CCM is a matched Cauchy-characteristic evolution designed to tackle such radiation problems [18)]. The two evolutions are matched across a worldtube, with the Cauchy domain supplying the boundary values for characteristic evolution and vice versa. Just as several coordinate patches are necessary to describe a spacetime with nontrivial topology, an effective attack on the binary black hole problem is to use CCM to patch together regions of spacetime handled by different algorithms.

Because of the singular time dependence of the compactified version of spatial infinity, a globally compactified approach to the Cauchy problem is not feasible numerically. Instead, the grid is terminated at a finite boundary, where the necessity of an artificial boundary condition, such as the Sommerfeld condition, leads to strong back reflection in the case of high asymmetry. Here the strengths and weaknesses of the Cauchy and characteristic approaches complement themselves in a fortuitous way.

The potential advantages of CCM over traditional artificial boundary conditions are: (1) Accurate waveform and polarization properties at infinity; (2) Computational efficiency for radiation problems in terms of both the grid domain and algorithm; (3) Elimination of an artificial outer boundary condition on the Cauchy problem, which eliminates contamination from back reflection and clarifies the global initial value problem; and (4) A global picture of the spacetime exterior to the horizon. These advantages have been realized in the following two tests of CCM.

First, CCM has been applied to nonlinear scalar waves propagating in a 3-dimensional Euclidean space, matching a spherical null grid to a Cartesian Cauchy grid [19)]. Performance of the matching algorithm was compared with the prime examples of both local and nonlocal radiation boundary conditions proposed in the computational physics literature. For linear problems, CCM outperformed all local boundary conditions and was about as accurate (for the same grid resolution) as the best nonlocal conditions. However, the computational cost of conventional nonlocal conditions is many times that of matching so that matching algorithm can be used with finer grids to yield higher accuracy. For strongly nonlinear problems, matching was *significantly more accurate* than all other methods tested. This is because all other currently available outer boundary conditions are based on linearizing the equations in the far field, while CCM consistently takes nonlinearity into account in both interior and exterior regions.

Second, a two-fold version of CCM has been used to evolve globally the spherical collapse of a self-gravitating scalar field onto a black hole [14)]. Null evolution on ingoing light cones, bounded on the inside by a dynamically tracked MTS, are matched at their outer boundary to the inner boundary of a Cauchy evolution. In turn, the outer boundary of the Cauchy evolution is matched to an exterior characteristic evolution extending to infinity. This study reveals further advantages of CCM for dealing with black holes. Because the null evolution is extended to infinity, the appearance of infinite redshifts acts as an automatic mechanism for stopping the evolution when the event horizon is reached. In addition, the null approach provides a mechanism for reducing extraneous incoming radiation from the initial data, thus sharpening the physical model behind the emitted waveforms.

The success of these scalar models suggests an ultimate application of CCM to the binary black hole problem. Two disjoint characteristic evolutions based upon ingoing light cones are matched across worldtubes Γ_1 and Γ_2 to a Cauchy evolution of the region between them, as illustrated in Fig. 1. The outer boundary Γ of the Cauchy region is matched to an exterior characteristic evolution based upon outgoing light cones extending to infinity, where the waveform is calculated. There

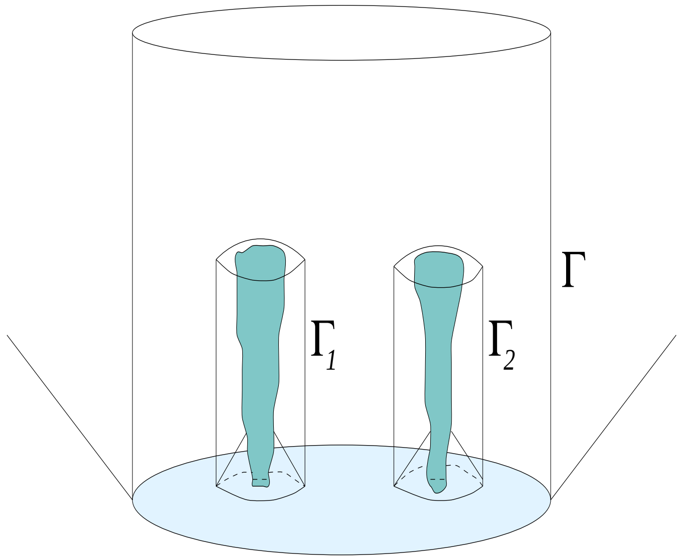

Fig. 1. A matching scheme for two orbiting black holes (in a co-rotating frame which eliminates the major source of time dependence). This global strategy has been successfully implemented for spherically symmetric self-gravitating scalar waves evolving in a single black hole spacetime.

