The Eikonal equation in asymptotically flat space–times

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In an arbitrary Lorentzian manifold we provide a description for the construction of null surfaces and their associated singularities, via solutions of the Eikonal equation. In particular, we study the singularities of the past light-cones from points on null infinity, the future light-cones from arbitrary interior points and the intersection of these with null infinity and unifying relationships between the different singularities. The starting point for this work is the assumption of a known family of solutions to the Eikonal equation. The work is based on the standard theory of singularities of smooth maps by Arnold and his colleagues. Though the work is intended to stand on its own, it can be thought of as being closely related to the recently developed null surface reformulation of GR. © *1999 American Institute of Physics.* [S0022-2488(99)01302-X]

I. INTRODUCTION

In a recent work[1] we studied properties of solutions of the flat-space–time Eikonal equation, namely,

$$\eta^{ab}\partial_a S \partial_b S = 0,$$

whose level ''surfaces'' [$S=S(x^a)=\text{const.}$], are, by definition, null (or characteristic) three-surfaces. These level surfaces, called by Arnold[2,3] ''big wave fronts,'' can have self-intersections and need not be smooth everywhere. In particular we were concerned with finding the analytic form (described parametrically) of the general solution to the equation, studying its level surfaces and the ''small (two-dimensional) wave fronts,'' (i.e., the intersection of a three surface with a big wave front) and then analyzing some of the resulting structures; the caustics of the full solution (three-dimensional), the singularities of the big wave front (two-dimensional), and the singularities of the ''small wave fronts'' (one-dimensional). These singularities are defined, respectively, by the intersection of a big wave front and the small wave front with the caustic surface. A special application of these ideas was to the study of 2-parameter families of solutions to the Eikonal equation from which it was possible to see an alternative analytic treatment of the structure of the singularities. This latter point of view plays an important role in a recent reformulation of GR known as the null surface formulation.[4,5]

In the present paper, we extend the ideas from the Minkowski case to,[1] first to arbitrary Lorentzian space–times and then specialize them to asymptotically flat space–times. There are two main reasons for doing this: (1) We want to understand in detail the structure of light-cones in the large, i.e., globally, in arbitrary space–times which are of great relevance to the general theory of gravitational lensing and (2) a recent reformulation of GR in terms of families of

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characteristic surfaces requires a deeper understanding of the singularities of the big and small wave fronts.

In Sec. II, we will show how from any arbitrary, but given, two parameter family of solutions of the curved-space Eikonal equation, any arbitrary characteristic surface can be constructed.

This construction will, in Sec. III, be specialized to an asymptotically flat space–time where the two-parameter family is chosen in a special way; namely, they are the family of past light cones from all the points on null infinity, $\mathfrak{I}^+$. Directly in terms of this fiducial family we can express any characteristic surface and in particular, we can express the light-cone, $\mathfrak{C}_x$, of any interior point x^a. Of particular interest is the singularity structure of the cones, $\mathfrak{C}_x$, which can be analyzed in terms of the variables of the fiducial family.

In Sec. IV, we will study the particular class of small wave fronts (two-dimensional) defined by the intersection of the three-dimensional cones, $\mathfrak{C}_x$, with the null surface $\mathfrak{I}^+$, i.e., the so-called light-cone cuts $\mathfrak{c}(x^a)$ of $\mathfrak{I}^+$. In particular we will be interested in finding (via Arnold's theory of Lagrange and Legendre maps[2,3,6,7]) the appropriate tools and variables to describe the singularities of these light-cone cuts.

Finally in Sec. V, we return to an issue that we deliberately postponed. We took, in Sec. III, a fiducial family of solutions of the Eikonal and used them to study the singularities of other characteristic surfaces but we avoided any discussion of the singularities of the fiducial family itself. The reason for the postponement is that this discussion is more complicated and difficult than the earlier ones and uses, in addition, different techniques; namely the equations of geodesic deviation.

The present work is partially intended to fill in the details of an earlier brief work in the Twistor Newsletter (TN43, 1997), where we anticipated some of these results.

II. SOLUTIONS OF THE EIKONAL EQUATION IN CURVED SPACE

In this section we will treat the Eikonal equation in a general curved Lorentzian space–time, $(g,\mathfrak{M})$, i.e.,

$$g^{ab}(x^a)\partial_a S\partial_b S=0 \tag{1}$$

and show how, if a special class of solutions is known, *any* solution can be easily constructed. An important special case of this will be the construction of any single characteristic surface, i.e., a level surface of *some* S, ''a big wave front.''

The difficult task (and it is very difficult, where perturbation techniques must be relied on) is to produce this special class. Specifically, the special class will be a two-parameter family of solutions, $S_0=Z(x^a,\zeta,\overline{\zeta})$ where the parameters are the complex stereographic coordinates on the sphere, S^2, and the null covector, $p_a=\partial_a Z(x^a,\zeta,\overline{\zeta})$ ranges over the entire light-cone at each point x^a as $(\zeta,\overline{\zeta})$ ranges over S^2. Later, in asymptotically flat space–times, we will make a unique choice of this family.

Now assuming that an allowable $Z(x^a,\zeta,\overline{\zeta})$ is known we can produce an arbitrary solution in the following fashion:[8] first, we rescale Z with a constant β and add to Z an arbitrary function, at least once differentiable, of $(\beta,\zeta,\overline{\zeta})$ and then extremize it with respect to the $(\beta,\zeta,\overline{\zeta})$, i.e., we have

$$S=\beta Z(x^a,\zeta,\overline{\zeta})-h(\beta,\zeta,\overline{\zeta}) \tag{2}$$

with

$$\beta\partial_\zeta Z-\partial_\zeta h=0,\quad \beta\partial_{\overline{\zeta}}Z-\partial_{\overline{\zeta}}h=0,\quad Z-\partial_\beta h=0. \tag{3}$$

For the simplicity of the immediate discussion (though the issue is an important one), we assume that the latter three equations can be solved for the $(\beta,\zeta,\overline{\zeta})$ as functions of the x^a, i.e., with

$$(\beta,\zeta,\overline{\zeta}) = (B(x^a),\Upsilon(x^a),\overline{\Upsilon}(x^a)), \tag{4}$$

then on substitution back into Eq. (2), the resulting function of x^a also satisfies the Eikonal equation. To see this we have, from (3) that

$$\partial_a S = \beta\partial_a Z + (Z - \partial_\beta h)\partial\beta/\partial x^a + (\beta\partial_\zeta Z - \partial_\zeta h)\partial\zeta/\partial x^a + (\beta\partial_{\overline{\zeta}} Z - \partial_{\overline{\zeta}} h)\partial\overline{\zeta}/\partial x^a = \beta\partial_a Z, \tag{5}$$

which satisfies the Eikonal equation, by the assumption on Z.

The important issue of how to deal with the case when Eq. (3) cannot be solved for $(\beta,\zeta,\overline{\zeta})$ is discussed later in this section.

Though we will not go into the proof one can show that given arbitrary Cauchy data, $S_C(x^i)$ for the Eikonal equation, i.e., a function of three arguments, then it determines the function $h(\beta,\zeta,\overline{\zeta})$. The construction, thus, allows for the general solution to the Eikonal equation.

