High Amplitude Behavior of Gravitational Energy

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Abstract

We demonstrate how an exponentially strong redshift creates a sharp boundary layer at null infinity for strong gravitational fields. This leads to saturation effects by which the gravitational energy depends only linearly on amplitude, as opposed to the quadratic dependence in the weak field case.

1 Introduction

Proof of the positive definiteness of the mass loss due to gravitational radiation [1] was a key theoretical step toward providing a physically useful picture of the nonlinear properties of isolated gravitational systems. It naturally led to the important question: Is gravitational mass itself a necessarily positive definite quantity? There are many ways in which Lou Witten played a seminal role in the pursuit of the ultimate answer to this question. It was in the informal discussions at the 1969 Cincinnati Conference [2], organized by Lou, that the question became a hot topic of research. These discussions led to the realization that preliminary results indicating a positive mass would need considerable tightening in order to resolve the issue. It was also Lou [3] who pioneered the use of spinors in general relativity, which eventually emerged as the crucial tool in reducing the analytical complexity of the positive mass problem [4] to geometric terms [5].

In this paper, we explore the high amplitude behavior of the mass of a purely gravitational system. At low amplitudes, there is nothing surprising or technically difficult about the above results. The positivity of the energy flux of weak gravitational waves is a standard result of linearized theory. Also, the positivity of the matter contribution to the total energy, follows immediately from the positivity properties of the stress-energy tensors for standard sources such as fluids, Maxwell fields and Klein-Gordon fields. Furthermore, the leading contribution from gravity to the mass is quadratic in the amplitudes, so that a positive gravitational energy density can be easily found by introducing appropriate gauge conditions.

This quadratic dependence of energy on field amplitudes underlies the positivity of energy in the weak field case. The positivity of the gravitational contribution to the total energy in the nonlinear case is much more subtle. Only in the spinor description does anything resembling a positive definite quadratic energy density emerge and this expression lacks the uniqueness to describe a physically well defined local energy distribution. In fact, it is not even obvious in general how to separate the gravitational contribution to the total energy from the matter contribution.

This can be put in perspective by considering a spherically symmetric massless Klein-Gordon scalar field with the standard coupling to gravity. Because of spherical symmetry, dynamical degrees of freedom can be excited in the scalar field but not in the gravitational field. However, even though the Klein-Gordon stress energy tensor is quadratic in amplitude, the total energy does not scale quadratically. In fact, in the extreme strong field limit, the energy scales **linearly** with amplitude! More precisely, if at retarded time u the scalar field possesses a monopole moment Q,

$$\Phi = Qr^{-1} + O(r^{-2}), \tag{1}$$

the Bondi mass has the high amplitude asymptotic behavior [6]

$$M_B \sim \pi|Q|/\sqrt{2} \tag{2}$$

at this time. Not only is this linear in the amplitude of Φ, it depends solely on the amplitude of the radiation part of the field. In this limit, the mass is completely independent of the interior structure of the field. This remarkable behavior is entirely the result of the self-coupling through gravity. Even if it could be argued that gravity makes no direct contribution to the mass it certainly has a drastic indirect effect.

In this simple model, the physical mechanism behind this effect can be traced to a version of gravitational redshifting. The redshift between any worldline at fixed luminosity distance and null infinity increases at an exponential rate with increasing field amplitude. In the extreme high amplitude limit, this effectively redshifts away all the interior contributions to the Bondi mass when compared with the contribution from the far field. In the compactified Penrose picture, the dominant contribution to the total mass lies in a narrow boundary layer at null infinity.

Such a basic mechanism as the redshift might be expected on physical grounds to have the a universal effect on all high amplitude fields and even on vacuum gravitational fields. The results of this paper, for axially symmetric fields, confirm this expectation for the pure gravitational case. In the strong field limit, the mass does scale linearly with amplitude. But in this case no simple asymptotic formula analogous to Eq. (2) emerges. These results are established by a combination of a numerical approach, described in Sec. 2, and an analytic treatment based upon new asymptotic techniques, described in Sec. 3.