are major computational advantages in posing the Cauchy evolution in a frame co-rotating with the orbiting black holes. Indeed, such a description may be necessary in order to keep the numerical grid from being intrinsically twisted. In co-orbiting coordinates the individual holes would be tracked as they wobble inside the inner matching worldtubes.

All the pieces for such an attack on the binary black hole problem are presently in place and tested except for the long term stability of 3-dimensional CCM for general relativity. A CCM module incorporating all the necessary geometric transformations between curved space versions of a Cartesian Cauchy grid and a spherical null grid has been written and thoroughly debugged [20)]. However, at present there are instabilities of a type not found in the simpler applications of CCM but similar to those arising from other choices of Cauchy boundary conditions.

§3. Characteristic treatment of colliding black holes

An event horizon is a special type of null hypersurface whose light rays emerge from an initial caustic-crossover region, where the horizon forms, and then expand and asymptotically "hover" at a finite constant surface area. In geometric optics, the chief consideration in the study of wavefronts is the 2-dimensional caustic surface where neighboring rays meet and classically the intensity would be infinite. The elementary caustics have been classified in terms of catastrophe optics and, equivalently, in terms of singular maps of dynamical systems. Because of the inherent structural stability of this classification, it is unchanged by the tidal distortions produced by spacetime curvature. A caustic set consisting of a single point is structurally unstable, i.e. a small perturbation can produce qualitative changes in the features. This applies to the point caustic associated with black hole formation in the Oppenheimer-Snyder model of spherically symmetric collapse. Only the elementary caustics can play a role in generic horizon formation.

In contrast to caustics, the classical intensity is finite on the crossover set, where distinct light rays traced back on the horizon collide. The crossover set is the dominant structure in horizon formation. Generically, it is a 2-dimensional surface bounded by a curve of caustics. The generic properties of crossover surfaces are nonlocal and have not been classified. After describing the geometry of axisymmetric horizons, I will discuss some generic properties of the simplest type of crossover set and show how they lead to topological features in the merger of black holes quite different from the axisymmetric head-on collision.

3.1. *Axisymmetric Horizons*

We have identified the caustic structure of the horizon found in simulations of the axisymmetric head-on collision of two black holes [21]. Cusps and folds are the only structurally stable caustics in the axisymmetric case. Although the theory of elementary caustics is local, we have used it to piece together a global model of the "trousers" shaped horizon obtained in the numerical simulations. Some surprising spacetime features emerge. As schematically illustrated in Fig. 2, the light rays generating the horizon originate on a spacelike seam $\mathcal{X}$ (a set of crossover points) running along the inside of the trouser legs. The trousers are bowlegged with no special sharpness at the crotch. For black holes formed at a finite time by the collapse of matter, the crossover seam extends around the bottom of each trouser and slightly up the outer side where it becomes asymptotically null and terminates at a cusp, where neighboring rays focus.

For vacuum black holes, the axisymmetry implies the vanishing of focusing along the umbilical (shear-free) rays on the axis of symmetry. As a result, the individual black holes in the head-on collision are eternal. The trouser legs extend forever into the past, although their cross-sectional area becomes vanishingly small (in the past) as all rays except those on the axis leave the horizon. Asymptotically, the initial black holes holes are spherical due to the umbilical nature of the axis.