Remark 1: Since the function $h(\beta,\zeta,\overline{\zeta})$ determines a single solution $S^*(x^a)$ of the Eikonal equation (i.e., a one parameter family of characteristic surfaces) by replacing the $h(\beta,\zeta,\overline{\zeta})$ by the function $h(\beta,\zeta,\overline{\zeta};\eta,\overline{\eta})$ the above construction then produces a two-parameter family of solution of the Eikonal equation, $S^* = Z^*(x^a,\eta,\overline{\eta})$. We thus have that from any special two-parameter family, Z we can construct any other two parameter family that could also be used as the ''special'' family.

We now specialize the construction so that we can obtain any single characteristic surface to be given by $S = u =$ constant; Eqs. (2) and (3) are replaced by the specialization, $\beta = 1$,

$$S = Z(x^a,\zeta,\overline{\zeta}) - h(\zeta,\overline{\zeta}) = u, \tag{6}$$

$$\partial_\zeta Z - \partial_\zeta h = 0, \quad \partial_{\overline{\zeta}} Z - \partial_{\overline{\zeta}} h = 0. \tag{7}$$

Assuming that Eq. (7) can be solved for

$$(\zeta,\overline{\zeta}) = (\Upsilon(x^a),\overline{\Upsilon}(x^a)), \tag{8}$$

that any characteristic surface can be obtained by a judicious choice of $h(\zeta,\overline{\zeta})$ can be seen from the argument that if we begin with any spacelike two-surface, $\mathfrak{G}$, parametrized by the same $(\zeta,\overline{\zeta})$, i.e., given by

$$x^a = x_0^a(\zeta,\overline{\zeta}), \tag{9}$$

we can choose

$$h(\zeta,\overline{\zeta}) = Z(x_0^a(\zeta,\overline{\zeta}),\zeta,\overline{\zeta}). \tag{10}$$

The resulting characteristic surface. $S = 0$, is formed by the null normals to $\mathfrak{G}$ and since any characteristic surface is formed by the null normals to some two-surface we have proven our contention.

Actually, in general, there are lower dimensional regions where Eqs. (7) cannot be solved for the $(\zeta,\overline{\zeta})$ pair. These regions (three dimensional) define the caustics of the solution. The intersection of these caustic regions with any particular level surface of S (big wave front), i.e., with $u = S =$ const defines the ''big wave front'' singularities[7] of Arnold. The intersection of $u = S$ with a generic three-surface, (e.g., a constant time surface) defines a ''small wave front'' while the intersection of the ''small wave front'' with the caustic three-surface, defines the ''small wave front'' singularities. Though for precise usage we should only refer to the full three dimensional caustic region as the ''caustics,'' we, however, will take the liberty of referring to the singularities of either the big or small wave front as the ''caustics.''

These caustic regions (or on the big or small wave front singularities) which are characterized by the inability to solve for the $(\zeta,\bar{\zeta})$ are simply determined from the implicit function theorem, by the condition

$$\hat{D} \equiv \begin{vmatrix} \dfrac{\partial^2(Z(x^a,\zeta,\bar{\zeta})-h(\zeta,\bar{\zeta}))}{\partial\zeta^2} & \dfrac{\partial^2(Z(x^a,\zeta,\bar{\zeta})-h(\zeta,\bar{\zeta}))}{\partial\zeta\partial\bar{\zeta}} \\ \dfrac{\partial^2(Z(x^a,\zeta,\bar{\zeta})-h(\zeta,\bar{\zeta}))}{\partial\zeta\partial\bar{\zeta}} & \dfrac{\partial^2(Z(x^a,\zeta,\bar{\zeta})-h(\zeta,\bar{\zeta}))}{\partial\bar{\zeta}^2} \end{vmatrix} = 0, \tag{11}$$

a condition that will later play a basic role.

To determine the solution S, there is an alternative to solving Eqs. (7) for the $(\zeta,\bar{\zeta})$ that is often more desirable and can be used even when $\hat{D}=0$. Equations (6) and (7) can be considered as defining families of three different 3-surfaces parametrized by the $(\zeta,\bar{\zeta})$ pair. Their intersection defines a family of curves (parametrized by the $(\zeta,\bar{\zeta})$) that are the null geodesics that rule the level surfaces of $u=S$. The equations can always be solved in the following manner: of the four x^a there will be a subset of three of them (say x^i) and the fourth one, say x^* such that

$$x^i = X^i(x^*,u,\zeta,\bar{\zeta}), \tag{12}$$

which are the null geodesics themselves. They define, parametrically, the level surfaces of $u=S$.

An alternative treatment of the null geodesics, Eq. (12), is to introduce a geodesic parameter (not in general an affine parameter) by

$$r = (1+\zeta\bar{\zeta})^2 \frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}(Z-h), \tag{13}$$

which, with Eqs. (6) and (7), can be solved for

$$x^a = X^a(r,u,\zeta,\bar{\zeta}) \tag{14}$$

yielding the parametric description of the null geodesics ruling the level surfaces of S. Unfortunately this description can break down at the caustics of S where r sometimes becomes infinite. It nevertheless is a means of treating the geodesics almost everywhere.

Remark 2: The description we have given here for the construction of solutions to the Eikonal equation involves the construction of envelopes of tangent lines to the original two-parameter family of solutions $Z(x^a,\zeta,\bar{\zeta})$ to form the $S(x^a)$. This description and the treatment of the caustics is an example of V. I. Arnold's theory of generating families.[2,7]

III. EIKONAL EQUATION IN ASYMPTOTICALLY FLAT SPACE–TIMES

Before the introduction of a special or fiducial family of null surfaces $S_0 = Z(x^a,\zeta,\bar{\zeta})$, we begin with a brief discussion of asymptotically flat space–times. These space–times allow a conformal rescaling of the space–time metric bringing null infinity (the end points of all future-directed null geodesics) into a finite region thereby defining a (null boundary) for the space–time. Though we will not be using the conformal rescaling explicitly, we will however use the language of the conformal boundary. The boundary, referred to as $\mathfrak{I}^+$, can be attained by limiting procedures in the unrescaled space–time. The boundary, which is a null three surface with topology

$R\times S^2$ can be given coordinates $(u,\zeta,\bar{\zeta})$, with u on the R part and the complex stereographic coordinates $(\zeta,\bar{\zeta})$ on the S^2 which label the null generators (geodesics) of $\mathfrak{I}^+$. It is this structure that we will use to obtain the fiducial family of null surfaces.

For *each* generator of $\mathfrak{I}^+$, i.e., for $(\zeta,\bar{\zeta})=(\zeta_0,\bar{\zeta}_0)=\text{constant}$, we choose the one parameter family of *past null cones* having their apexes on that generator. This yields the special solution of the Eikonal equation, $u=Z(x^a,\zeta_0,\bar{\zeta}_0)$. Doing the same for each generator defines for us our unique fiducial family of solutions. $S_0\equiv u=Z(x^a,\zeta,\bar{\zeta})$, the past cones of each point of $\mathfrak{I}^+$. We emphasize that we are describing these null surfaces in the language of the conformal compactification—in the language of the physical space–time they describe the family of all asymptotic plane waves and in the case of flat space they *are* the family of *all* plane waves. As the concepts described here are conformally invariant, the choice of language is at our discretion.

