1
Geometric Constructions

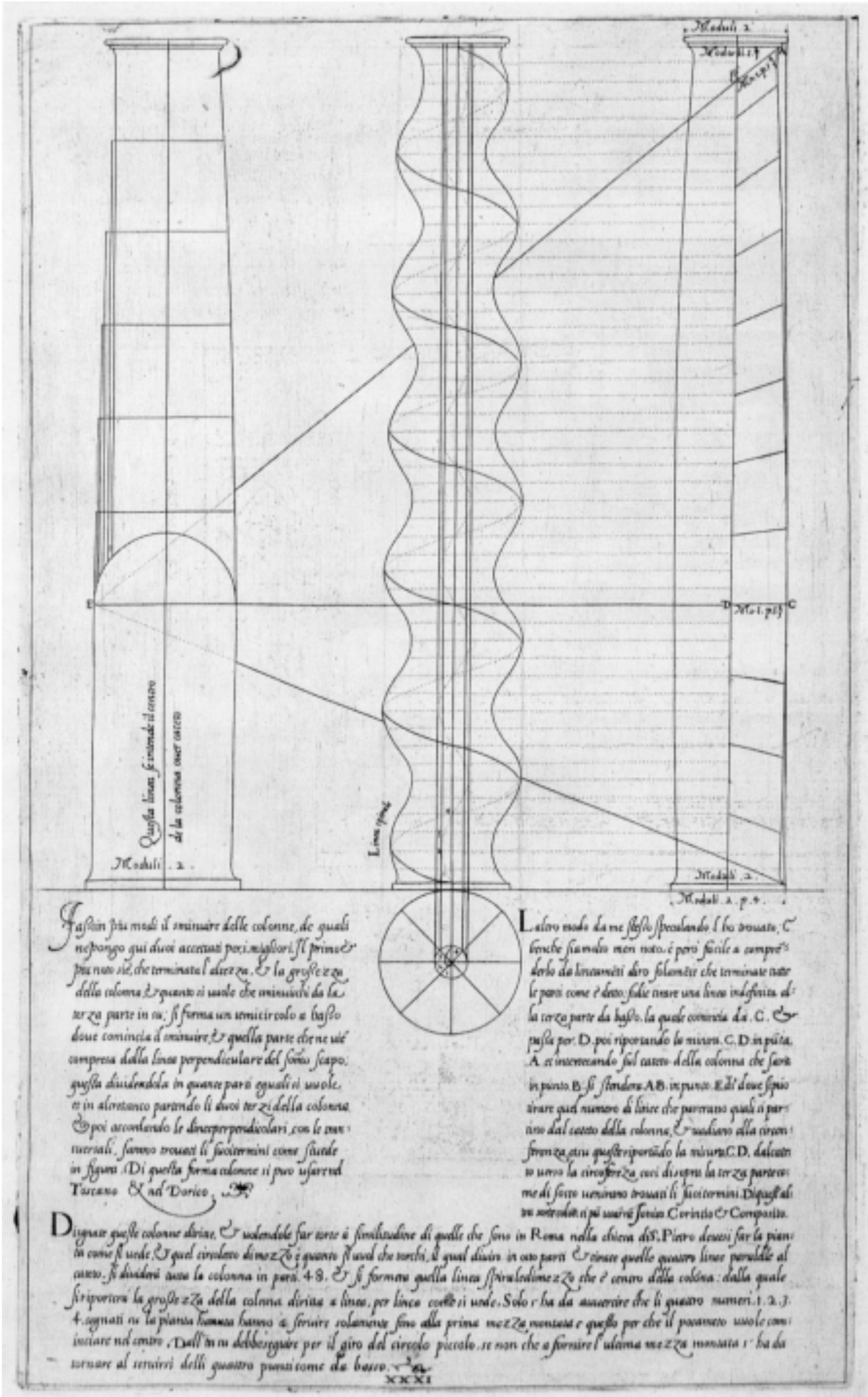
Everyone knows something about geometry and about certain basic entities such as lines, angles, arcs, etc. Geometry is used in a very practical way in the design fields.

1.1 CLASSICAL COLUMNS

Let us begin with constructions that are at once historical, practical and motivational.

Giacomo Barozzi da Vignola in the mid-16th century wrote one of the more influential treatises on Renaissance architecture. In it, he describes manual ways of constructing architectural views. I highlight two constructions from his text. Each deals with an elevation of a classical column of a specific profile. The first produces a tapered shaft, the second, also called an *enthasis,* a profile that diminishes towards both the top and bottom. Figure 1-1 shows a plate from his treatise, *Canon of the Five Orders of Architecture.* Each construction yields a finite number of points on the shaft profile. These points are connected by a smooth curve to produce the desired profile.

Giacomo Barozzi da Vignola
(Author) *Gli Ordini D'Architettura Civile*
(Canon of the Five Orders of Architecture)



1-1
"Entasis"
Plate 31 from Giacomo Barozzi Da Vignola, Canon of the Five Orders of Architecture,
Translated by Branko Mitrovic (New York: Acanthus Press, 1999)

Construction 1-1
Constructing the profile of a classical tapered column

There are five steps.

1. Determine the height and largest diameter of the column, *d*. There are clear rules about preferred proportions between the height and diameter of various types of classical columns (doric, ionic etc.) These measures are normally related to each other as integral multiples of a common module, *m*.

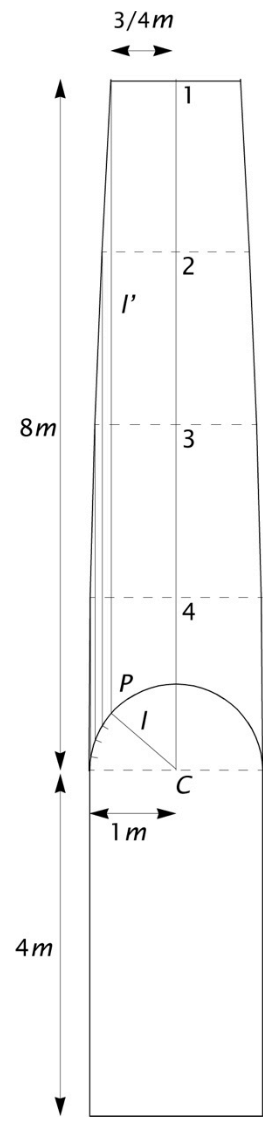
   Figure 1-2 shows the shaft of a column with diameter *2m* and height 12*m*. That is, the proportion between diameter and height is 2:12 or 1:6

2. At $^1/_3$ of the shaft's height, draw a straight line, *l*, across the shaft and draw a semi- circle, *c*, about the center point of *l*, *C*, with radius *d* (1*m* in the figure). The shaft will have the uniform diameter *d* below line *l*.

3. Determine the smallest diameter at the top of the shaft (1.5*m* in our case). Draw a perpendicular, *l'*, through an end-point of the diameter. *l'* intersects *c* at a point *P*. The line through *P* and *C* defines together with *l* a segment of *c*.

4. Divide the segment into segments of equal size and divide the shaft above *l* into the same number of sections of equal height.

5. Each of these segments intersects *c* at a point. Draw a perpendicular line through each of these points and find the intersection point with the corresponding shaft division as shown in Figure 1-2. Every intersection point is a point of the profile.



1-2
Classical column with diminishing diameter

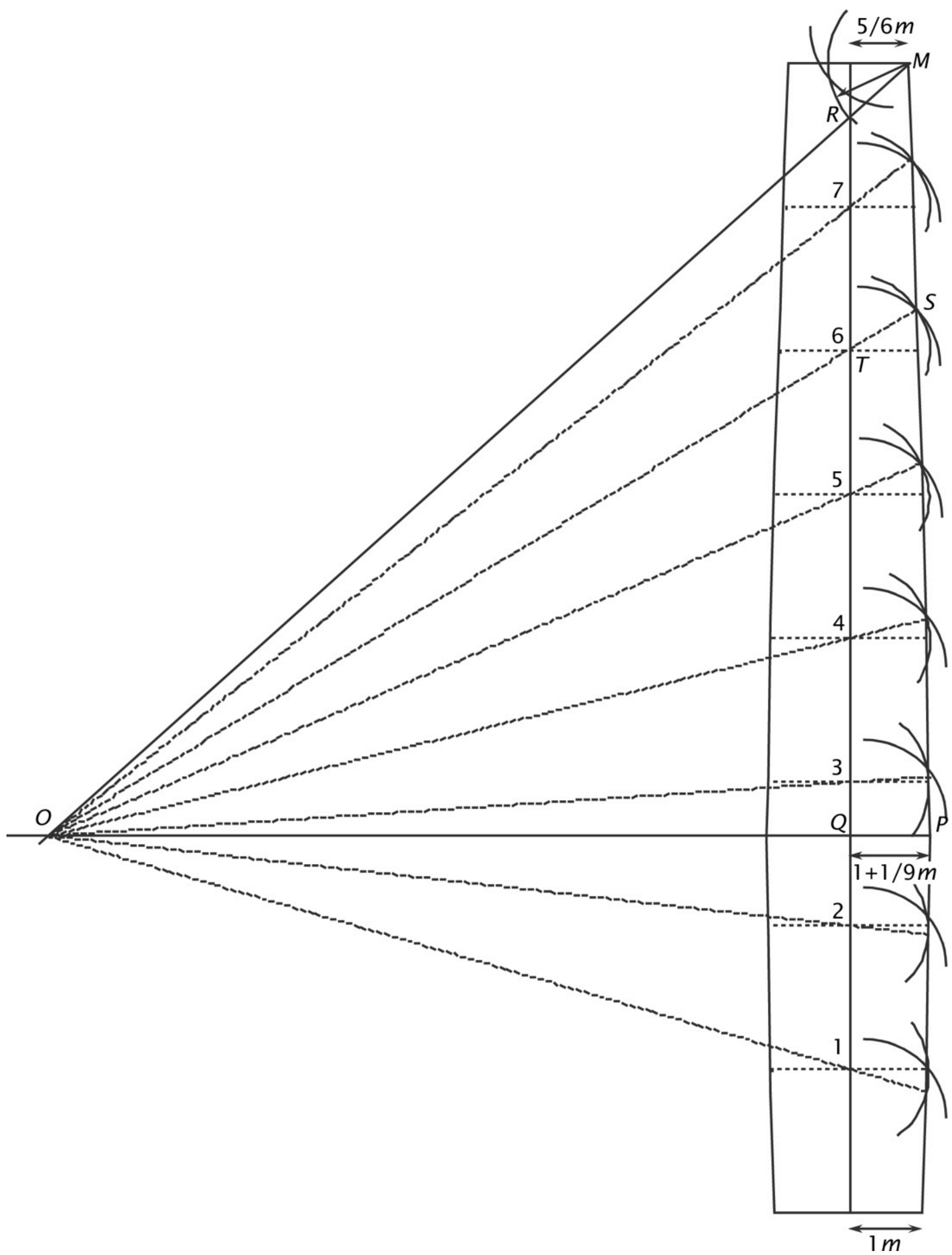
Construction 1-2
Constructing the profile of a classical column with entasis

Here, again, there are five steps.

1. Determine the height and its diameter (or radius) where it is widest and at the top. Following Vignola, the base is again assumed to be 2*m* wide, and the height is 16*m*;

that is, the proportion of the diameter to height is 1:8. The widest radius occurs at $\frac{1}{3}$ of the total height and is $1+\frac{1}{9}m$. The radius at the top is $\frac{5}{6}m$.