Surprisingly, a trousers shaped horizon also appears in the axisymmetric gravitational collapse of a rotating cluster [22), 23)], except now the rotational symmetry axis in Fig. 2 must be identified differently. In the head-on collision, the symmetry axis lies in the x-direction of Fig. 2. After replacing the t-direction in the figure by the suppressed spatial dimension (the y-direction), a rotation about the x-axis produces two spherical black holes at time T_1. Also, in the head-on collision, the crossover points $\mathcal{X}$ in Fig. 2 lie on the rotation axis so that they generate a crossover line under rotation.

For the horizon formed by the rotating cluster, the symmetry axis in Fig. 2 lies the z-direction. Now replacing the t-direction in the figure by the suppressed y-direction, a rotation about the symmetry axis produces a single toroidal black hole at time T_1. Furthermore, the crossover points generate a disc with a circular

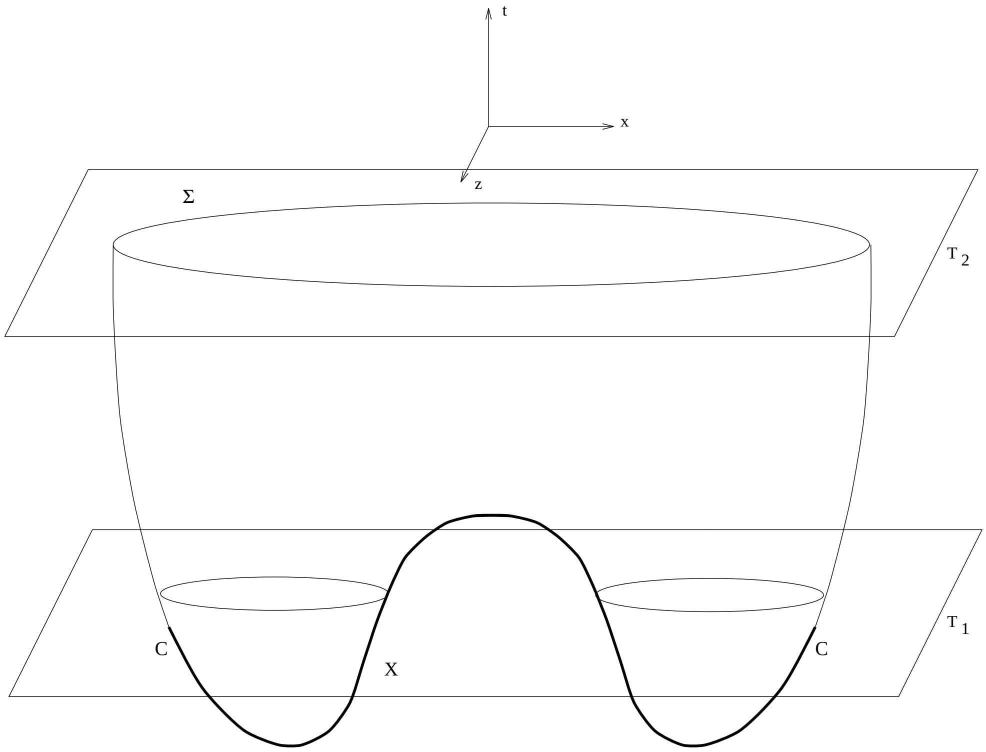

Fig. 2. Spacetime picture of a pair-of-pants horizon emerging from the (thicker shaded) crossover points $\mathcal{X}$ bounded by cusps C.

boundary of cusps. In both models, at the later time T_2 the horizon consists of a single spherical black hole Σ, which in one case results from the collision of two black holes and in the other from the hole in the torus closing up.

The discovery of temporarily toroidal black holes raised the concern [24)] of a potential mechanism for violating the topological censorship theorem, which requires that any two causal curves extending from past to future null infinity be deformable into each other [25)]. The key issue is whether two twins starting at the same spacetime point outside the horizon could travel to another spacetime point outside the horizon by homotopically inequivalent causal paths. A light ray traveling from the infinite past through the hole in the torus and back out to future null infinity would not be causally deformable to a light ray that altogether skirts the horizon if the hole in the torus were *too long lived.* However, because the intersection of two null hypersurfaces is spacelike, the crossover set $\mathcal{X}$ in Fig. 2 must also be spacelike so that the the hole closes up superluminally. Consequently, a causal curve passing inside the hole at a given time can be slipped below the bottom of a trouser leg to yield a causal curve lying entirely outside the hole in the torus.