Our special family of solutions

$$u=Z(x^a,\zeta,\bar{\zeta}) \tag{15}$$

has the two important dual meanings: (1) As we just mentioned for fixed point. $(u,\zeta,\bar{\zeta})$, on $\mathfrak{I}^+$, as x^a varies, it defines the past cone of the point and (2) for a fixed value of the x^a, as $(\zeta,\bar{\zeta})$ are varied over the S^2, $u=Z$ defines a two-surface on $\mathfrak{I}^+$, the end points of all the null geodesics leaving x^a. This two-surface is referred to as the *light-cone cut* of the point x^a and is denoted by $\mathfrak{c}(x^a)$. The function $Z(x^a,\zeta,\bar{\zeta})$ will be referred to as the *light-cone cut function*.

Both meanings to $u=Z(x^a,\zeta,\bar{\zeta})$ play a fundamental role in the remainder of this work. The actual determination of $u=Z(x^a,\zeta,\bar{\zeta})$ is quite difficult and up to the present, depends on perturbation arguments that have not yet been completed. We nevertheless will assume that the function $Z(x^a,\zeta,\bar{\zeta})$ is known; we then study several consequences of this knowledge.

Remark 3: Though we will not be concerned with it here, we mention that the $Z(x^a,\zeta,\bar{\zeta})$ codes all conformal information of the space–time metric[4,5,9] *and in fact determines a conformal metric. Furthermore $Z(x^a,\zeta,\bar{\zeta})$, with a scalar function $\Omega(x^a,\zeta,\bar{\zeta})$ that acts as a conformal factor, can be used as the basic variables, replacing the metric, in a reformulation of the Einstein equations.*[4,5,9]

Our goal here is somewhat simpler (though some of the calculations themselves are not simple); we want to study the structure of the singular regions of different surfaces. First, we will show how to construct from the $Z(x^a,\zeta,\bar{\zeta})$, using the techniques of the previous section, the entire light cone $\mathfrak{C}_{x_0}$ of an arbitrary interior point x_0^a and then study its singularities. The light-cone cut $\mathfrak{c}(x_0^a)$ is the intersection of $\mathfrak{C}_{x_0}$ with $\mathfrak{I}^+$, defining a small wave front; its singularities will then be studied. Finally we return to and study the singular regions of the fiducial family of null surfaces, defined by the light-cone cut function, $Z(x^a,\zeta,\bar{\zeta})$ itself.

We first define, in the case of asymptotically flat spaces, several variables that play an important later role. Instead of using the notation of ∂_ζ and $\partial_{\bar{\zeta}}$ for the $(\zeta,\bar{\zeta})$ derivatives, we make use of the *edth* notation, e.g., $\eth Z=(1+\zeta\bar{\zeta})\partial_\zeta Z$. $\eth\bar{\eth}Z=(1+\zeta\bar{\zeta})^2\partial_\zeta\partial_{\bar{\zeta}}Z$ or $\eth^2 Z=\partial_\zeta\{(1+\zeta\bar{\zeta})^2\partial_\zeta Z\}$, etc. We then have by direct calculation from the $Z(x^a,\zeta,\bar{\zeta})$,

$$(1)\quad \omega\equiv\eth Z,\quad \bar{\omega}\equiv\bar{\eth} Z,\quad \text{the tangent directions to the light-cone cuts,}$$

$$(2)\quad \Lambda\equiv\eth^2 Z,\quad \bar{\Lambda}\equiv\bar{\eth}^2 Z,\quad \text{``accelerations'' along the } (\bar{\zeta},\zeta) \text{ constant curves,}$$

$$(3)\quad R\equiv\eth\bar{\eth}Z \quad \text{extrinsic curvature of the light-cone cuts.} \tag{16}$$

Using this notation, the determination of the caustics, i.e., the vanishing of the determinant $\hat{D}$ from Eq. (11) is equivalent, using Eq. (7), to

$$D \equiv \begin{vmatrix} \eth^2[Z(x^a,\zeta,\bar{\zeta}) - h(\zeta,\bar{\zeta})] & \eth\bar{\eth}[Z(x^a,\zeta,\bar{\zeta}) - h(\zeta,\bar{\zeta})] \\ \eth\bar{\eth}[Z(x^a,\zeta,\bar{\zeta}) - h(\zeta,\bar{\zeta})] & \bar{\eth}^2[Z(x^a,\zeta,\bar{\zeta}) - h(\zeta,\bar{\zeta})] \end{vmatrix} = 0. \tag{17}$$

IV. LIGHT-CONES AND THEIR SINGULARITIES

In this section we consider the future light-cone of a point x_0^a, namely, the set of all (future directed) null geodesics that pass through x_0^a. As a three-surface in the four-dimensional space–time, the light-cones in general have singularities that are caused by the focusing effect of the space–time curvature. These singularities are characterized by the vanishing of the geodesic deviation vector associated with neighboring geodesics on the light-cone and are what we have been referring to as the caustics of the null surface. It is our purpose here to first find these light-cones and then describe their singularities in terms of the light-cone cut function, $Z(x^a,\zeta,\bar{\zeta})$.

As we pointed out earlier (Sec. II), given a two-parameter family of solutions to the Eikonal equation, $Z(x^a,\zeta,\bar{\zeta})$, any characteristic surface can be constructed by adding a term that depends only on the parameters $(\zeta,\bar{\zeta})$; i.e.,

$$S(x^a,\zeta,\bar{\zeta}) = Z(x^a,\zeta,\bar{\zeta}) - h(\zeta,\bar{\zeta})$$

and extremizing with respect to the two parameters. If we choose

$$h(\zeta,\bar{\zeta}) = Z(x_0^a,\zeta,\bar{\zeta}) \equiv Z_0(\zeta,\bar{\zeta}), \tag{18}$$

then the level surface obtained from

$$S = 0 = Z(x^a,\zeta,\bar{\zeta}) - Z(x_0^a,\zeta,\bar{\zeta}) \tag{19}$$

with the extremal conditions

$$\begin{aligned} &\eth[Z(x^a,\zeta,\bar{\zeta}) - Z(x_0^a,\zeta,\bar{\zeta})] = 0, \\ &\bar{\eth}[Z(x^a,\zeta,\bar{\zeta}) - Z(x_0^a,\zeta,\bar{\zeta})] = 0, \end{aligned} \tag{20}$$

describes the light-cone from the point x_0^a. To see this, we first remember that this construction yields characteristic surfaces, then we see that the surface does go through the point $x^a = x_0^a$ and coincides on $\mathfrak{I}^+$, with the light-cone cut of x_0^a. Finally if we take its gradient, i.e.,

$$\partial_a S|_{x=x_0} = \partial_a[Z(x^a,\zeta,\bar{\zeta}) - Z(x_0^a,\zeta,\bar{\zeta})]|_{x=x_0} = \partial_a Z(x^a,\zeta,\bar{\zeta})|_{x=x_0} \equiv p_a(x_0^a,\zeta,\bar{\zeta}), \tag{21}$$

we see that it ranges over the entire light-cone at x_0^a.