2. Draw a line, l, through the column where it is widest. Call the center point of the column on that line Q and the point at distance $1+\frac{1}{9}m$ from Q on l, P.



1-3
Constructing a classical column with entasis

3. Call the point at distance $\frac{5}{6}m$ from the center at the top. On the same side as *P, M*, draw a circle centered at *M* with radius $1+\frac{1}{9}m$; that is, the circle with the widest radius of the column. This circle intersects the centerline of the column at point *R*.
4. Draw a line through *M* and *R* and find its intersection, *O*, with *l*.
5. Draw a series of horizontal lines that divide the shaft into equal sections. Any such line intersects the centerline at a point *T*. Draw a circle about each *T* with radius *m*. The point of intersection, *S*, between this circle and the line through *O* and *T* is a point on the profile.

The construction is illustrated in Figure 1-3.

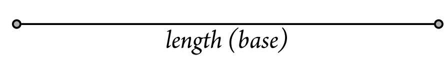
Both constructions illustrate architectural forms – that is, spatial forms – that can be produced using simple mechanical tools, in this case, a compass to produce circular arcs and a ruler with measurements marked on it. Of course, the construction is augmented with freehand sketching. The steps assume an ability to construct particular lines, for example, lines parallel and perpendicular to the given line using just such tools. Such 'mechanical' constructions are the subject matter of this course.

1.2 MEASUREMENTS

There are other kinds of useful practical geometric constructions. Among these are those that involve measurements, specifically, to calculate length, area, volume and so on.

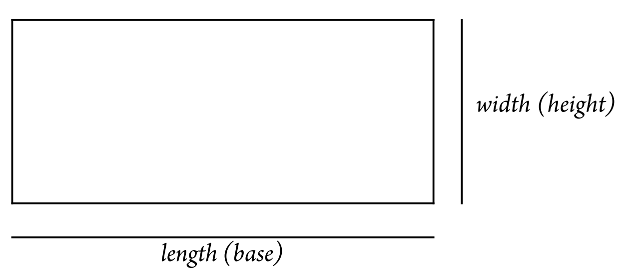
We use a line, or more specifically the *length* between the end-points of the line as a representation of one kind of measurement — namely, a linear measurement.

1-4
Length of a line as a measure



When a geometric figure is drawn on a sheet or a surface it occupies a certain portion of the surface, which is referred to as its *area*. A rectangle is representative of area and is specified by its *length (base)* × *width (height)*. The length and width represent two distinct linear measurements.

1-5
Area represented by a rectangle



If height = 1, then **area** = **length**

If height = 10, then **area** = **length** *plus* a positional addition of **a zero** at the end of the number or by moving the decimal point to the right by **one position**

If height = 100,then **area** = **length** *plus* a positional addition of **two zeroes** at the end of the number or by moving the decimal point to the right by **two positions**

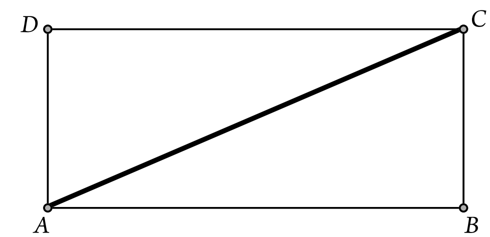
And so on …

That is,

The length of a line can represent area

1.2.1 Area of a triangle

We can use rectangles and a theorem that states that the diagonal divides a rectangle into identical triangles.

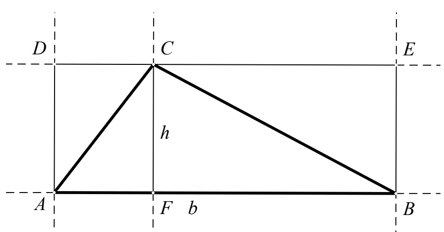


1-6
The diagonal of a rectangle divides it into two identical triangles

$$\square_{area} ABCD = \triangle_{area} ABC + \triangle_{area} ADC$$

$$\therefore \triangle_{area} ABC = ½\ \square_{area} ABCD$$

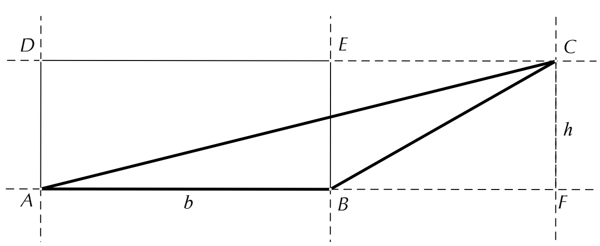
Consider the triangles shown in Figure 1-7.



$$\triangle_a ABC = \triangle_a ACF + \triangle_a CFB$$

$$\therefore\ \triangle_a ABC = ½\ \square_a ADCF + ½\ \square_a CEBF$$

$$= ½\ \square_a ABED$$



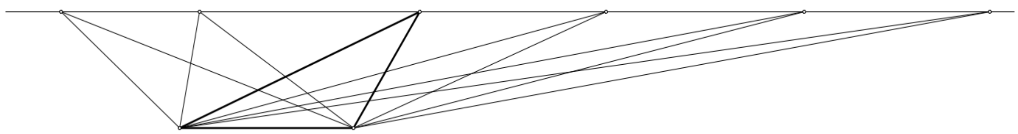
$$\triangle_a ABC = \triangle_a ACF - \triangle_a CFB$$

$$\therefore\ \triangle_a ABC = ½\ \square_a ADCF - ½\ \square_a CEBF$$

$$= ½\ \square_a ABED$$

1-7
Two triangles with the same base and height have the same area

Consider Figure 1-8. What can we say about these triangles?



1-8
What can we say about these triangles?

☛ ANSWER

1.3 A SHORTHAND NOTATION

Many of you, perhaps, tweet. You almost certainly text. You probably invoke some kind of abbreviation, or shorthand. We employ something similar when describing constructions. In fact, you may have already noticed their use in the constructions above. For consistency of description, I employ the following shorthand convention (with more added as we move through the subject):

Points are identified by a single uppercase letter: *A, B, C*, …, *P, Q, R*

Measurements such as lengths are given in lowercase italics: *a, b, c, …*

Common geometrical figures are denoted in shorthand

–*A*–*B*–, –*AB*–	A *line* passing through points *A* and *B*.
–*A*–	A *line* passing through point A.
A–	A *ray* emanating from point *A*
$\overline{AB}$	The *line segment* between points *A* and *B*
AB	Length of the line segment between points *A* and *B*
$\|AB\|$	The signed length of the segment *AB*. $\|AB\| = -\|BA\|$
⊥	Is perpendicular, e.g., –*AB*– ⊥ –*C*–
‖	Is parallel, e.g., –*AB*– ‖ –*C*–
O(*r*)	Circle centered at point O with radius *r*.
O(*AB*)	Circle centered at O with radius equal to the length *AB*
O(*P*)	Circle centered at *O* with radius = *OP*
△	A triangle
□	A rectangle, parallelogram, rhombus, or quadrilateral
$\square_n$	An *n*-sided geometrical figure, e.g., $\square_6$ for a hexagon

$\triangle ABC$	Triangle with corners A, B and C
$\square\ ABCD$	Quadrilateral with corners A, B, C and D
▫$ABCDEF\ \ldots$	Polygon with corners A, B, C, D, E, F, …
$\angle BAC$	Angle at A defined by sides AB and AC
$\triangle_a, \square_a, \ldots$	Area of triangle, etc.

Construction 1-3

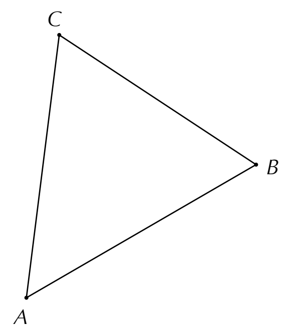
Construct a triangle with the same area as a given triangle and with a given base

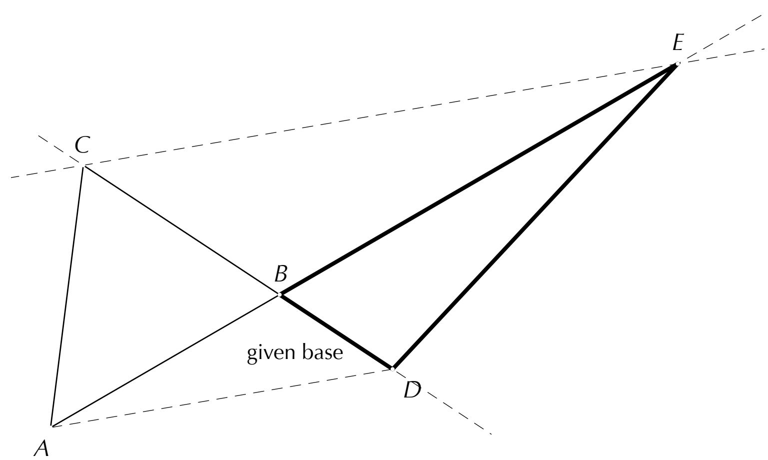
Let $\triangle ABC$ be the given triangle.
(See figure on the right.)

1. Extend –CB– to –CBD– so that BD = given base.
2. Draw a line –C– parallel to –AD–, that is, –C– ‖ –AD–; and extend –AB– to intersect it at E

$\triangle\ BED$ is the required triangle.

The construction is shown in Figure 1-9.



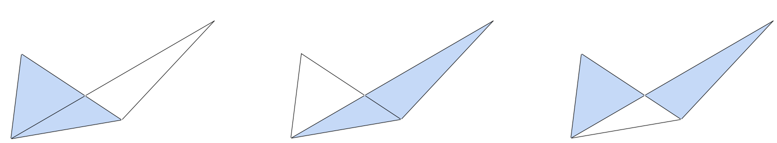


1-9
$\triangle_a BED = \triangle_a ABC$

☛ *QUESTION: Why is $\triangle_a BED = \triangle_a ABC$?*

☛ *ANSWER*

Explanation will be given in class but these diagrams should help.

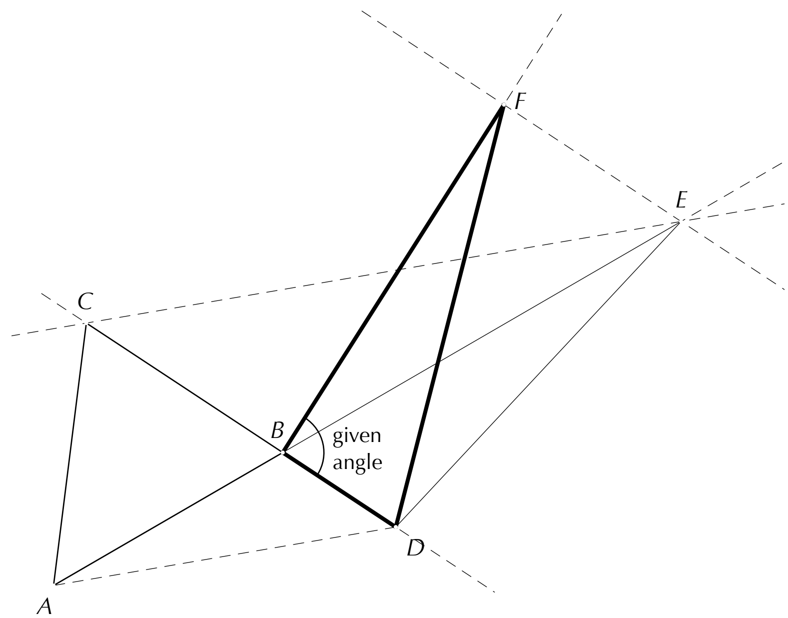


Construction 1-4
Construct a triangle with the same area as a given triangle and with a given base and angle

Suppose additionally we are given an angle as well.

