15-459: Undergraduate Quantum Computation September 8, 2022

Lecture 4 : Nature of quantum operations

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Overview

- *Amplitude trees can be compressed*
- *Quantum operations can be represented by bipartite DAGs or matrices.*
- *Reverses and inverses of operations*
- *What are the* allowed *quantum operations?*

1 Probabilistic analogue

To get the probability of ending at a particular computational state, use the "sum of path-products" rule. This can be done by matrix multiplication, since each product summed in the matrix product corresponds to the probability of a given path.

In summary, every probabilistic operation on 1 (2, 3, 4, ...) bits can be encapsulated by

1. "Bipartite probability DAG" with 2 (4, 8, 16, ...) input and output nodes
2. A 2×2 (4×4, 8×8, ...) matrix, with columns labeled by input strings, rows by output strings

To composite two operations,

1. Glue together DAGs, use sum-of-path-products recipe.
2. Matrix multiplication.

To get final probabilities given an initial probabilistic state,

1. Fork from an initial node to initial states with edge probability equal to state probability, and glue to next DAG.
2. Multiply by a column vector of the initial probabilities.

2 Quantum models, reverse, and inverse

All the computational rules are **just the same** for quantum, except:

- Answers are quantum states, amplitudes whose squares add up to 1, as opposed to probability states, non-negative probabilities whose sum is 1.

Recall the "Hat" operation, given by the matrix $\begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{pmatrix}$. What properties of this transformation preserve the sum of the squares of the amplitudes?

Reversing an operation

1. reverses the arrows of the DAG
2. transposes the matrix

In the probabilistic case, the result may not be a valid instruction, i.e. the probabilities going out of a node/in a column may not sum to one.

Inverting an operation undoes the effect of that operation, i.e. $f \circ f^{-1} = f^{-1} \circ f = 1$.

In the probabilistic case, inverses almost never exist: you cannot recover the previous state of a flipped coin. The only exception is deterministic instructions that happen to be permutations.

Recall linear algebra: $(AB)^T = B^T A^T$, and if A, B are invertible, $(AB)^{-1} = B^{-1}A^{-1}$.

In the quantum case, everything is (mostly) the same.

For reversal, you reverse the arrows (and taking the complex conjugate of the weights), i.e. $A \mapsto A^\dagger$.

For inversion, you take the inverse of the matrix, which must exist: $A \mapsto A^{-1}$.

For Hat, $\text{Hat}^{-1} = \text{Hat}^\dagger = \text{Hat}$.

Let $\text{Rot} = \text{Incr} \circ \text{Hat}$. The corresponding matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}$. Note that this is the rotation matrix for rotation by $\frac{\pi}{4}$ counterclockwise.

Naturally, $\text{Rot}^{-1} = \text{Rot}^T = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}$, rotation by $-\frac{\pi}{4}$.

3 Quantum mechanics

Law: An operation U is physically allowable if and only if $U^{-1} = U^\dagger$, i.e. U is unitary.

Theorem: A matrix is "unitary" if and only if it maps quantum states to quantum states, where a quantum state is a vector where the squares of entries sum to 1.