A Caveat: We have assumed that the cut function, $u = Z(x^a,\zeta,\bar{\zeta})$, for fixed x^a is a single valued function on $\mathfrak{I}^+$. In fact, in general, this is not true; most often there will be regions on $\mathfrak{I}^+$ where it will be multivalued and it must be given as several different ''sheets'' in different $(\zeta,\bar{\zeta})$ patches. Though this does not present obstacles in principle, it does present technical difficulties in implementation. Then Eqs. (19) and (20) must be repeated on the different sheets. A way to avoid this difficulty is to describe the light-cone cut function and the light-cone cut itself parametrically, i.e., to write it as $u = U(x^a,\lambda,\bar{\lambda})$ and $\zeta = \Gamma(x^a,\lambda,\bar{\lambda})$ with single-valued functions. For simplicity of presentation we will, for the moment, continue to treat the cut function as if it were single valued.

If, to the set of Eqs. (19) and (20), we add, from Eq. (13), the equation

$$r = (1+\zeta\bar{\zeta})^2 \frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}(Z - Z_0) \equiv \eth\bar{\eth}(Z(x^a,\zeta,\bar{\zeta}) - Z(x_0^a,\zeta,\bar{\zeta})), \tag{22}$$

they implicitly define all the null geodesics of the light-cone $\mathfrak{C}_{x_0}$, i.e., they determine

$$x^a = X^a(x_0^a, r, \zeta, \overline{\zeta}). \tag{23}$$

If the geodesic goes from x_0^a to $\mathfrak{I}^+$ without encountering a caustic then r goes from 0 to infinity along that geodesic; if however it does encounter a caustic before $\mathfrak{I}^+$, r then becomes infinite before $\mathfrak{I}^+$.

The location of the caustics of $\mathfrak{C}_{x_0}$ (or the conjugate points to x_0^a) is given by the vanishing of D from Eq. (17), with $h = Z(x_0^a, \zeta, \overline{\zeta})$;

$$D \equiv \begin{vmatrix} \eth^2[Z(x^a,\zeta,\overline{\zeta}) - Z(x_0^a,\zeta,\overline{\zeta})] & \eth\bar{\eth}[Z(x^a,\zeta,\overline{\zeta}) - Z(x_0^a,\zeta,\overline{\zeta})] \\ \eth\bar{\eth}[Z(x^a,\zeta,\overline{\zeta}) - Z(x_0^a,\zeta,\overline{\zeta})] & \bar{\eth}^2[Z(x^a,\zeta,\overline{\zeta}) - Z(x_0^a,\zeta,\overline{\zeta})] \end{vmatrix} = 0, \tag{24}$$

or, with definitions (16),

$$D = (\Lambda - \Lambda_0)(\overline{\Lambda} - \overline{\Lambda}_0) - (R - R_0)^2 = 0. \tag{25}$$

We have thus been able to express the location of the caustics of an arbitrary light-cone in terms of derivatives of the cut function $Z(x^a, \zeta, \overline{\zeta})$. Given a fixed point x_0^a and a particular null geodesic (labeled by $(\zeta, \overline{\zeta})$), the curvature and ''acceleration'' of its light-cone cut is given by $(R_0(x_0^a, \zeta, \overline{\zeta}), \Lambda_0(x_0^a, \zeta, \overline{\zeta}))$ while for an arbitrary point along that geodesic it would be $(R(x^a, \zeta, \overline{\zeta}), \Lambda(x^a, \zeta, \overline{\zeta}))$. D which begins as zero at $r = 0$, does not vanish any other place along a geodesic that does not encounter a caustic but does go to zero at the caustic. There are special geodesics $(\zeta, \overline{\zeta})_c$ which meet the caustic on $\mathfrak{I}^+$. For this limiting case, it is difficult to study the behavior of Eq. (25) since $\Lambda \Rightarrow 0$ and $R \Rightarrow \infty$, the flat-space limits, which applies here since the points x^a near $\mathfrak{I}^+$ are in the very weak field region and R_0 and Λ_0 are infinite (see next section). Other techniques for this study are needed. (See Fig. 1, the light-cone with the crossovers and cusps.)

V. THE LIGHT-CONE CUTS AND THEIR SINGULARITIES

As we saw earlier, the cut function, $u = Z(x^a, \zeta, \overline{\zeta})$ has the dual meaning of being the past light-cones of the points $(u, \zeta, \overline{\zeta})$ of $\mathfrak{I}^+$ and representing the light-cone cut of an interior point, x^a. Fixing the interior point $x^a = x_0^a$, we studied, in the last section, its light-cone and saw that we could locate its caustics but as the caustics approached $\mathfrak{I}^+$ difficulties developed. We wish to study the singularities of the light-cone cuts by an alternative method.

First of all, if we assume that *all* the null geodesics leaving x_0^a arrive at $\mathfrak{I}^+$ without encountering a caustic then the cut function, $u = Z(x^a, \zeta, \overline{\zeta})$, will describe a single-valued smooth 2-surface on $\mathfrak{I}^+$. If however some did encounter caustics then the cut-surface will only be piecewise smooth and will have, in general, self-intersections. The appropriate way to describe the cut is not through the cut function but instead via the mapping of the space of null directions at x_0^a, i.e., at $S^2(x_0^a)$, coordinatized by $(\lambda, \overline{\lambda})$, onto $\mathfrak{I}^+$. It would be given by the relations

$$(u, \zeta, \overline{\zeta}) = (U(x_0^a, \lambda, \overline{\lambda}), \Gamma(x_0^a, \lambda, \overline{\lambda}), \overline{\Gamma}(x_0^a, \lambda, \overline{\lambda})), \tag{26}$$

which are just the ''end-points'' or boundary points of the null geodesics originating at x_0^a in the $(\lambda, \overline{\lambda})$ directions. (If the $(\lambda, \overline{\lambda}) \Rightarrow (\zeta, \overline{\zeta})$ were invertible, then one would have the smooth case, $u = U(x_0^a, \lambda, \overline{\lambda}) = Z(x_0^a, \zeta, \overline{\zeta})$.)

To obtain a clearer picture the light-cone cut can, in some sense, be thought of as an infinitely late ''small wave front.'' The ''early'' wave fronts on the future lightcone of x_0^a are smooth deformations of spheres, but they may become singular at sufficiently late times, from the focusing

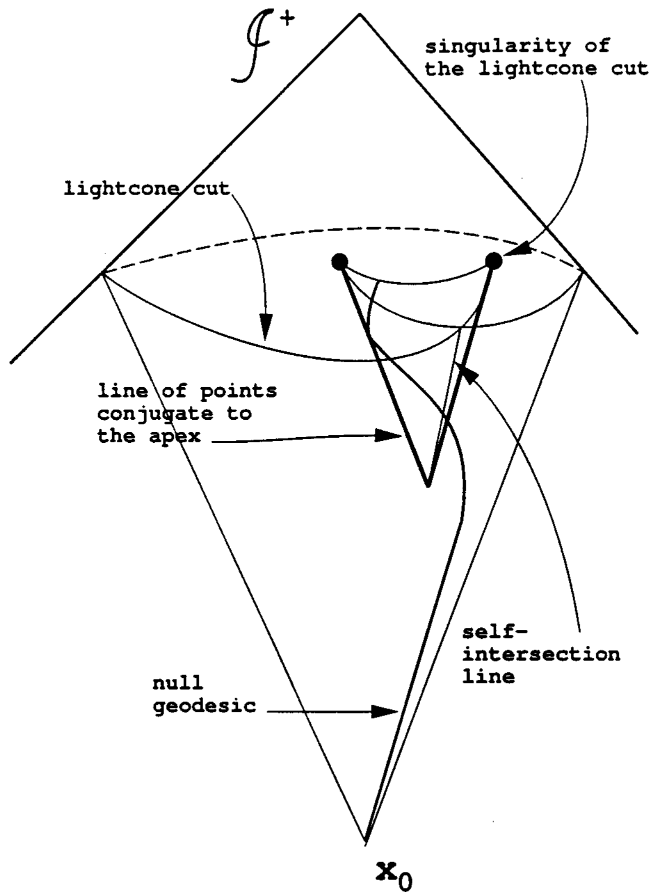

FIG. 1. The future light-cone and light-cone cut of a point x_0.