1. Construct $\triangle BED$ as before.
2. Draw a line –*E*– parallel to –*CD*–
3. Draw a line at the given angle to –CBD– at B to intersect –*E*– at *F*

$\triangle BDF$ is the required triangle with given base *BD* and $\angle DBF$, the given angle. The construction is shown in Figure 1-10.



1-10
Construction 1-3 with an added angle constraint

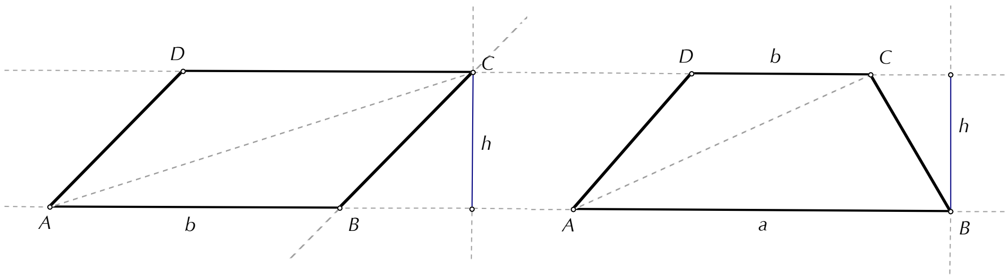
☛ *QUESTION: Can you find a single line whose length equals the area of a triangle based on what we have done so far?*

☛ *ANSWER*

Left as an exercise

(HINT: What should the length *BD* be?)

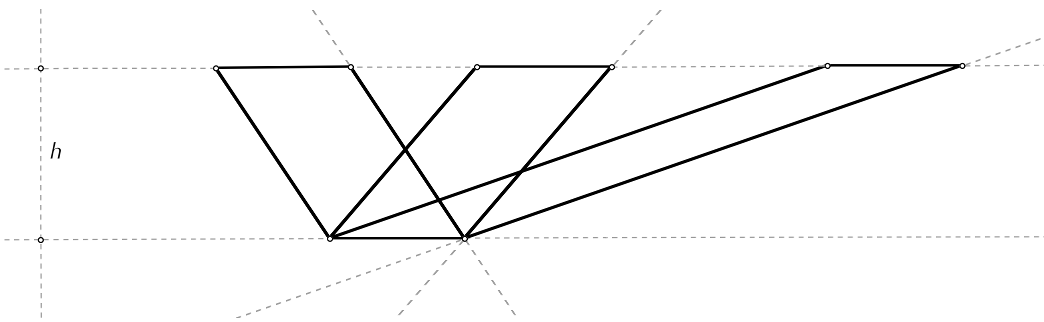
1.4 PARALLELOGRAMS AND TRAPEZIUMS



$\square_a ABCD = \triangle_a ABC + \triangle_a ADC$
$= ½\ bh + ½\ bh = bh$

$\square_a ABCD = \triangle_a ABC + \triangle_a ADC$
$= ½\ ah + ½\ bh = ½(a+b)h$

1-11
Area of a parallelogram and trapezium



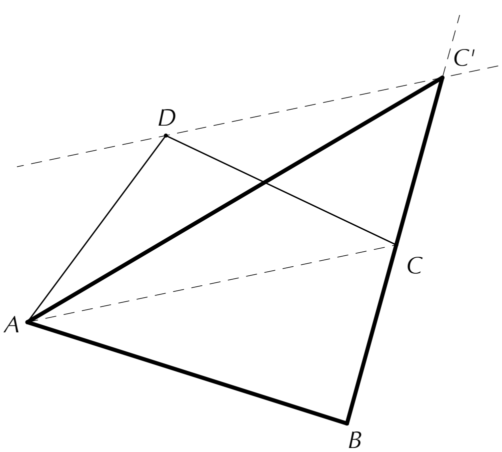
1-12
Parallelograms with the same base and height have the same area

Construction 1-5
Construct a triangle with the same area as a quadrilateral

Let ▱ *ABCD* be the given quadrilateral

1. Draw a line –*D*– through *D* parallel to the diagonal –*AC*–
2. Extend –*BC*– to meet this line at *C′*.

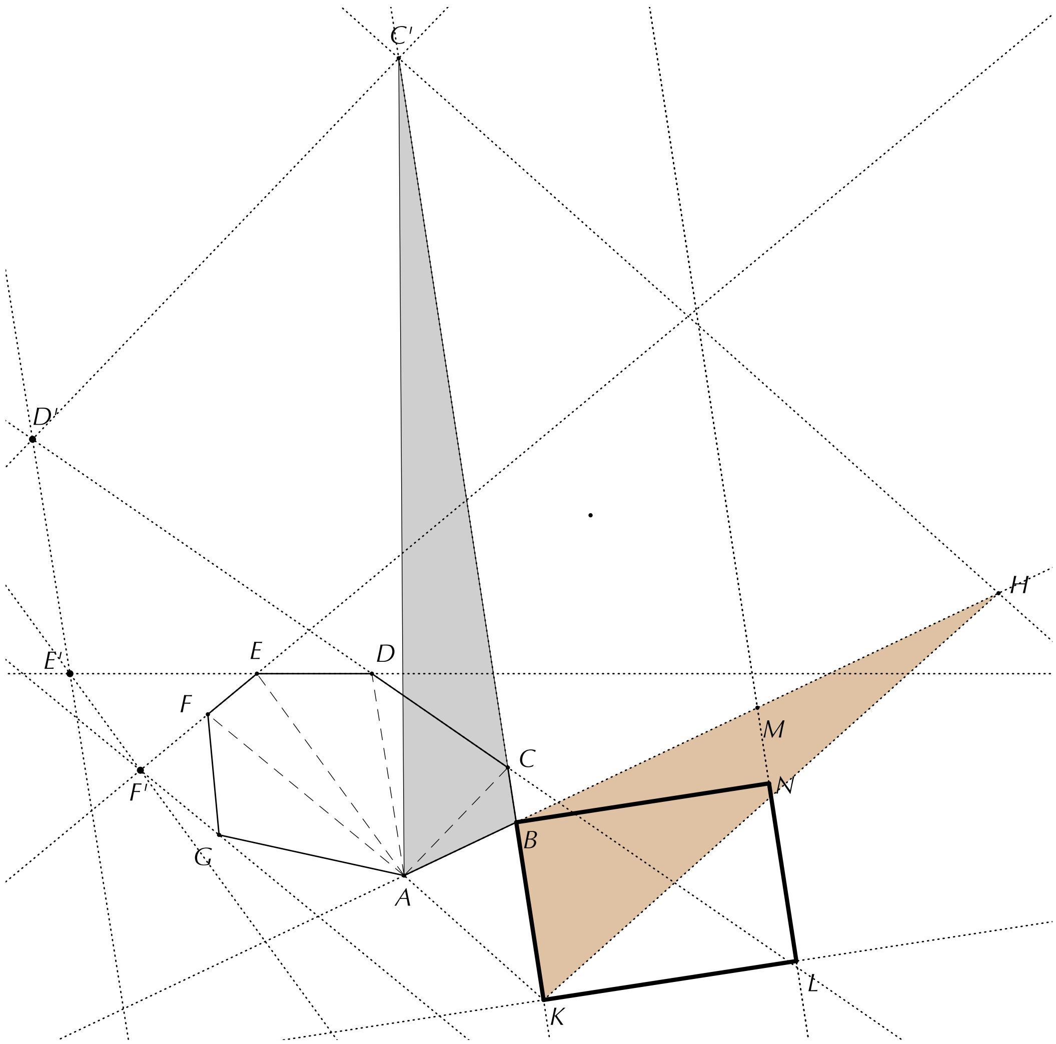
△ *ABC′* is the required triangle



1-13
Converting a quadrilateral to a triangle with equal area

Construction 1-6
Constructively calculate the area of a polygon

The construction is shown in Figure 1-14.



1-14
Area of a polygon

Note that constructively $\triangle_a ABC' = \triangle_a BKH = \square_a BKLN$ (Why?)

If $BK = 1$, then area of polygon $\square_a\, ABCDEFG$ is given by the length, *BN*.

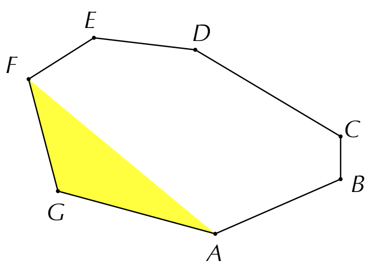
$\square$ABCDEFG is a polygon with area identical to $\triangle$ *ABC'*.

☛ *QUESTION*: *What is the construction?*

☛ *ANSWER*

We successively eliminate triangles until we only have one triangle

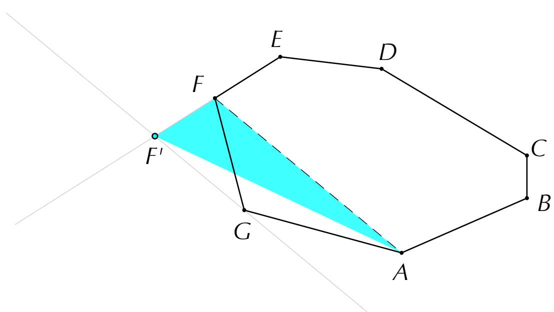
In the example polygon ▱$ABCDEFG$ above, consider $\triangle AGF$.



Draw a line –G– parallel to –AF–

Extend –EF– to meet this line at F'.

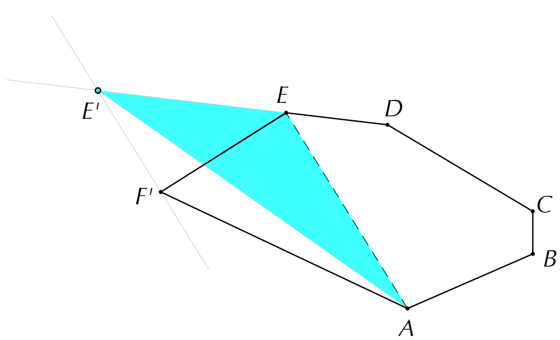
Then, $\triangle AF'F = \triangle AGF$ and polygon ▱$ABCDEF'$ has the same area has polygon ▱$ABCDEFG$.



We repeat this for $\overline{AF}$. That is,

Draw a line –F'– parallel to –AE–

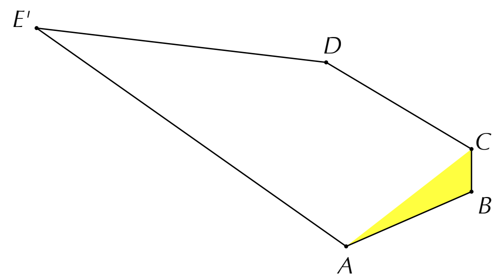
Extend –DE– to meet this line at E'.



But order is unimportant.