2 Computation of the Bondi Mass

We limit our numerical investigation to axial and reflection symmetric, vacuum spacetimes, at retarded times admitting a nonsingular asymptotically flat null cone. The geometry at this retarded time is described by the Bondi metric

$$\begin{aligned} ds^2 &= (r^{-1}Ve^{2\beta} - r^2U^2e^{2\gamma})\,du^2 + 2e^{2\beta}du\,dr \\ &\quad +2r^2Ue^{2\gamma}du\,d\theta - r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2). \end{aligned} \tag{3}$$

The free gravitational initial data γ describes the conformal geometry of the cross-sections of this null cone. Specification of γ uniquely determines the future evolution as well as the initial Bondi mass M.

Our coordinates are chosen to reduce to a local inertial frame at the vertex whereas standard Bondi coordinates are chosen to lead to an inertial frame at null infinity. Smoothness requires that γ and β vanish at the vertex and that they have the asymptotic form at infinity

$$\gamma = K + c/r + O(1/r^2) \tag{4}$$

$$\beta = H - c^2/(4r^2) + O(1/r^4). \tag{5}$$

In a standard Bondi frame, $K = H = 0$.

Historically, accurate numerical calculations of the Bondi mass have been frustrated by technical difficulties arising from the necessity to pick off nonleading terms in an expansion about null infinity. We have developed a numerical algorithm with two key ingredients which avoids these problems [7]: (i) the use of Penrose compactification, which allows null infinity to be represented as a finite boundary to the numerical grid. and (ii) the introduction of renormalized variables in which the Bondi mass aspect appears as the leading term. The resulting accuracy enables us to obtain highly accurate numerical results even in the high amplitude regime. In this algorithm, the luminosity distance r is replaced by the coordinate $x = r/(1+r)$, with compact range from $x = 0$ at the vertex to $x = 1$ at null infinity, and the angular coordinate is replaced by $y = -\cos\theta$.

The Bondi mass is given by the surface integral at null infinity

$$M_B = \frac{1}{4\pi}\oint \omega^{-1}\mu \sin\theta d\theta d\phi. \tag{6}$$

Here μ is a generalized mass aspect and ω is the conformal factor relating the asymptotic 2-geometry to the unit sphere geometry in Bondi coordinates,

$$e^{2K}d\theta^2 + \sin^2\theta e^{-2K}d\phi^2 = \omega^{-2}d\theta_B^2 + \sin^2\theta_B\, d\phi_B^2. \tag{7}$$

In terms of the renormalized variables, the mass aspect is obtained from γ through radial integrations of the sequence of equations

$$\beta_{,r} = \frac{1}{2}r(\gamma_{,r})^2, \tag{8}$$

$$(r\tau)_{,r} = r(1-y^2)^{-1}e^{2\gamma}[(1-y^2)e^{-2\gamma}(r^2\gamma_{,r})_{,r}]_{,y} \tag{9}$$

and

$$\begin{aligned} 2\mu = \quad & \textstyle\int_0^\infty dr e^{2(\beta-H)}\{r^{-2}(1-y^2)e^{-2\gamma}\psi^2 \\ & +2\beta_{,r}[((1-y^2)e^{-2\gamma}\tau)_{,y} - r^3(\mathcal{K}/r)_{,r}] \\ & -2[(1-y^2)e^{-2\gamma}(\psi\gamma_{,r} - r\beta_{,ry})]_{,y}\}, \end{aligned} \tag{10}$$

where

$$\psi = \tau + \frac{1}{2}r^2(1-y^2)^{-1}e^{2\gamma}[(1-y^2)e^{-2\gamma}]_{,ry} \tag{11}$$

and where

$$\mathcal{K} = -\frac{1}{2}[(1-y^2)e^{-2\gamma}]_{,yy} \tag{12}$$

is the Gaussian curvature of the angular metric

$$e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2. \tag{13}$$

With these renormalizations, the mass can be determined to second order accuracy in grid size by numerical integration.