due to curvature. Therefore, light-cone cuts of (early) points in space–time are generically singular two-surfaces in three-dimensions, and they must exhibit the standard stable singularities, the cusp ridges, and swallowtails. In this view, for a fixed point x_0^a, a singularity in the light-cone cut would be a conjugate point to x_0^a. Generically, because singularities of two-surfaces lie on curves, the singularities of a light-cone cut would single out a one-parameter set $(\zeta(s),\overline{\zeta}(s))$ of null geodesics in the future light-cone for which the apex is a focal point.

Because Eq. (26) arises from the Hamiltonian evolutions (null geodesic flow) the map is a Legendre map and we can use the general theory of Legendre submanifolds and Legendre maps of Arnold and his colleagues in order to have a description of the location of the singularities of the light-cone cuts.[3,6] A two-dimensional surface in a three-manifold which is obtained by a Legendre map can always be represented as the projection of a smooth 2-surface (a Legendre submanifold) in a five-dimensional space, with the singularities located by the singularities of the projection. In other words, there exists a way to ''unfold'' a singular surface by adding two dimensions to the space where the surface lives. In this view, one of the three original dimensions (the u coordinate of $\mathfrak{I}^+$) is singled out from the remaining two-dimensional space; the two-dimensions, $(\zeta,\overline{\zeta})$, are to be considered as a configuration space. The two added dimensions consist of the two-dimensional cotangent space over the configuration space. Thus the enlarged five-dimensional space on which our surfaces ''unfold'' consists of points $(\zeta,\overline{\zeta},p_\zeta,p_{\overline{\zeta}},u)$, a contact bundle over the sphere. It is preferable to use *real* coordinates, and later translate the results in terms of our complex coordinates. Thus, in the following we assume that we have real coordinates (q^1,q^2) on the sphere, which can be taken to be the real and imaginary parts of ζ, and their corresponding momenta as (p_1, p_2).

A smooth ''unfolding'' is generically represented in terms of a smooth generating function $G(q^1, p_2)$. The points (q^1, q^2, p_1, p_2, u) that lie on such unfolding are given by

$$q^2 = -\frac{\partial G(q^1, p_2)}{\partial p_2}, \tag{27a}$$

$$p_1 = \frac{\partial G(q^1, p_2)}{\partial q^1}, \tag{27b}$$

$$u = G(q^1, p_2) + p_2 q^2, \tag{27c}$$

and arbitrary values for q^1 and p_2. This is the expression of a two-dimensional surface within a five-dimensional space, parametrized by (q^1, p_2).

Remark 4: Note that from the general theory, there must be an invertible relationship between the parametrization (q^1, p_2) *of the Legendre submanifold and the directions* $(\lambda, \overline{\lambda})$.

A projection of this surface down to the space (q^1, q^2, u) is parametrically represented by a map $(q^1, p_2) \rightarrow (q^1, q^2(q^1, p_2), u(q^1, p_2))$ which breaks down at points where the Jacobian matrix

$$\left\| \begin{matrix} \dfrac{\partial q^1}{\partial q^1} & \dfrac{\partial q^1}{\partial p_2} \\ \dfrac{\partial q^2}{\partial q^1} & \dfrac{\partial q^2}{\partial p_2} \\ \dfrac{\partial u}{\partial q^1} & \dfrac{\partial u}{\partial p_2} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ -\dfrac{\partial^2 G}{\partial q^1 \partial p_2} & -\dfrac{\partial^2 G}{\partial p_2^2} \\ \dfrac{\partial G}{\partial q^1} - p_2 \dfrac{\partial^2 G}{\partial q^1 \partial p_2} & -p_2 \dfrac{\partial^2 G}{\partial p_2^2} \end{matrix} \right\| \tag{28}$$

drops rank, from 2 to 1 or 0. The drop in rank takes place where the three 2×2 determinants vanish, namely, where

$$\frac{\partial^2 G}{\partial p_2^2} = 0, \tag{29}$$

$$p_2 \frac{\partial^2 G}{\partial p_2^2} = 0, \tag{30}$$

$$\frac{\partial G}{\partial q^1} \frac{\partial^2 G}{\partial p_2^2} = 0. \tag{31}$$

Clearly all three equations can be satisfied if and only if

$$K(q^1, p_2) \equiv \frac{\partial^2 G(q^1, p_2)}{\partial p_2^2} = 0. \tag{32}$$

Thus Eq. (32) locates the curve $K(q^1, p_2) = 0$ in the (q^1, p_2) parameter space and hence, via Eqs. (27a) and (27c), it locates the singular points on the surface. Equation (32) also expresses the location of points where Eq. (27a) fails to be invertible; namely, if we think of Eq. (27a) as implicitly defining $p^2 = h(q^1, q^2)$, then h fails to be differentiable there. (From the drop in rank, it is straightforward to see that $\partial h / \partial q^2$ blows up. See Eq. (39a) below.)

In order to translate this treatment into our complex notation, we pass from (q^1, q^2) to the complex coordinates $\zeta = \frac{1}{2}(q^1 + i q^2)$ and reinterpret

$$u = G(x^a, q^1, h(q^1, q^2)) + q^2 h(q^1, q^2)$$

as our cut function $u = Z(x^a, \zeta, \overline{\zeta})$, where x^a are fixed parameters and play no role in the discussion of this section. We can then express Eq. (32) in terms of derivatives of Z in the following manner.