3.2. *The conformal model of black hole coalescence*

Remarkably, the horizon geometries found in the axisymmetric numerical simulations can be independently obtained from an analytic model based upon the conformal rescaling of a flat space null hypersurface [26]. For a null hypersurface emanating from a prolate spheroid (cigar shaped ellipsoid of revolution), the conformal model reproduces the pair-of-pants found in the head-on collision of black holes. For an oblate spheroid (pancake shaped ellipsoid of revolution), it yields the temporarily toroidal horizon found in the collapse of a rotating cluster. But the model is not confined to axisymmetry and reveals features of the axisymmetric head-on collision which are not generic. In a generic black hole merger there is always a toroidal phase [27], [28]. The analytic nature of the model also provides insight into the saddle shape geometry at the crotch of the trousers and implications for the hoop conjecture.

The conformal model treats the horizon in stand-alone fashion as a 3-dimensional manifold endowed with a degenerate metric γ_{ab} and affine parameter t along its null rays. The metric is obtained from the conformal mapping $\gamma_{ab} = \Omega^2 \hat{\gamma}_{ab}$ of the intrinsic metric $\hat{\gamma}_{ab}$ of a flat space null hypersurface emanating from a convex surface $\mathcal{S}_0$ embedded at constant time in Minkowski space. The horizon is identified with the null hypersurface formed by the inner branch of the boundary of the past of $\mathcal{S}_0$, and its extension into to the future to $\mathcal{I}^+$. The flat space null hypersurface expands forever as its affine parameter $\hat{t}$ (given by Minkowski time) increases but the conformal factor is chosen to stop the expansion so that the cross-sectional area of the black hole approaches a finite limit in the future. At the same time, the Raychaudhuri equation (which governs the growth of surface area) forces a nonlinear relation between the affine parameters t and $\hat{t}$ which introduces the nontrivial topology of the affine slices of the black hole horizon.

The *number* of black holes or the topology of a black hole at a given time is not conventionally defined in terms of such a stand-alone model but in terms of the number and type of disjoint intersections with a Cauchy hypersurface. In the conformal model of a black hole collision the notion of "two holes" and their merger arises intrinsically from the affine foliation of the horizon. It is the relative distortion between the affine parameters t and $\hat{t}$ brought about by curved space focusing which gives rise to the trousers shape.

In the conformal model, the caustic-crossover set for the black hole horizon inherits properties from its flat space counterpart. The generic features of the model stem from the structurally stable properties of this set which are preserved under arbitrary smooth perturbations of $\mathcal{S}_0$. Classification of the generic properties of the crossover set $\mathcal{X}$ is a global problem. The simplest case to consider is when $\mathcal{X}$ is a double-crossover set consisting purely of points at which precisely two rays intersect. This includes the horizon formed when $\mathcal{S}_0$ is ellipsoidal, which we have used to study black hole coalescence analytically.

Consider then the caustic-crossover structure of the wavefront emanating backward in time from a smooth convex surface $\mathcal{S}_0$ embedded at constant time $\hat{t} = 0$

in Minkowski space. The rays tracing out the wavefront generate a smooth null hypersurface until they reach past endpoints on the boundary of the past of $\mathcal{S}_0$. The endpoints consist of a set of caustic points $\mathcal{C}$, where neighboring rays focus, and a set of crossover points $\mathcal{X}$, where distinct null rays collide. We have established the following generic caustic-crossover properties of such flat space null hypersurfaces.[27)]

Generic Property 1: *A caustic point is not also a crossover point.*

Generic Property 2: *Considered as a subset of Minkowski space, the double-crossover set is a smooth, open, spacelike 2-surface.*

Generic Property 3: *In the absence of triple or higher order crossovers, the caustic set forms a compact boundary to the crossover set (considered as a subset of the horizon). The tangent space of the crossover set joins continuously to the tangent space of the null portion of the horizon at this caustic boundary.*

Generic Property 4: *A crossover point lies at the intersection of at most four rays.*