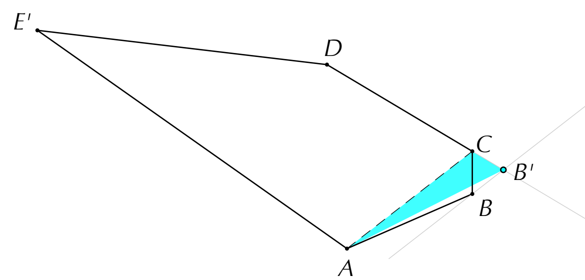
We can choose any triangle to eliminate at any time.

Suppose we choose $\triangle ABC$.

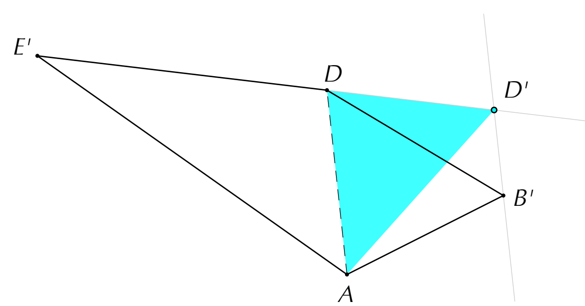


Draw a line –B– parallel to –AC–

Extend –DC– to meet this line at B'.



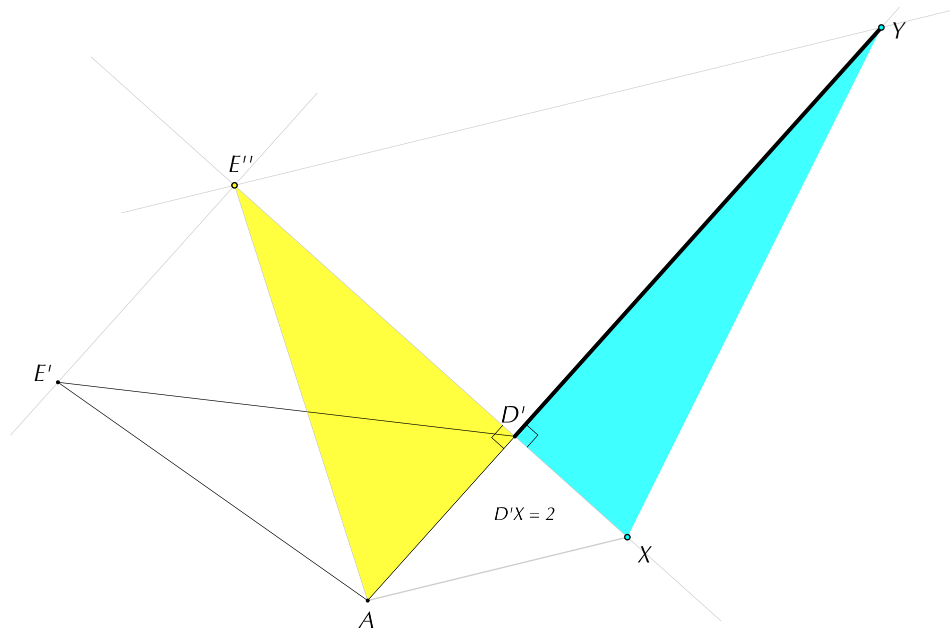
We now reduce the quadrilateral $\square\ AB'DE'$ to the $\triangle AD'E'$



The last step is to reduce triangle $\triangle AD'E'$ to an equivalent right-angled triangle with base equal to two.

Below

$\triangle AD'E' = \triangle AD'E'' = \triangle D'XY$



As $D'X = 2$ and $\angle XD'Y = 90^{\circ}$, $D'Y$ measures the *required area*

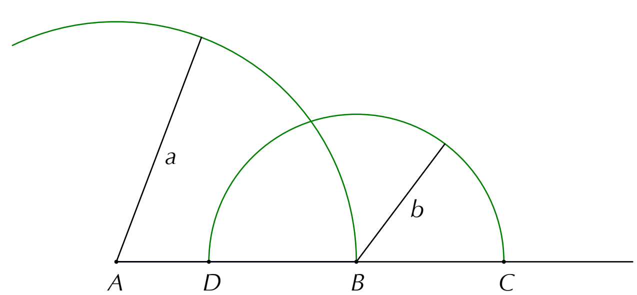
1.5 CONSTRUCTIBLE NUMBERS

Adding and subtracting numbers is trivially simple

Construction 1-7
Addition and subtraction

To add or subtract two numbers, say, a and b, we draw a line and mark a point A on it. Construct the circular arc $A(a)$ to meet the line at B. The circular arc $B(b)$ meets the line at two points C and D as shown in Figure 1-15. Then, $AC = a + b$ and $AD = a - b$.

1-15
Constructively adding and subtracting two numbers

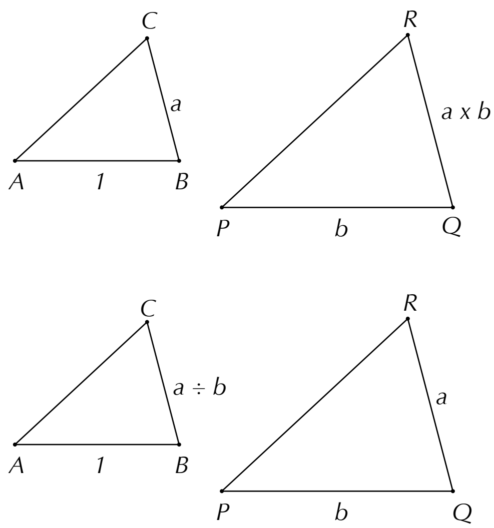


Construction 1-8
Multiplication and division

Similar triangles preserve proportion between corresponding sides

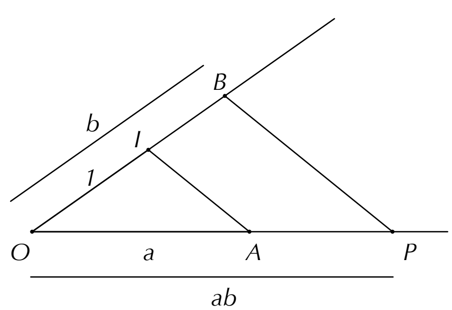
That is, if ΔABC and ΔPQR are similar triangles, then the ratio AB:PR is the same as BC:QR.

Suppose $AB = 1$, $PQ = b$. If $BC = a$, then QR must equal $a \times b$ (see Figure 1-16). If $QR = a$, then BC must equal $a \div b$.

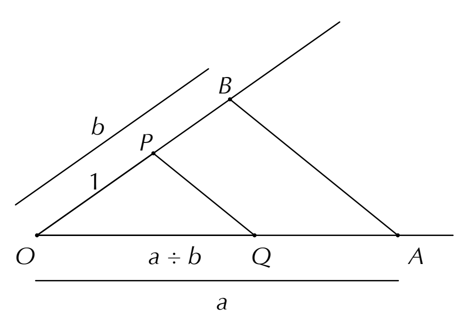


1-16
Similar triangles preserve proportion

The constructions for multiplication and division are shown in Figure 1-17. In each case we construct an arbitrary but convenient angle; from the corner O draw circle $O(a)$ to meet one side at A and draw circles $O(1)$ and $O(b)$ to meet the other side at P and B.



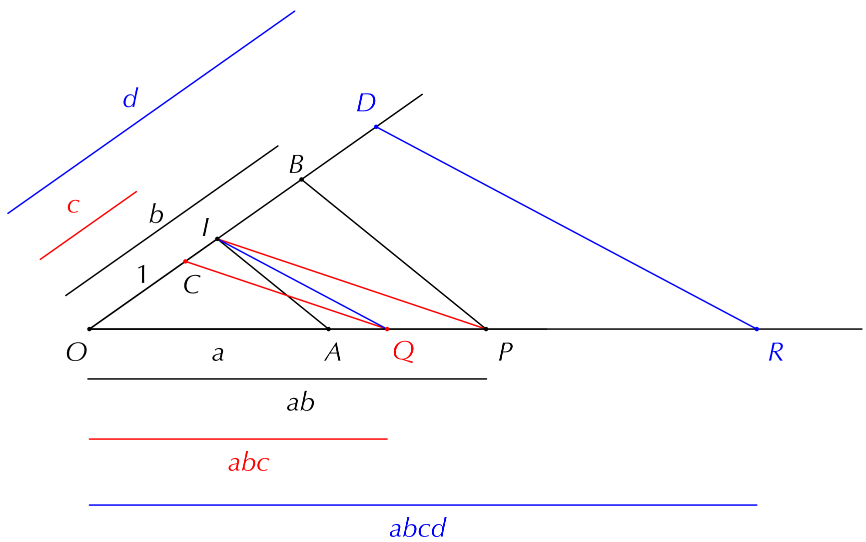
1-17
Multiplying (*above*) and dividing (*right*) numbers



For multiplication, draw a line through B parallel to meet the angle at Q. OQ is the required multiplication.

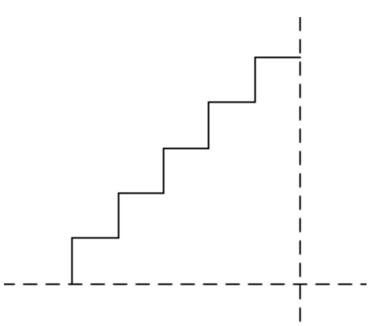
For division, draw a line through P parallel to meet the angle at Q. PQ is the required division.

We can clearly extend this to multiply (or divide) several quantities together. Figure 1-18 shows how to multiply four numbers together.



1-18
Multiplying four numbers constructively

☛ *QUESTION*: *Can you use of any of these arithmetical constructions to calculate the riser and tread for a staircase?*

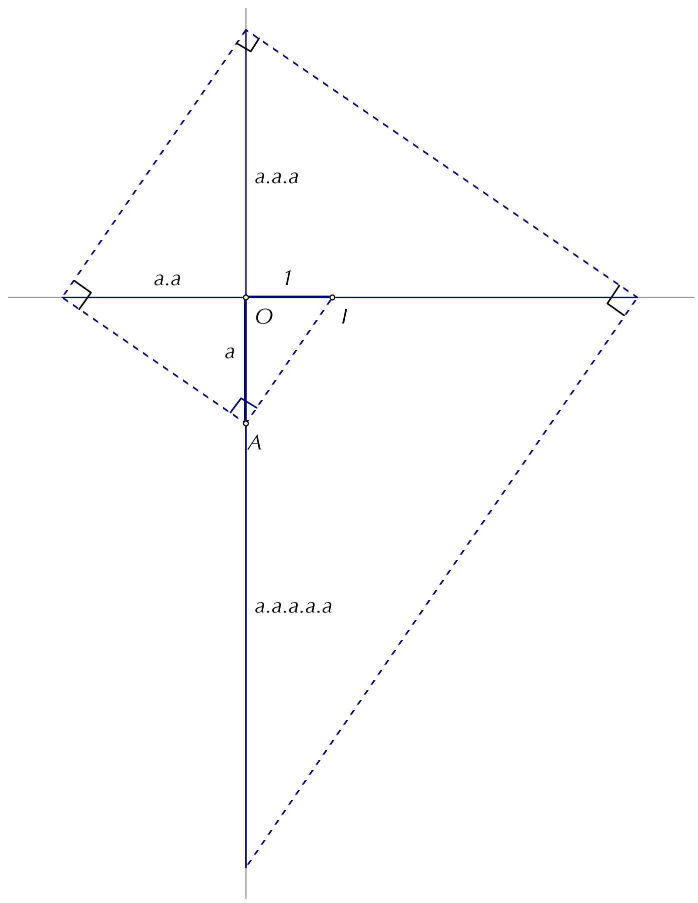


☛ *ANSWER*

Left as an exercise

Construction 1-9
Powers of quantities

The above constructions can be used to obtain the powers of numbers such as a^2, a^3, a^4 and so on. A simpler construction based on similar right (angled) triangles is shown in Figure 1-19.