Beginning with the function $\Lambda \equiv \eth^2 Z$, we express it parametrically in terms of (q^1, p_2),

$$\Lambda = 2(1+\zeta\overline{\zeta})\overline{\zeta}\frac{\partial Z}{\partial \zeta} + (1+\zeta\overline{\zeta})^2 \frac{\partial^2 Z}{\partial \zeta^2}, \tag{33}$$

where

$$\frac{\partial}{\partial \zeta} = \left.\frac{\partial}{\partial q^1}\right|_{q^2} - i \left.\frac{\partial}{\partial q^2}\right|_{q^1}. \tag{34}$$

Carrying through the calculation, which involves several implicit differentiations, we first arrive at

$$\omega = (1+\zeta\overline{\zeta})\left(\frac{\partial G}{\partial q^1} - i p_2\right), \quad \overline{\omega} = (1+\zeta\overline{\zeta})\left(\frac{\partial G}{\partial q^1} + i p_2\right), \tag{35}$$

where

$$\zeta = \frac{1}{2}\left(q^1 - i \frac{\partial G}{\partial p_2}\right), \quad \overline{\zeta} = \frac{1}{2}\left(q^1 + i \frac{\partial G}{\partial p_2}\right). \tag{36}$$

Then

$$\Lambda = 2\overline{\zeta}\omega + (1+\zeta\overline{\zeta})^2 \left\{ \frac{\partial^2 G}{\partial (q^1)^2} - \left(\frac{\partial^2 G}{\partial q^1 \partial p_2} - i\right)^2 \left(\frac{\partial^2 G}{\partial p_2^2}\right)^{-1} \right\}. \tag{37}$$

Similarly, we obtain a parametric expression for $R \equiv \bar{\eth}\eth Z = (1+\zeta\overline{\zeta})^2 (\partial^2 Z/\partial \zeta \partial \overline{\zeta})$ in the form

$$R = (1+\zeta\overline{\zeta})^2 \left\{ \frac{\partial^2 G}{\partial (q^1)^2} - \left(1 + \left(\frac{\partial^2 G}{\partial q^1 \partial p_2}\right)^2\right) \left(\frac{\partial^2 G}{\partial p_2^2}\right)^{-1} \right\}. \tag{38}$$

In deriving (37) and (38), the following were needed:

$$\frac{\partial h}{\partial q^1} = -\frac{\partial^2 G}{\partial q^1 \partial p_2}\left(\frac{\partial^2 G}{\partial p_2^2}\right)^{-1}, \quad \frac{\partial h}{\partial q^2} = -\left(\frac{\partial^2 G}{\partial p_2^2}\right)^{-1}, \tag{39}$$

which are obtained by taking derivatives $\partial/\partial q^1|_{q^2}$ and $\partial/\partial q^2|_{q^1}$ of Eq. (27a).

From (37) and (38) we can see that both Λ and R diverge at points where Eq. (32) is satisfied, and only at those points, since G is assumed to be smooth. Therefore, we can locate the singular points (a curve), $(\zeta(s), \overline{\zeta}(s))$ of light-cone cuts by either of the conditions,

$$P(x_0^a, \zeta, \overline{\zeta}) \equiv \frac{1}{\bar{\eth}\eth Z(x_0^a, \zeta, \overline{\zeta})} = 0, \tag{40}$$

$$L(x_0^a, \zeta, \overline{\zeta}) \equiv \frac{1}{\eth^2 Z(x_0^a, \zeta, \overline{\zeta})} = 0 \tag{41}$$

for given values of x_0^a.

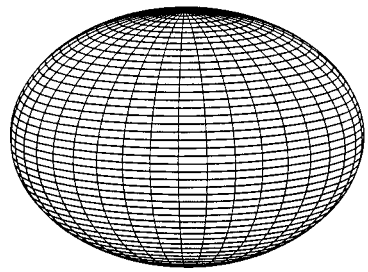

FIG. 2. A regular light-cone cut.

We interpret this result as follows. The light-cone cut represents a wave front that has progressed out to infinity, tracing the future light-cone of the point x_0^a. For a class of points x_0^a (at least sufficiently early), the wave front starts out spherical, but there is a time at which it becomes self-intersecting. Late wave fronts have singularities which represent the location of points conjugate to x_0^a. When the wave front reaches infinity, the points conjugate to the apex lie at infinity and form the singularities of the light-cone cut (See Fig. 2, for a smooth light-cone cut and Fig. 3, for a generic light-cone cut with cusp ridges and swallowtails.)

Finally note that the vanishing of $P(x_0^a,\zeta,\overline{\zeta})$ and $L(x_0^a,\zeta,\overline{\zeta})$ are not inconsistent with Eq. (25) of the previous section where as $\mathfrak{I}^+$ is approached. $\Lambda\to 0$. $R\to\infty$ and the $\Lambda_0\to\infty$ and $R_0\to\infty$.

VI. SINGULARITIES OF THE PAST LIGHT-CONES FROM $\mathfrak{I}^+$

Up to this point we have simply assumed that we had the three parameter family of null surfaces (or equivalently the two parameter family of solutions to the Eikonal equation) that we called the fiducial family or the light-cone cut function, namely, $u=Z(x^a,\zeta,\overline{\zeta})$, with $(u,\zeta,\overline{\zeta})$ constant. We never raised the issue of the location of their singularities until now. The reason was that, to locate them, requires a different technique, namely the use of pairs of geodesic deviation vectors (Jacobi fields) and their associated area element. It will be the vanishing of the area element (obtained from the Jacobi fields) along a geodesic that locates the singularities. We begin by returning to certain structures obtainable from the light-cone cut function $Z(x^a,\zeta,\overline{\zeta})$ that were defined earlier; namely,

$$u=Z(x^a,\zeta,\overline{\zeta}), \tag{42}$$

which represents the past light-cones from all points on $\mathfrak{I}^+$,

$$\omega=\eth Z(x^a,\zeta,\overline{\zeta}), \quad \overline{\omega}=\bar{\eth} Z(x^a,\zeta,\overline{\zeta}), \tag{43}$$

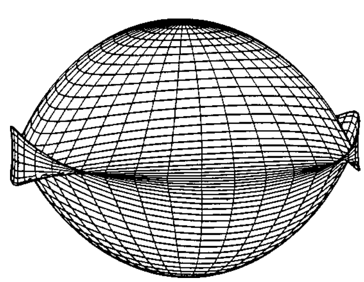

FIG. 3. A singular light-cone cut, showing cusp ridges and swallowtails.

which label the null geodesics leaving the point $(u,\zeta,\overline{\zeta})$ of $\mathfrak{I}^+$. From the compactified point of view they are the (stereographic) angles labeling the directions from the past light-cone while from the physical space point of view they are the ''distance'' between the asymptotic parallel geodesics,

$$R=\bar{\eth}\eth Z(x^a,\zeta,\overline{\zeta}), \tag{44}$$

which defines an ''optical distance'' or geodesic parameter (not affine) along the geodesics $(u,\omega,\overline{\omega})=\text{const}$. As we mentioned earlier, geometrically R is a curvature of the cut.

Using the notation

$$\theta^i=\theta^i(x^a,\zeta,\overline{\zeta})\equiv(u,\omega,\overline{\omega},R), \quad \text{with} \quad (i=0,+,-,1), \tag{45}$$

Eq. (45) can be interpreted either as a coordinate transformation, $\theta^i\Leftrightarrow x^a$, for every fixed value of the two parameters $(\zeta,\overline{\zeta})$ or simply as the introduction of four scalar functions parametrized by the $(\zeta,\overline{\zeta})$. We will make extensive use of the transformation interpretation though care must be taken in the regions where the Jacobian either vanishes or diverges. One might even expect that the troublesome region will be where the (big) wave front singularities develop.

In generic space–times, the presence of the curvature, Weyl or Ricci-type, has a *focusing effect* on parallel beams of light.[10] Thus, generically, two neighboring null geodesics in our asymptotically parallel congruences meet at some point, which means that our coordinate system breaks down by assigning two different labels to the same space–time point.