These generic properties are violated in the case of a spherical wavefront, where the crossover set and caustic set both degenerate to a common point. They are also violated in the prolate spheroidal case, where the crossover set is a curve of non-generic caustics, which is bounded at each end by a non-degenerate cusp type caustic[23)]. These generic properties are satisfied when $\mathcal{S}_0$ is a triaxial ellipsoid (no degeneracies in the lengths of the major axes). For this case, the crossover set is smooth and connected, consists purely of double crossover points and goes asymptotically null at an elliptical caustic boundary where it joins smoothly to the null hypersurface. More complicated examples would allow higher order crossovers. However, Property 4 limits the complexity of a generic crossover set arising in Minkowski space from a smooth, convex surface $\mathcal{S}_0$.

The conformal construction of the curved space event horizon preserves the structure of the underlying flat space crossover set if appropriate identifications can be made. This is the case in the triaxial ellipsoid model because reflection symmetry of the ellipsoid supplies the consistency conditions for the conformally transformed metric to be single valued at the crossover points. The generic properties of an arbitrary event horizon are important features of black hole physics. Certain of the flat space properties generalize easily to curved space, e.g. the spacelike nature of the crossover set. However, for those properties established using specifically Euclidean constructions, e.g. Generic Property 4, the generalization is not obvious.

The black hole event horizon associated with a triaxial ellipsoid reveals new features which are not seen in the axisymmetric head-on collision of two black holes corresponding to the degenerate case of a prolate spheroid. If the degeneracy is slightly broken, the physical picture of a black hole collision remains. The individual black holes form with spherical topology but, as illustrated in Fig. 3, as the holes approach, tidal distortion produces sharp pincers just prior to merger. At merger, the pincers join to form a single temporarily toroidal black hole, as illustrated in Fig. 4. The inner hole of the torus subsequently closes up (superluminally) to produce first a peanut shaped black hole and finally a spherical black hole. In the degenerate axisymmetric limit, the pincers reduce to a point so that the individual holes have teardrop shape and they merge without a toroidal phase. Details of this merger can be viewed at http://artemis.phyast.pitt.edu/animations.

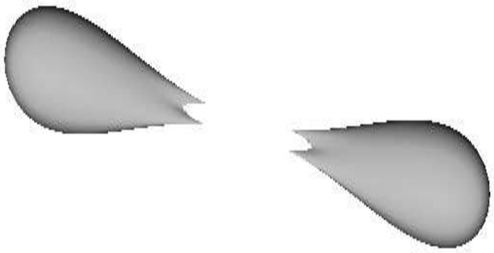

Fig. 3. Prelude to a generic collision of black holes.

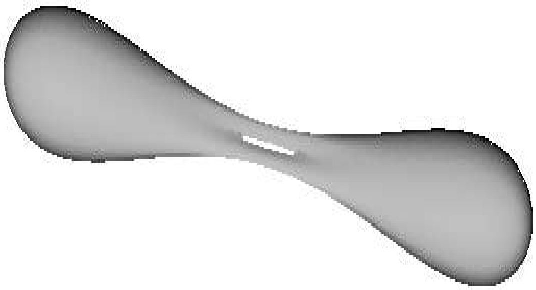

Fig. 4. Temporarily toroidal black hole produced by the merger of two black holes.

§4. Waveforms from coalescing black holes

A prime application of the conformal horizon model is the calculation of the waveform emitted by coalescing black holes using the null code. The model supplies the data on the horizon necessary for an evolution of the exterior spacetime. The evolution is carried out along a family of ingoing null hypersurfaces which intersect the horizon in topological spheres. The evolution is restricted to the period from merger to ringdown, for otherwise the ingoing null hypersurfaces would intersect the horizon in disjoint pieces. In the strategy being pursued, the remaining data necessary for the evolution is the conformal geometry of the *final* ingoing null hypersurface in the family, which is taken to approximate $\mathcal{I}^+$; i.e. the missing piece of data is essentially the outgoing waveform! The evolution then proceeds *backward* in time to determine the exterior spacetime in the post-merger era. Since the outgoing waveform is a necessary piece of data, it might seem that this is a circular approach. However, as explained below, a variant of this strategy supplies the physically correct waveform.