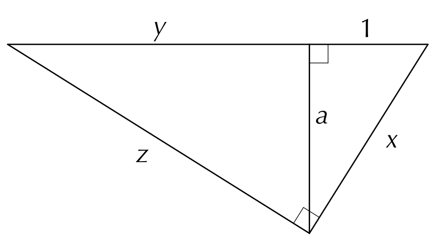


1-19
Powers of a number
obtained constructively

☛ *QUESTION*: *Why does this construction work?*

☛ *ANSWER*

Consider any two successive triangles in the sequence as shown below.



From three right angled triangles

$x^2 = a^2 + 1$, $z^2 = a^2 + y^2$ and

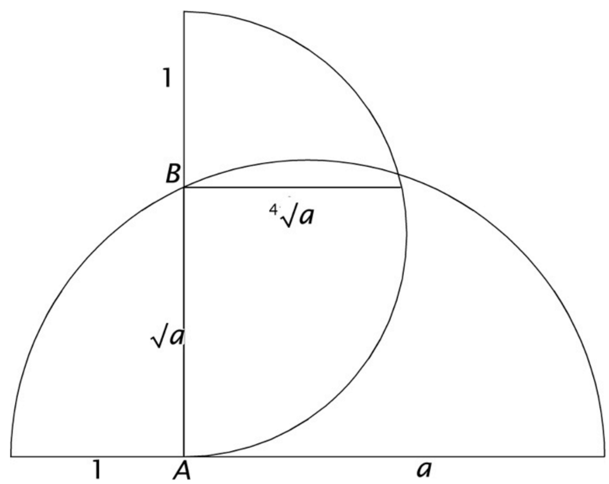
$(y+1)^2 = x^2 + z^2 = 2a^2 + y^2 + 1$

$\therefore\ y = a^2$

Construction 1-10
Square Roots

Like powers, square roots of numbers can be calculated graphically. On a line mark off a unit length and a length = a. On the combined length draw a semi-circle as shown in Figure 1-20. At A, draw a perpendicular to meet the semi-circle at B. $AB = \sqrt{a}$.

As Figure 1-20 indicates, we can continue the construction to produce $\sqrt[4]{a}$ and so on.



1-20
Even roots of a number

1.5.1 Theoretical versus practical constructions

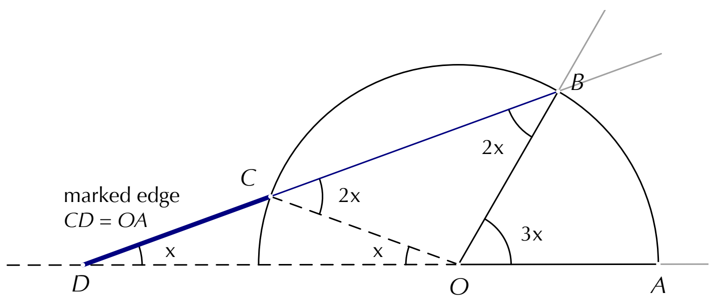
There are geometrical problems that cannot be solved using conventional geometric tools such as a compass or ruler in the pure sense. Among the well known 'impossible' constructions are the following problems:

- *Squaring the circle*. Constructing a square having the same area as a given circle.
- *Duplicating a cube.* Constructing the edge of a cube having twice the volume of a given cube.
- *Trisecting an angle*. Constructing two lines that divide a given angle into three equal parts.

However, some of these so-called "impossible" constructions can be solved, by tweaking the tools a fraction. Here are two examples. The deceptive nature of the impossibility of the constructions is illustrated by the following trisection constructions.

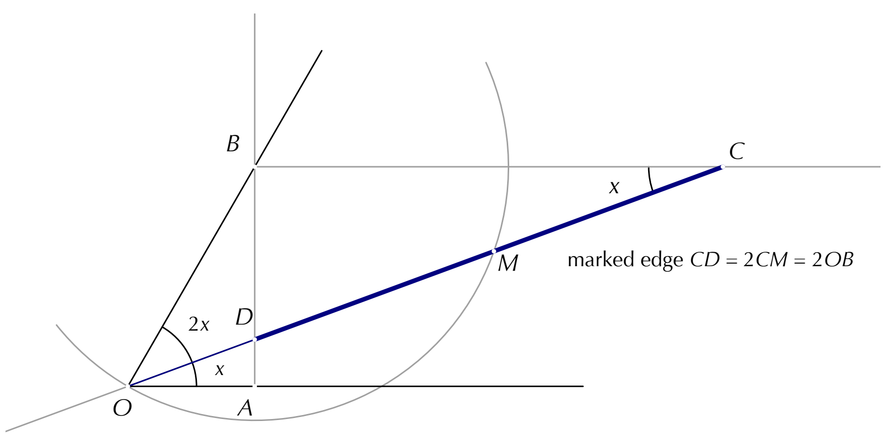
Trisecting an angle			
Given	$\angle AOB$ (without loss in generality, let $OA = OB$)		
Draw	$O(OA)$, $–BCD–$, $CD=OA$, $OC=OA$	$–OA–$, $–BD–$	$\angle ADB$
Points of intersection	C	D	

The proof for angle trisection is straightforward (see Figure 1-21). As can be seen from the figure, the construction requires a 'marked' straightedge, where the distance CD ($=OA$) is marked on the straightedge. This is distinct from Euclidean constructions, which rely solely on a compass and an *unmarked* straightedge. Clearly, in this construction the straightedge has to be aligned so that marked points C and D are coincident with the semi-circle and horizontal line respectively.



1-21
Trisecting an angle using a 'marked' straightedge

The above construction creates an angle 1/3rd the measure of the given angle. The following construction, illustrated in Figure 1-22, creates a trisected angle using one of the sides of the given angle, again using a 'marked' straightedge.



1-22
Another method for trisecting an angle using a 'marked' straightedge

1.5.2 The insertion principle

A marked straightedge is an example of a mechanical construction device known as the insertion principle.

The above constructions are each an example of a construction drawn by the insertion principle.

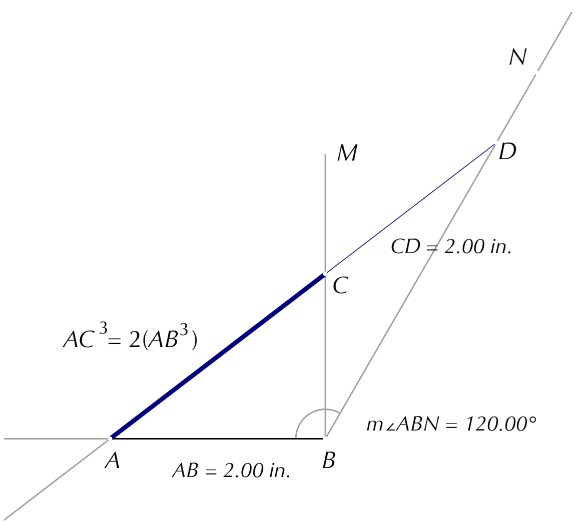
> **Insertion Principle**
>
> Given two curves and a point, the line passing through the given point, touching the two curves at points corresponding to marked locations on a straight-edge, is said to be drawn by the *insertion principle*.

We can use the insertion principle to **duplicate a cube.**

If the length of a segment, say a, represents the side of a cube, then we are interested in finding a segment the length of which, say b, satisfies $b^3 = 2(a^3)$.

On a given line segment, AB, construct a perpendicular, BM–, and another ray, BN–, which meets $\overline{AB}$ at 120º.[1] Using a 'marked' straightedge construct a line that meets line –BM– at C and line –BN– at D such $AB = CD$. Then, $AC^3 = 2(AB^3)$.

See Figure 1-23.



1-23
Duplicating the cube by the insertion principle

The existence of 'impossible' constructions, and perhaps, to explore 'possible' constructions prompted early geometers to investigate and, potentially, invent new and different 'mechanical' devices. Some are the forerunners of modern drawing equipment. We illustrate three devices in light of the angle trisection problem.

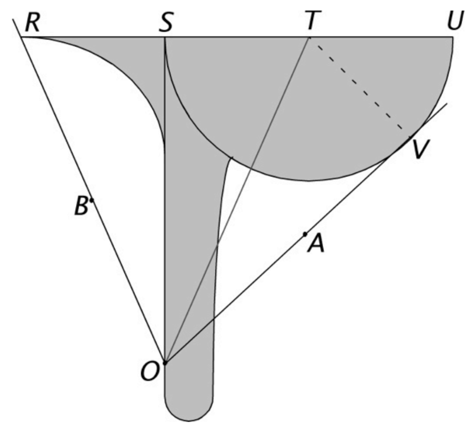
---

[1] An angle of 120° can be easily constructed with an unmarked straightedge and a compass. We leave this as an exercise to the reader.

1.5.3 Practical tools

Tomahawk

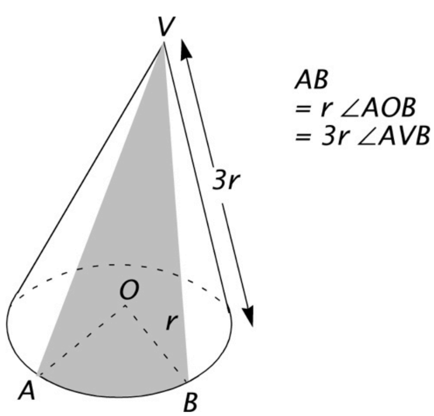
The Tomahawk was first described in 1835, although its author is unknown. The device is based on the following construction. RU is a trisected line such that $RS = ST = TU$. At S draw a line perpendicular to RU. With T as center construct a semicircle with radius ST. To use the device, align the given angle $\angle AOB$ such that ray OB– touches R and ray OA–is tangential to the semicircle and O lies on the perpendicular at S. Then, $\overline{OT}$ trisects angle $\angle AOB$. The proof follows from the fact that triangles ΔRSO, ΔTSO and ΔVTO are all congruent.



1-24
Tomahawk
See Eves,
An Introduction to the History of Mathematics, 1983

Aubrey Right Circular Cone

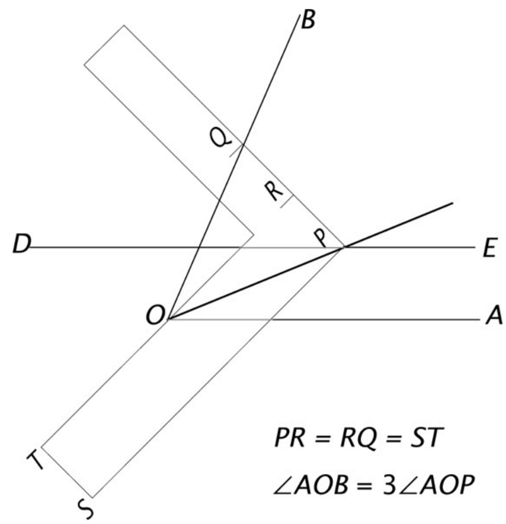
The following tool offers a simple 'practical' way to solving the trisection problem. Construct a right circular cone (say, of wood) with center O and apex V, with slant height thrice its radius. Mark points A and B on the circumference of the base so that the given angle is $\angle AOB$. Wrap a paper around the cone and mark points corresponding to A, B and V. Flatten the paper. $\angle AVB$ is the required trisected angle. Proof follows by considering the length of the chord AB.