We will describe two alternative approaches to the region of breakdown.

(1) We can calculate the Jacobian of Eq. (45) most easily by returning to the description of the cut function Z by the generating function, $G(x^a,q^1,p_2)$ of the previous section, $Z=G+q^2p_2$. By a completely straightforward calculation (using MATHEMATICA to calculate the determinant) we find that

$$\left|\frac{\partial\theta^i}{\partial x^a}\right|\propto\left(\frac{\partial^2G(x^a,q^1,p_2)}{\partial p_2^2}\right)^{-1}, \tag{46}$$

$$\left|\frac{\partial x^a}{\partial\theta^i}\right|\propto\left(\frac{\partial^2G(x^a,q^1,p_2)}{p_2^2}\right) \tag{47}$$

so that the Jacobian breaks down precisely at the comparable point where the light-cone cut had its singularities.

Remark 5: In the previous section we saw that for fixed x^a, but varying the $(\zeta,\overline{\zeta})$, the functions $R(x^a,\zeta,\zeta)$ and $\Lambda(x^a,\zeta,\overline{\zeta})$ both diverged at the singularities of the light-cone cut. We can now see that for fixed $(\zeta,\overline{\zeta})$ but varying the point x^a along a null geodesic, the same functions diverge at the caustic of the past light-cone.

(2) In the second approach, we derive an explicit algebraic condition to locate these regions, by finding the points where a geodesic deviation vector vanishes. Our present derivation is in great measure a reinterpretation of an earlier derivation due to Kozameh and Newman,[11] reproduced here in current notation in order to maintain the unity of the present work.

By (in principle) inverting Eq. (45) we obtain

$$x^a=X^a(\theta^i,\zeta,\overline{\zeta})=X^a(u,\omega,\overline{\omega},R,\zeta,\overline{\zeta}), \tag{48}$$

which for fixed values of $(u,\zeta,\overline{\zeta})$ is the parametric form of the past cone of $\mathfrak{I}^+$ and for fixed values of $(u,\omega,\overline{\omega},\zeta,\overline{\zeta})$ it is the parametric form for the null geodesics on the cone each labeled by $(\omega,\overline{\omega})$.

Of prime importance to us are the connecting vectors to the null geodesics that are on the past null cone. Two connecting vectors (from which all others can be constructed) are given by

$$M^a = \frac{\partial X^a}{\partial \omega}, \quad \overline{M}^a = \frac{\partial X^a}{\partial \overline{\omega}}. \tag{49}$$

We are interested in the area A constructed from M^a and $\overline{M}^a$. Taking a pair of (complex) spacelike unit vectors m^a and $\overline{m}^a$ ($g_{ab}m^a m^b = 0$, $g_{ab}\overline{m}^a\overline{m}^b = 0$, $g_{ab}\overline{m}^a m^b = -1$), that are parallel transported along the null geodesics. M^a and $\overline{M}^a$ can be written as

$$M^a = \xi m^a + \overline{\eta m}^a, \quad \overline{M}^a = \overline{\xi}\overline{m}^a + \eta m^a, \tag{50}$$

so that the ''area'' form is

$$M^{[a}\overline{M}^{b]} = (\xi\overline{\xi} - \eta\overline{\eta}) m^{[a}\overline{m}^{b]} \equiv A m^{[a}\overline{m}^{b]}. \tag{51}$$

From this we see that

$$A^2 \equiv (g_{ab}M^a\overline{M}^b)^2 - (g_{ab}M^a M^b)(g_{ab}\overline{M}^a\overline{M}^b) \equiv (M\cdot\overline{M})^2 - (M\cdot M)(\overline{M}\cdot\overline{M}). \tag{52}$$

Our task (which requires a bit of preparation) is to express the $M\cdot\overline{M}$ and $M\cdot M$ in terms of $Z(x^a,\zeta,\overline{\zeta})$ and its derivatives. We choose the one-form basis

$$\theta^i_a \equiv \partial_a \theta^i = (\partial_a Z, \partial_a \omega, \partial_a \overline{\omega}, \partial_a R) \equiv (\theta^0_a, \theta^+_a, \theta^-_a, \theta^1_a) \tag{53}$$

and the dual vectors

$$\theta^a_i = (\theta^a_0, \theta^a_+, \theta^a_-, \theta^a_1) = (\partial X^a/\partial u, \partial X^a/\partial\omega, \partial X^a/\partial\overline{\omega}, \partial X^a/\partial R) \tag{54}$$

which satisfy

$$\theta^a_i \theta^j_a = \delta^j_i, \quad \theta^a_i \theta^i_c = \delta^a_c. \tag{55}$$

From the one-form basis θ^i_a, using the space–time metric, g^{ab}, one can express the dual basis set by

$$\theta^a_i = g^{ac}\theta^j_c \eta_{ji} \quad \text{or} \quad \theta^i_c = g_{ac}\theta^a_j \eta^{ij}, \tag{56}$$

where

$$\eta_{ij} = \theta^a_i \theta^c_j g_{ac}, \quad \eta^{ij} = \theta^{\,i}_a \theta^{\,j}_c g^{ac}. \tag{57}$$

Returning to the computation of the area, we have for the tangent vector to the geodesics,

$$\theta^a_1 \equiv L^a = \partial X^a/\partial R, \tag{58}$$

and from the geodesic deviation vectors, $M^a = \partial X^a/\partial\omega = \theta^a_+$ and $\overline{M}^a = \partial X^a/\partial\overline{\omega} = \theta^a_-$ that

$$M\cdot\overline{M} = \eta_{+-}, \quad M\cdot M = \eta_{++}, \quad \overline{M}\cdot\overline{M} = \eta_{--}. \tag{59}$$

Remark 6: Note that $l^a \equiv \Omega^2 L^a$ is the affine parametrized tangent vector to the geodesics. [See (Eq. (63) below for the definition of Ω.]

The calculation of the three η's though lengthy, is fairly straightforward; It is found from the inverse of η^{ij} [i.e., from the second version of Eq. (57)]. The components of η^{ij} are found by beginning with

$$\eta^{00}=\theta^0_a\theta^0_c g^{ac}=g^{ac}\partial_a Z\partial_c Z=0 \tag{60}$$

which vanishes by definition. By applying the operators ð and $\bar{ð}$ several times to Eq. (60) one finds[4] for the relevant components of η^{ij} [see Eq. (16) for definitions]

$$\eta^{00}=0, \tag{61}$$

$$\eta^{0+}=0, \quad \eta^{0-}=0, \tag{62}$$

$$\eta^{01}\equiv\Omega^2=g^{ac}\partial_a Z ð\bar{ð}(\partial_c Z)=g^{ac}\partial_a Z\partial_c R, \tag{63}$$

$$\eta^{++}=-\Omega^2 L^a ð^2(\partial_a Z)=-\Omega^2 L^a\partial_a\Lambda=-\Omega^2\partial\Lambda/\partial R, \tag{64}$$

$$\eta^{--}=-\Omega^2 L^a \bar{ð}^2(\partial_a Z)=-\Omega^2 L^a\partial_a\overline{\Lambda}=-\Omega^2\partial\overline{\Lambda}/\partial R, \tag{65}$$

$$\eta^{-+}=-\Omega^2, \tag{66}$$

which in turn leads to

$$M\cdot M=\eta_{++}=-\frac{1}{\Omega^2\left(1-\dfrac{\partial\Lambda}{\partial R}\dfrac{\partial\overline{\Lambda}}{\partial R}\right)}, \tag{67}$$

$$M\cdot\overline{M}=\eta_{+-}=-\frac{\dfrac{\partial\overline{\Lambda}}{\partial R}}{\Omega^2\left(1-\dfrac{\partial\Lambda}{\partial R}\dfrac{\partial\overline{\Lambda}}{\partial R}\right)}. \tag{68}$$

The area then is

$$A^2=\frac{1}{\Omega^4\left(1-\dfrac{\partial\Lambda}{\partial R}\dfrac{\partial\overline{\Lambda}}{\partial R}\right)}. \tag{69}$$

This expression for A tells us several things; first of all to keep A real we must have the inequality

$$\left|\frac{\partial\Lambda}{\partial R}\right|\leqslant 1, \tag{70}$$

and we learn that Ω must diverge at the singularity given by $A=0$.