The evolution is an implementation of the construction of a vacuum space-time based upon the characteristic initial value problem posed on an intersecting pair of null hypersurfaces [29)-31)]. Here one of the null hypersurfaces is the event horizon $\mathcal{H}$. The other is the ingoing null hypersurface J^+ which intersects $\mathcal{H}$ in the topologically spherical surface $\mathcal{S}_0$. The conformal model provides the geometry of $\mathcal{S}_0$ and induces the necessary data on the horizon for, say, a black hole collision, in terms of the geometry of a flat space ellipsoidal null hypersurface. We locate $\mathcal{S}_0$ as a cross-section of the event horizon at a late quasi-stationary time so that the ingoing null hypersurface J^+ approximates $\mathcal{I}^+$, as illustrated in Fig. 5. Part of the horizon data is a quantity, called the twist [30), 31)], which is analogous to the extrinsic curvature in the Cauchy problem. The twist need only be specified on $\mathcal{S}_0$ and then a component of Einstein's equations propagates it along the generators of the horizon by a simple ordinary differential equation. The initial value of the twist can be specified in terms of its asymptotic value in the final Kerr black hole.

The data is completed by specifying the conformal geometry of J^+ to the past of $\mathcal{S}_0$. There are existence theorems for solutions to this double null initial value problem [32)-34)]. Although global issues remain unresolved, these theorems guarantee existence in some neighborhood of the initial null hypersurfaces. Thus, referring to Fig. 5, the above data determines a solution of the vacuum equations in the a domain of dependence D^- to the past of S_0. (In the figure we assume the absence of singularities in the past of S_0.)

This double null version of the characteristic initial value problem is equivalent to the world tube - null cone problem [9)] in the case where the world tube is null. This is precisely the problem for which the PITT code provides a stable evolution which is highly accurate even when run on a single processor. The code also determines the space-time in a subregion of the domain of dependence D^-, thus providing the local existence of a space-time satisfying Einstein's equations in the finite difference approximation. Ideally, the numerical evolution would extend throughout the domain of dependence D^- but that is more problematical. The evolution algorithm requires the foliation of D^- by a one parameter family of null hypersurfaces J_v. However, referring to Fig. 5, such a foliation becomes singular at J_M in the portion of the horizon corresponding to the black hole merger. Thus only the post-merger space-time is determined by the evolution. This problem is due to a coordinate singularity, not a physical singularity, and from a mathematical point-of-view the space-time is extendable to earlier times; but just how much earlier cannot be answered by the numerical evolution. Irregardless, the conformal model of the horizon for a black hole collision can be used as characteristic initial data to construct a vacuum space-time covering a very interesting nonlinear domain from merger to ringdown.

To the extent that S_0 lies in the late quasi-stationary era of the horizon, the null hypersurface J^+ approximates future null infinity. So, for a horizon describing a binary black hole, the physically appropriate characteristic data on J^+ describes the sought after outgoing radiation emitted during the black hole merger. However, unlike the Cauchy problem, the data on a null hypersurface for a solution of Einstein's equations can be posed in a constraint free manner and any such data leads to a vacuum solution. This allows us to compute the merger-ringdown waveform in the following manner. Since the outgoing waveform is a necessary but arbitrary part of the data, we first proceed by setting the outgoing waveform to zero. Next we evolve backward in time to calculate the incoming radiation entering from $\mathcal{I}^-$. There is no problem setting up the null data on J^+ to be asymptotically flat so that $\mathcal{I}^-$ exists, at least for some finite evolution time. This radiation entering from $\mathcal{I}^-$ is eventually absorbed by the black hole. The idea is to cancel it out by adding the right amount of outgoing radiation on J^+. If this calculation were carried out in the perturbative regime of a a Schwarzschild or Kerr background geometry, as in the close approximation[35], this can be accomplished by taking advantage of the time reflection symmetry.