1-25
Aubrey Cone to trisect an angle

Carpenter Square

This tool is readily available in any hardware store. Although the carpenter square was not specifically developed for the trisection problem, however, otherwise, it does come in handy. $\angle AOB$ is the given angle. Construct a line –*DE*– parallel to line –*AO*– at distance equal to wider arm of the carpenter's square. Mark points P, Q and R on the other arm such that R is the mid-point of $\overline{PQ}$, which equals twice the height of the wider arm. Align point P on –*DE*– and the other edge at O as shown. $\angle POR$ is the required trisected angle. Proof follows from considering similar triangles.



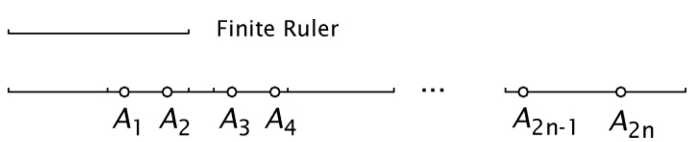
1-26
Carpenter Square to trisect an angle

1.6 CONSTRUCTIONS FROM PROJECTIVE GEOMETRY

Construction 1-11
Drawing a line between two points using a short ruler

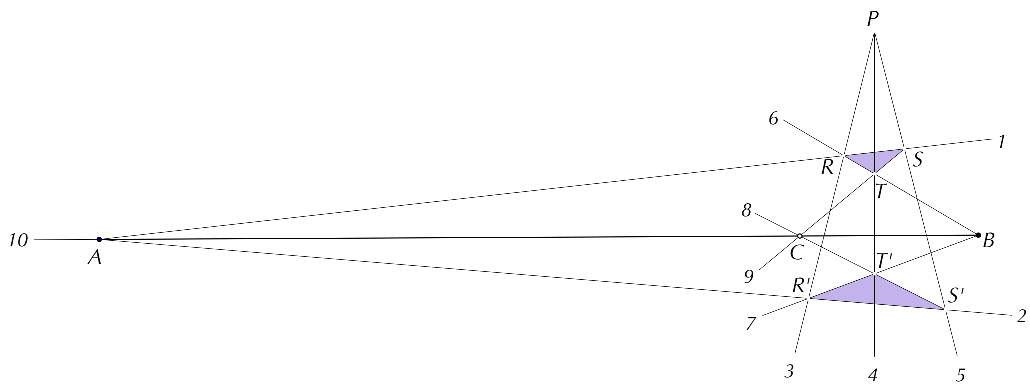
Interestingly, the following construction borrows from the theory of perspectives (or rather projective geometry). We will see more on perspective in this class.

Consider the following trial and error construction. Construct any line with the finite ruler. Mark two points on the line the distance of which is less than the length of the ruler. Align the ruler with these points and extend the line. This process can be repeated indefinitely.



1-27
Constructing an indefinite line using a finite ruler by 'trial and error'

Let the length of the ruler be ε (*epsilon*) units. (It is a very short ruler!) Two points are said to be ε-*near* (pronounced *epsilon near*) if the ruler can span them. Let A and B be two points that are not ε-near. By trial and error perhaps, repeatedly, it is possible to construct two lines through A, say 1 and 2, such that the lines 1, 2 and AB are ε-near. That is, the ruler spans the three lines. Choose a point P ε-near B such that by using the finite ruler, we can construct lines 3, 4 and 5 through P that cut the lines 1 and 2. See Figure 1-28.



1-28
Construction to join two points far enough apart by a line using a finite ruler

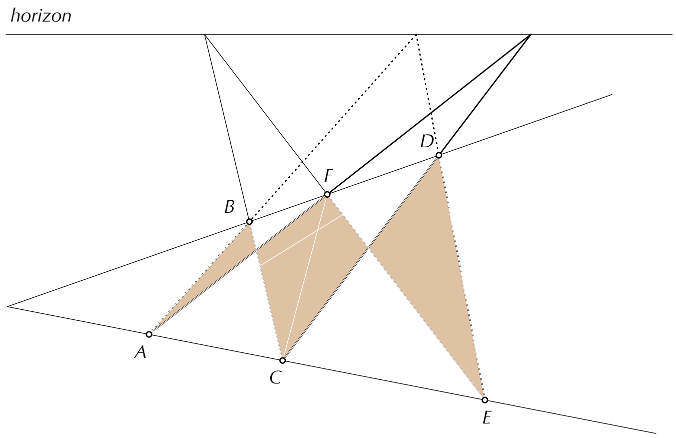
Points R and R' are joined by lines 6 and 7 through B. The lines intersect line 4 at T and T'. Join T by line 8 through S. Likewise, join T' by line 9 through S'. Then, lines –ST– and –ST'– meet at C, and points A, B and C are collinear. Moreover, C is ε-near B. It follows that by the trial and error construction described above, it is possible to extend BC to meet A (line 10).

[The proof is omitted here. For the interested reader, it follows from Desargues configurations. See Section 1.6.1 below.]

1.6.1 Desargues configuration

It is well known that if two particular pairs of lines are parallel then a third pair is likewise parallel. There is a counterpart to this notion in projective geometry where parallel lines meet at the horizon, namely, if two particular pairs of lines meet at the horizon so does a third pair. The Pappus and Desargues theorems express this differently about three pairs of lines having their intersection on the *same* line. The projective Pappus configuration shown in Figure 1-29 illustrates six points lying alternatively on two straight lines forming a hexagon whose opposite sides meet on a line, namely, the horizon.

The projective Desargues configuration states that if two triangles are in perspective from a point, then their corresponding sides meet on a line. See Figure 1-30. In Figure 1-28, the two triangles $\triangle RST$ and $\triangle R'S'T'$ are in perspective from P and the required line –AB– corresponds to the horizon of the perspective projection.



1-29
Projective Pappus configuration

Co-polar and co-axial triangles

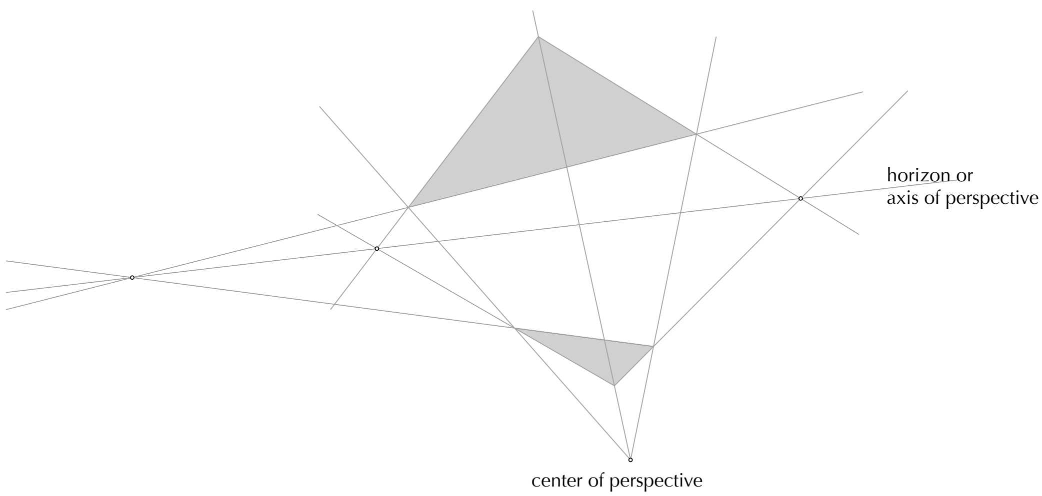
Two triangles are *copolar* if the lines joining corresponding vertices are concurrent.

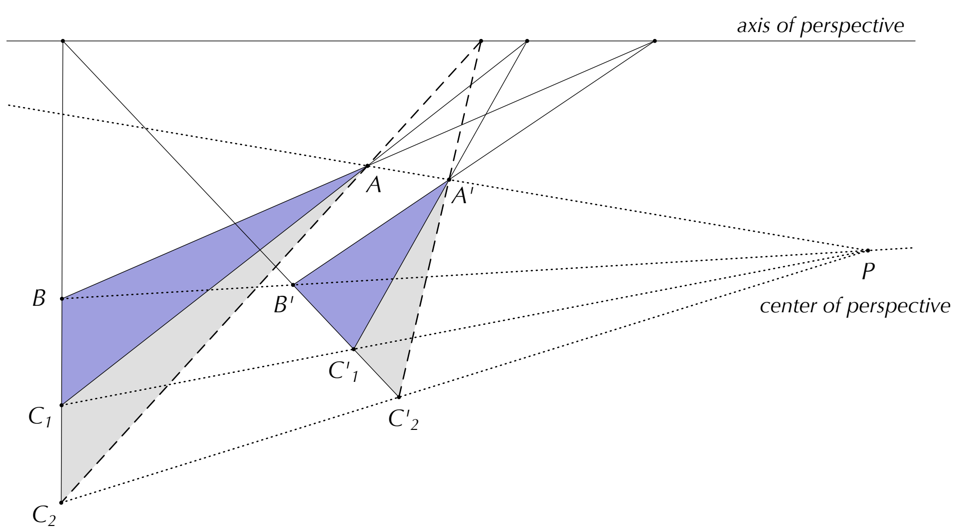
The triangles are *coaxial* if the points of intersection of corresponding sides of the
triangles are collinear.

The concurrent point is called the *center* of perspective and coaxial line is the *horizon* or *axis* of perspective.