We have thus learned in this section that the singularities of the past light-cones from $\mathfrak{I}^+$ can be characterized by one of several methods:

(1) Using the generating function $G(x^a,q^1,p_2)$, the singularities are given by the vanishing of the Jacobian of the transformation (45), i.e., by

$$\frac{\partial^2 G(x^a,q^1,p_2)}{\partial p_2^2}=0. \tag{71}$$

(2) This, in turn, tells us (from the previous section) that both $R(x^a,\zeta,\overline{\zeta})$ and $\Lambda(x^a,\zeta,\overline{\zeta})$ diverge as the singularities are approached.

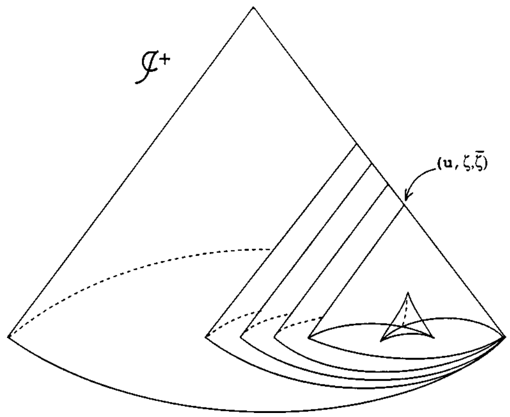

FIG. 4. The foliation of space–time by past light-cones from points at null infinity.

(3) From the geodesic deviation argument

$$\Omega \to \infty \tag{72}$$

as the singularity is approached.

(4) From Eqs. (69) and (70) we learn that $|\partial\Lambda/\partial R|$ must be bounded but we can not see what is its behavior is as the singularity is approached. However on the basis of several examples, e.g., Ref. 1, where $|\partial\Lambda/\partial R| \to 1$ it appears to be reasonable to expect that this result might be true in general. If so, then we would have that $[1-(\partial\Lambda/\partial R)(\partial\overline{\Lambda}/\partial R)] \to 0$ as the singularity is approached. In turn, from Eq. (69), we would gain some information about how fast both Ω and $[1-(\partial\Lambda/\partial R)(\partial\overline{\Lambda}/\partial R)]$ approach their limits. (See Fig. 4, a past light-cone from $\mathfrak{I}^+$.)

VII. SUMMARY AND CONCLUSIONS

In this work we have studied the kinematics or general structure of several different classes of surfaces (associated with surface forming null geodesic congruences) in asymptotically flat Lorentzian space–times, namely, the future light cones of interior points, $\mathfrak{C}_x$; the intersection of $\mathfrak{C}_x$ with $\mathfrak{I}^+$, i.e., the light-cone cuts, $\mathfrak{c}(x^a)$; and the past light-cones from points $(u,\zeta,\overline{\zeta})$ on $\mathfrak{I}^+$.

These surfaces, which in general have singularities, are closely related to each other; in particular there is a close association between their singularities. As was pointed out earlier, for the future light cones $\mathfrak{C}_{x_0}$ with an apex x_0^a that is sufficiently early in time, the small wave fronts begin spherical but as they evolve they become self-intersecting and develop singularities (the stable one being cusp ridges and swallowtails[6]) which represent the conjugate points to x_0^a. The limit, in the asymptotic future, of these small wave fronts is the light-cone cut $\mathfrak{c}(x_0^a)$; the singularities of $\mathfrak{c}(x_0^a)$ being the points conjugate to the apex. They are also the intersection of the singularities of $\mathfrak{C}_{x_0}$ with $\mathfrak{I}^+$ (see Figs. 1, 2, and 3.)

Alternatively (an example of the reciprocity theorem of Penrose and Sachs[11,12]), the singularities of light-cone cuts must be related to the singularities of the past light cones from points at infinity. The singularities of light-cone cuts are interpreted as singling out the null geodesics leaving $\mathfrak{I}^+$ which are conjugate to or focus at x_0^a. These null geodesic belong to two congruences of interest to us. First, they belong to the future light cone of the point x_0^a, and second, they belong to the past light cone of the point $(u,\zeta,\overline{\zeta})$ of $\mathfrak{I}^+$ reached by the first set. The light-cone cut function, with the vanishing of either $P(x^a,\zeta,\overline{\zeta})$ or $L(x^a,\zeta,\overline{\zeta})$, locate both the singularities of the light-cone cut and the interior points conjugate to points on $\mathfrak{I}^+$ (see Fig. 4).

Most of the kinematic issues raised here are, we believe, now reasonably well understood. {It still would be of considerable interest to determine the behavior of $[1-(\partial\Lambda/\partial R)(\partial\overline{\Lambda}/\partial R)]$, in the neighborhood of the caustics.} Our interest now is to apply these kinematic insights to the study of null surfaces (specifically, light-cones) in vacuum Einstein spaces. Though there is a formalism[4,5] in which the Einstein equations have been rewritten as differential equations for the cut function, $Z(x^a,\zeta,\overline{\zeta})$ and $\Omega(x^a,\zeta,\overline{\zeta})$ (aside from some very special cases), the equations have been difficult to deal with because of the difficulty of treating the caustics, which are ubiquitous. We feel that the situation has changed; we now know how to identify the presence of the caustics in terms of both R and Λ. {The reason for our interest in the term $[1-(\partial\Lambda/\partial R)(\partial\overline{\Lambda}/\partial R)]$ is that it arises frequently in denominators of the field equations and we would like to know if it always tends to zero at a caustic.} We have also realized that it probably will be very advantageous to use the representation of $Z(x^a,\zeta,\overline{\zeta})$ by

$$Z(x^a,\zeta,\overline{\zeta})=G(x^a,q^1,p_2)+q^2p_2 \tag{73}$$

with $q^2=-\partial G/\partial p_2$, $\zeta=1/2(q^1+iq^2)$ (see Secs. V and VI). Our immediate goals are first to find the behavior of $[1-(\partial\Lambda/\partial R)(\partial\overline{\Lambda}/\partial R)]$ near caustics and then rewrite the field equations in terms of the $G(x^a,q^1,p_2)$ rather than $Z(x^a,\zeta,\overline{\zeta})$.

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