In order to illustrate the simplicity of this approach, consider the null evolution of a perturbation of a Schwarzschild background. The numerical evolution provides the complex metric quantity $J(v, r, \theta, \phi)$ which determines the conformal null geometry of the ingoing foliation. The incoming news function $N_{in}(v, \theta, \phi)$ is obtained from the asymptotic dependence of J near $\mathcal{I}^+$. By construction, there is no outgoing radiation in this perturbation. Application of the time reflection symmetry of the Schwarzschild background provides the alternative perturbation given by

$$J_{out}(-t + r^*, r, \theta, \phi) = J_{in}(t + r^*, r, \theta, \phi) \qquad (4{\cdot}1)$$

where $v = t + r^*$ and $u = t - r^*$ are advanced and retarded Schwarzschild times.

The perturbation J_{out} contains outgoing radiation but no incoming radiation. Thus the perturbation J_{out} satisfies appropriate boundary conditions for the collision of two black holes, with no incoming fields at least in the post-merger period. The corresponding news function N_{out} for the outgoing waveform is obtained from N_{in} (supplied by the simulation) by a substitution analogous to Eq. (4·1).

In the same perturbative regime as the close approximation, this numerical procedure supplies the outgoing waveform under the physically appropriate condition of no ingoing radiation. More generally, beyond the perturbative regime, the appropriate outgoing waveform can be obtained by a more complicated inverse scattering procedure which minimizes the incoming radiation with respect to variations of the null data on J^+.

Thus the conformal horizon model combined with the null evolution code offers a new way to calculate the merger-ringdown waveform from coalescing black holes. Because this is an unexplored area of binary black hole physics we are beginning this study in the simple case of a head-on collision, where the close approximation waveform has been calculated. This will provide some preliminary physical checks for extending the work into the nonlinear and nonaxisymmetric case where inspiraling black holes can be treated. Preliminary calculations for the head-on-collision show

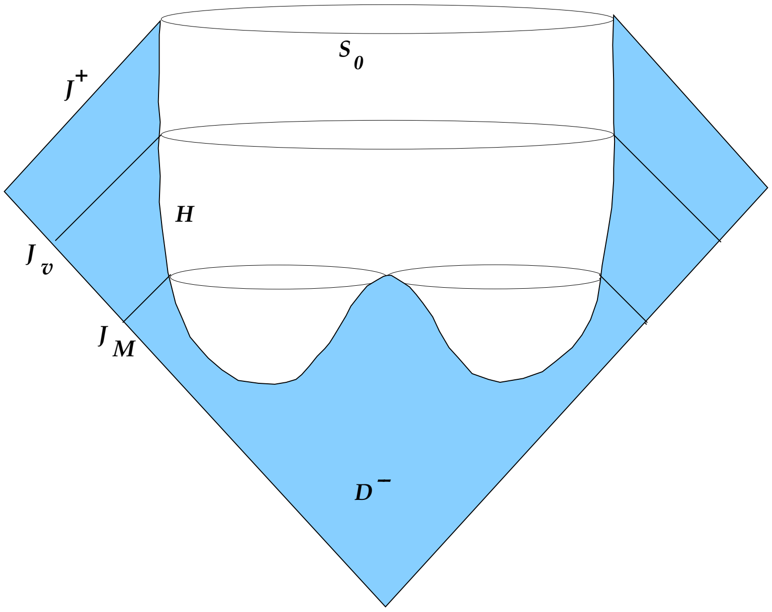

Fig. 5. A portion of the space-time prior to the coalescence of two black holes. The parameter v labels the advanced time on a family of incoming null hypersurfaces. D^- is the domain of dependence of characteristic data given on the event horizon $\mathcal{H}$ and on J^+.

that at late times the waveform is entirely quadrupole ($\ell = 2$) in agreement with the close approximation, but that a strong $\ell = 4$ mode exists just after merger.

It is interesting to note that turned upside down (time reversed), Fig. 5 describes a white hole fission. In that case, setting the null data to zero on J_v corresponds to no ingoing radiation and the numerical calculation supplies the outgoing waveform from the fission with the physically correct boundary condition.. It is this white hole fission waveform that is used to generate the physically correct waveform for a black hole merger, but the solution to the fission problem is of itself at least of academic interest.

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