Desargues theorem can restated as:

Copolar triangles are coaxial and conversely

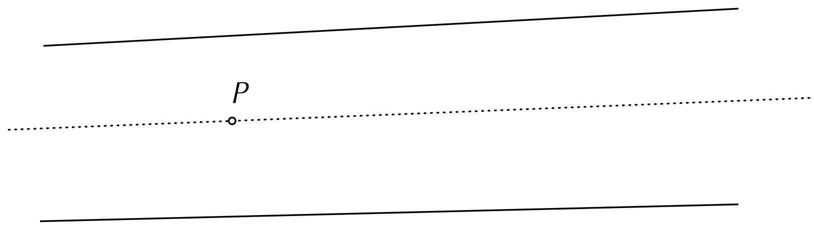




1-30
Desargues configurations

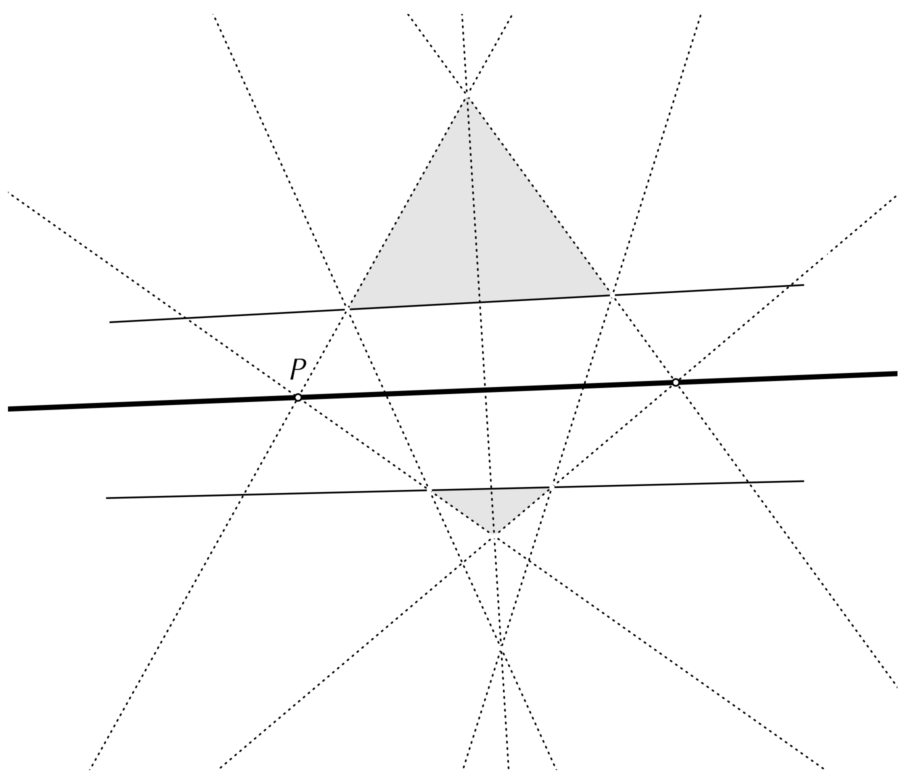
☛ *A TYPICAL QUESTION*

Suppose we are given a point *P* and two lines whose point of intersection is inaccessible (i.e., not shown on the drawing), construct the line through *P* that meets the inaccessible point of intersection (see the figure below indicating the desired line shown dotted).



☛ *ANSWER*

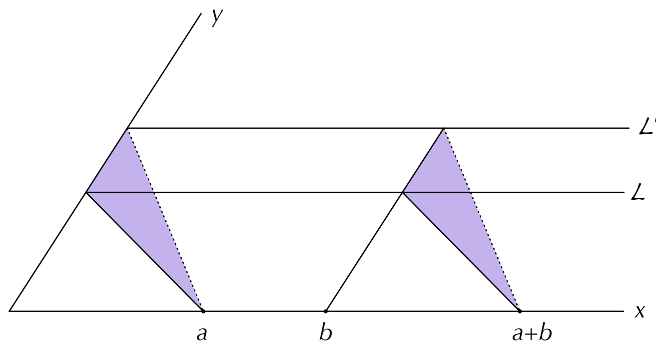
[Left as an exercise – HINT – construct a Desargues configuration involving point *P* and the two lines as shown below in Figure 1-31.]

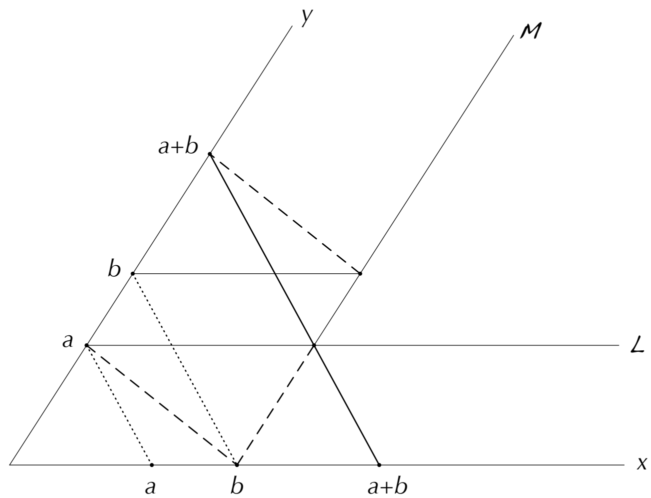


1-31
Finding a line through a point and the inaccessible point of intersection of two lines

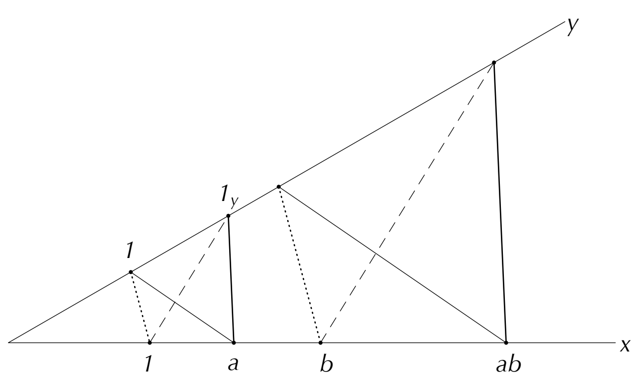
1.6.2 Projective arithmetic

We can employ the projective Desargues configuration to add two numbers and a variation to multiply two numbers as Figures 1-32 and 1-33 show.



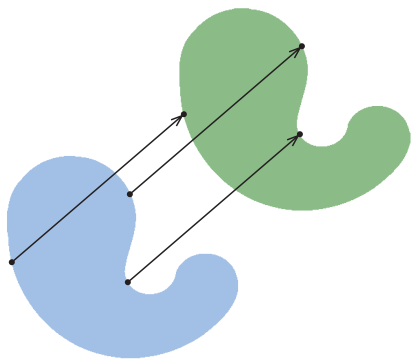


1-32
Addition by applying Desargues configuration
(*Top*) Sum is independent of the choice for *L*
(*Bottom*) Sum is independent of the choice of axis



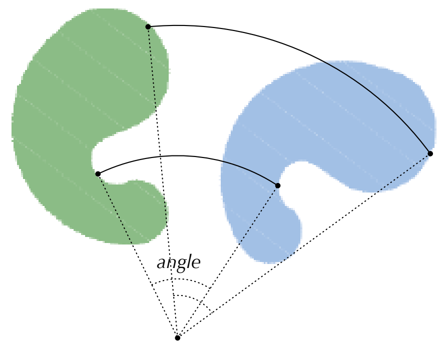
1-33
Multiplication by applying a variation of Desargues theorem
(The construction is independent of the unit in both axes)

1.7 GEOMETRIC TRANSFORMATIONS IN THE PLANE



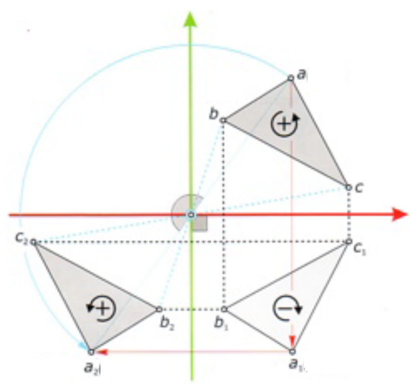
Translation:

Each point on the figure moves to a corresponding point on the *translated figure* by the same distance in the same direction.



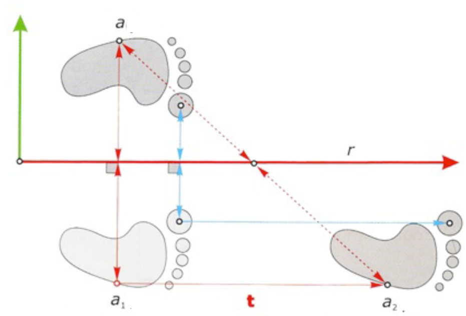
Rotation:

Each point on the figure moves to a corresponding point on the *rotated figure* by the same angle and angular direction about a fixed center of rotation.



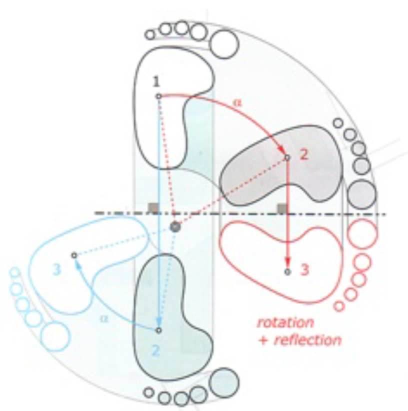
Reflection:

Each point on the figure moves to a corresponding point on the *reflected figure* by the same distance about an axis of reflection in a direction perpendicular to the axis.



Glide-reflection

Each point on the figure moves to a corresponding point on the *reflected figure* by a reflection and a translation.



Rotated Reflection:

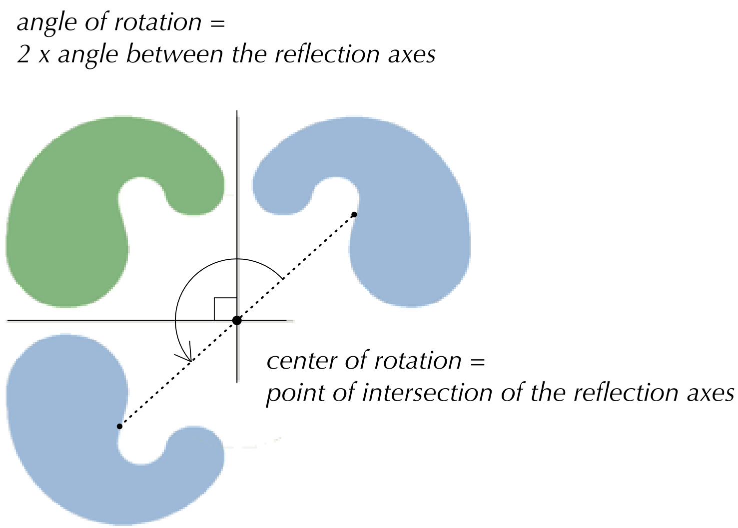
Each point on the figure is moved to a corresponding point on the *reflected figure* by a reflection and a rotation.

1-34
Basic geometric transformations in the plane

1.7.1 Rotating a geometric figure without using a compass

This construction relies upon properties relating to geometric transformations, symmetry and congruence. A little too much to explain here, but I will explain the basis of the construction in class. Figure 1-35 illustrates the construction.