3 High Amplitude Asymptotic Limits

Given initial data $\gamma(u,r,\theta)$, the substitution $\gamma \to \lambda\gamma$ generates a one parameter family of spacetimes ranging from the linearized regime (λ small) to the high amplitude limit ($\lambda \to \infty$). We define this family in terms of a luminosity parameter r so that amplitude scaling automatically preserves asymptotic flatness. For large λ, the exponential factor $e^{2(\beta-H-\gamma)}$, common to each term in the integrand of the mass aspect in Eq. (10), leads to the formation of a boundary layer at null infinity. Equation (8) implies that β is a monotonically increasing function of r which scales quadratically with λ. As a result this exponential factor is very small except very close to null infinity, where $\beta = H$. Accordingly, the contribution to the mass is very small except in a region where the gravitational field can be represented by leading terms in a $1/r$ expansion.

In analogy with the high amplitude formula Eq. (2) for a scalar field, this suggests that the mass might only depend on the radiation part of the data γ, represented by the terms explicitly exhibited in Eq. (4). The leading term K can be thought of as pure gauge, since it can be transformed to zero by a conformal transformation to a standard Bondi frame at null infinity. As a result, one might expect that the mass be dominated by the Bondi amplitude c_B, given by the field c in that frame.

To explore this possibility, consider data with $K = 0$, for which the mass can be expressed simply as the integral

$$M = \frac{1}{4}\int_0^\pi \int_0^\infty \sin\theta d\theta dr\, e^{-2H} rV_{,rr}. \tag{14}$$

By integrating the Bondi hypersurface equations [1] while dropping curl terms which integrate to zero over the sphere, this gives

$$M = -\frac{1}{8}\int_0^\pi \int_0^\infty \sin\theta d\theta dr\, e^{-2H} r\partial_r \{\frac{1}{2}(r^2 e^{(\gamma-\beta)}U_{,r})^2 \\ +2e^{2(\beta-\gamma)}[2\beta_{,\theta}\gamma_{,\theta} - (\beta_{,\theta})^2 - 2\beta_{,\theta}\cot\theta]\}. \tag{15}$$

In the high amplitude limit, we want to isolate the terms generated by the radiation amplitude c. For β and γ these terms are displayed in Eq. (4) and Eq. (5). The hypersurface equation for U, gives for these terms

$$r^2 e^{2(\gamma-\beta)}U_{,r} \sim \frac{2}{r\sin^2\theta}(c\sin^2\theta)_{,\theta} - \frac{6N}{r^2}, \tag{16}$$

where the dipole-moment aspect [1] N is included for future reference. Retaining only the c-terms, Eq. (15) leads to the high amplitude asymptotic behavior

$$M \sim -\frac{1}{4}\int_0^\pi \int_0^\infty \sin\theta d\theta dr\, r\partial_r \{e^{2(\beta-H-\gamma)}\{r^{-2}[(c\sin^2\theta)_{,\theta}/\sin^2\theta]^2 \\ +r^{-2}cc_{,\theta}\cot\theta - r^{-3}c(c_{,\theta})^2 - \frac{1}{4}r^{-4}(cc_{,\theta})^2\}\}. \tag{17}$$

The exponential factor, determined by the c-terms in Eq. (4) and Eq. (5), creates a boundary layer between $r \approx c$ and ∞ (or between $x \approx 1-1/c$ and 1) which dominates the integral and guarantees its convergence in the interior except possibly along rays where c vanishes. On the naive assumption that this leads to no difficulty, the radial integration can be carried be carried out analytically,

$$\begin{aligned} M \;\sim\; & \frac{A}{4}\int_0^\pi \sin\theta d\theta\, c^{-1}\{[(c\sin^2\theta)_{,\theta}/\sin^2\theta]^2 + cc_{,\theta}\cot\theta + \frac{3}{4}(c_{,\theta})^2\} \\ & -\frac{1}{8}\int_0^\pi \sin\theta d\theta\, c^{-1}(c_{,\theta})^2, \end{aligned} \tag{18}$$

where

$$A = \int_0^\infty ds\, e^{-(s^2+4s)/2} \approx .42137 \tag{19}$$

can be evaluated with the help of a standard table of error integrals.

4 Comparison of Numerics and Analytics

The high amplitude asymptotic expression for the mass Eq. (18) clearly predicts linear scaling $M \to \lambda M$ under linear scaling of the Bondi amplitude. The appearance of the factor $1/c$ cautions that this naive formula might break down. However, this is possibly a technical difficulty with the present calculation which might be avoided by eliminating a different choice of curl terms before the high amplitude limit is taken.

Since c represents a spin-2 field, zeros are unavoidable. However, at least in the pure quadrupole case $c = \lambda \sin^2\theta$ $(\lambda > 0)$, the steps leading from Eq. (17) to Eq. (18) can be rigorously justified and Eq. (18) yields

$$M \sim \lambda(\frac{7}{2}A - \frac{1}{3}) \approx 1.14145\lambda. \tag{20}$$

This analytic result, based upon the dominance of the radiation amplitude c, can be checked against the results of the numerical code by choosing data $\gamma = \lambda \sin^2\theta F(r)/r$, where $F = 1$ outside a region of compact support and is chosen to make γ appropriately smooth in the interior. Care must be taken to coordinate the size of these regions, the range of λ and the grid size so that the boundary layer may be modeled numerically without loss of accuracy from machine roundoff due to small values of the exponential factor.

Figures 1 and 2 are surface plots of the exponential factor $e^{2(\beta-H-\gamma)}$ in Eq. (17) for $\lambda = 1$ and $\lambda = 10$, respectively. The development of the boundary layer is already apparent for $\lambda = 1$. It is fully formed at $\lambda = 10$, which is representative of the high amplitude regime. Besides consisting of a neighborhood of null infinity, the boundary layer also includes a neighborhood of the axis, where β, H and γ vanish. However, the remaining factor in the mass aspect vanishes on the axis, so that the axis plays no dominant role.

Figure 3 is a graph of the numerically computed mass versus λ, showing the quadratic dependence in the weak field case. In Figure 4, the extension of this graph to extremely high λ shows the transition to linear dependence in the vicinity of $\lambda = 5$. The persistence of this linear dependence in the limit $\lambda \to \infty$ is apparent. In this high λ regime, the slope of the graph is 1.325 as opposed to the value 1.14145 predicted analytically in Eq. (20). The contribution of numerical error to this discrepancy is negligible. Instead, the difference stems from the assumption made in the analytic calculation that only the radiative amplitude, i.e. c-terms, contribute to the mass at high amplitudes. A more detailed analysis shows an additional contribution, which is also linear in λ, that arises from the dipole-moment aspect N in Eq. (16). Although the dipole-moment aspect is itself one of the leading terms in a $1/r$ expansion of the metric, it cannot be determined solely from the radiation amplitude c but depends upon the details of the interior geometry. In the above example, it depends on the details of the smoothing function F. This precludes the possibility of any simple asymptotic formula relating the mass to the radiation amplitude analogous to Eq. (2) for the scalar case. These results are predicated on the assumption $K = 0$. For $K \neq 0$, the value of ω increases at too fast a rate with increasing amplitude to obtain reliable numerical results. However, since such a K can always be transformed to zero at the expense of introducing a coordinate singularity at the vertex of the null cone, if the boundary layer is a genuine physical effect then our conclusions should remain valid if interpreted in terms of a standard Bondi frame. However, amplitude scaling does not commute with time evolution so that they can only be expected to hold for a very short time interval, after which the field would leave the high amplitude regime.

Finally, we have assumed that the mass is calculated for a null cone with point vertex which is the initial data surface for a nonsingular spacetime. This assumption appears necessary to avoid potential contradictions with the existence of flat spacetimes for which c does not vanish on supertranslated null cones which have more general caustic structure. Such examples of "bad cones" in Minkowski space caution that our physical picture here may be too naive. They raise the possibility that there might be internal contributions to the mass from caustic structure that also survive the redshift effect to first order in λ.

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Figure 1: Plot of the exponential factor for $\lambda = 1$ *vs* $y = -\cos\theta$ and the compactified radial coordinate $x = r/(1+r)$.

Figure 2: Plot of the exponential factor for $\lambda = 10$.

FIGURES

Figure 3: Graph of M *vs* λ for $0 < \lambda < 1$.

Figure 4: Graph of M *vs* λ for $0 < \lambda < 100$.