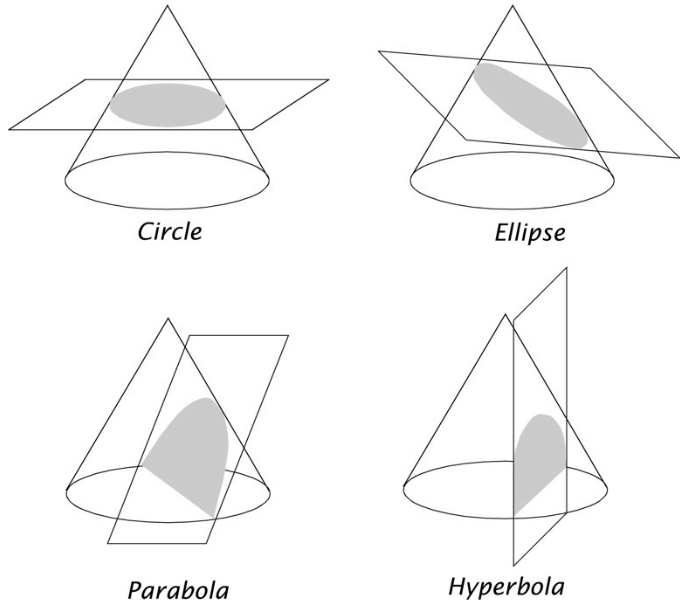
☛ *ANSWER*



1-35
Rotating a geometric figure without a compass

1.8 CONIC SECTIONS

I began this chapter by describing two practical constructions with application to Renaissance architecture. I conclude it by describing practical constructions for geometric objects of importance in architectural design. Specifically, we take a look at basic curved figures that are normally called *conic sections*, so called because all of these geometrical figures derive from a cone. If we take a right cone and intersect it by a '3-dimensional plane', *we obtain various geometrical figures depending on the inclination of the plane to the cone*. For instance, if the plane is parallel to the base of the cone, the cross section is a *circle*. If the plane slices the cone at an angle to the base, the resulting figure is an *ellipse.* If the plane cuts through both the cone and its base at an angle, the resulting figure is a *parabola.* If the plane is vertical to the base, the cross section is a *hyperbola.*

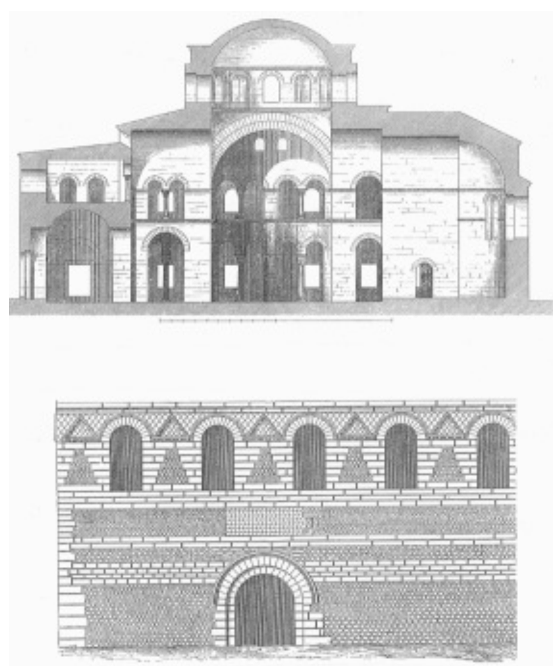


1-36
The conic sections

Circles are the most common figures to appear in architectural drawings. The reasons are manifold.

In traditional masonry construction, circles (more accurately, half- or semi-circles or other portions of a circle) appear in section as domes or vaulted ceilings over spaces too wide to be covered by wooden beams; or they occur in regions where stone and brick are more readily available as building materials than wood.

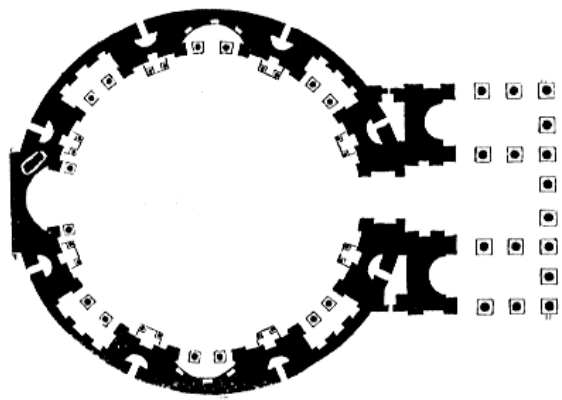
For the same reasons, we may find arches in elevations over openings instead of straight lintels (see Figure 1-37). The structural reason for this use is that bricks and cut stones arranged in semi-circles or in related curves are able to cover wide spans without being subjected to too much tensile stress (the tensile strength of these materials is rather weak).



1-37
Arches, walls and domes
in section or elevation

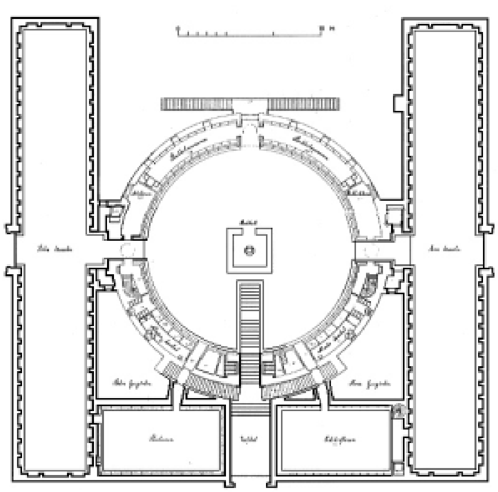
Over centuries of use, arches and vaults have become so universally accepted as forms that they are even employed in the absence of structural or economic reasons; that is, they have become elements of decoration and are, in fact, frequently abused in this manner. One should note that stress lines of these types of structures are rarely circular, and we find a variety of curves used besides the circle. The latter curve nevertheless dominates in practice because it is the easiest form to layout and construct.

Circles also appear in plan, but not so much for structural than for formal or expressive reasons. Most obvious is this use in buildings that have a circular plan overall. The prototype for this type of building in the western tradition is the Roman Pantheon (Figure 1-38), one of the best-preserved buildings to survive from antiquity.2It has inspired a host of very distinguished buildings in the classical tradition, for example, the library in Thomas Jefferson's campus for the University of Virginia at Charlottesville.



1-38
Plan of the Roman Pantheon

Circular buildings are generally occupied by a major space to which at most some small ancillary spaces have been added. For obvious reasons, circular buildings are rare when the functions become more complex and the resulting programs more diverse in terms of the spaces to be allocated (although there exist some notable attempts, for example by the French architect Viollet Ledoux). But in buildings with such programs, a circular room frequently marks the geometric center of the plan and is consequently reserved for the most important function in the building. Such rotundas are also noticeable in libraries, where they house the main reading room (see Figure 1-39).

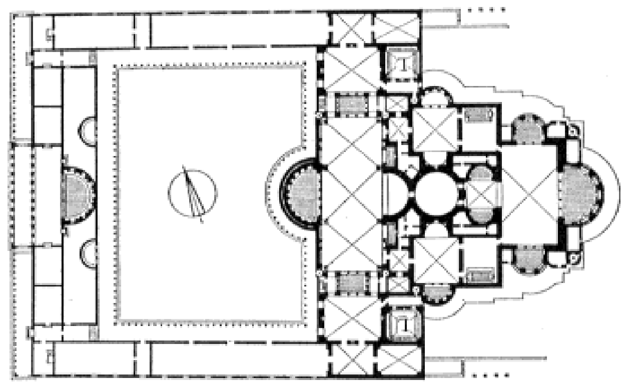


1-39
Plans with central rotundas:
Stockholm Public Library by Gunnar Asplund

Another prominent use of the circle in plan is that of a 'knuckle' or joint that turns a major circulation axis into another direction.

Semi-Circles also occur in similar circumstances in many plans.

Prominent is their use as the end of an axis, possibly combined with a special focus, as demonstrated by the apse at the end of a traditional church plan, which is also the place for locating the most important piece of furniture, the altar. Figure 1-40 shows the plan of a bath with a profusion of semi-circles.



1-40
Plans with semi-circles: Imperial baths, Trier, Germany

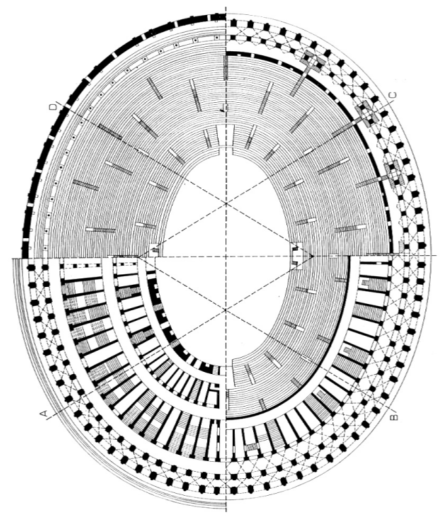
Parabolas are of special interest because the stress line of an arch forms a parabola under special loading conditions.

This means that the arch can be rather thin, whereas circular and other forms require a greater thickness to ensure that the stress line does not fall outside the arch. This is demonstrated in the famous arch spanning the central space of the palace at Ctesiphon in present-day Iraq that dates back to around 3rd century AD (see Figure 1-41). The arch is a beautiful example of a freestanding parabola.

1-41
Palace at Ctesiphon, Iraq (circa 3rd century AD)

Ellipses appear in circumstances similar to those in which circles appear; this is true for plans, sections and elevations.

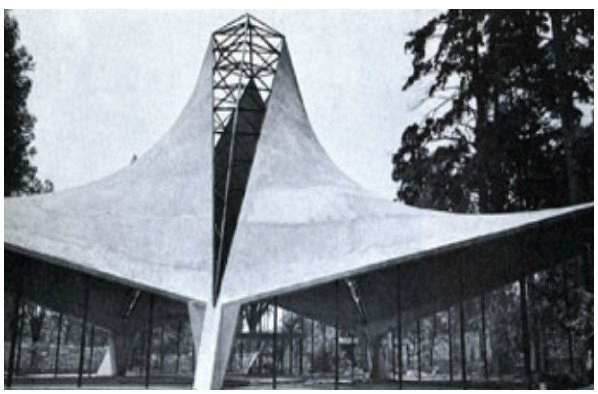
It is not always clear that the curve in question is a true ellipse (which is more difficult to construct than a circle); it is therefore often referred to as an *oval* of an unspecified nature. Figure 1-42 shows the most famous building inscribed in an oval, the Roman Colosseum.



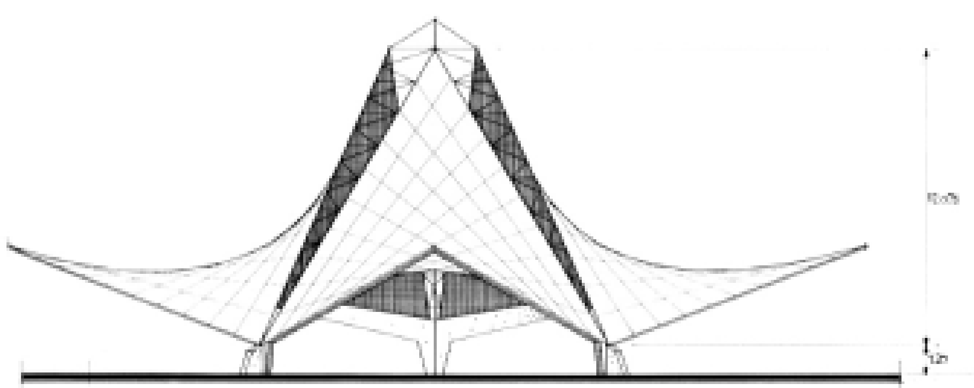
1-42
Plans of the Roman Colosseum

Hyberbolas come into their own in building design in connection with certain three dimensional surfaces called ‘shells’.

Figure 1-43 shows the elevation of the chapel of S. Vicente de Paul at Coyoacan, Mexico, which uses such surfaces for its roof structure.



1-43
S. Vicente de Paul at Coyoacan,
Mexico (1959-60) Enrique de la Mora and
Fernando Lopez Carmona, architects, and
Félix. Candela, structural engineer



1.9 CONSTRUCTING CONICS

1.9.1 Circle

Constructing circles is trivial given that a compass is one of our mechanical tools.

However, there are useful practical constructions that involve circular arcs. I give two that relate to *rectification*. The first determine the (approximate) length of a circular arc and its inverse problem, namely, to construct a circular arc of a given length for a given radius. The second relates to rectifying the circumference of a circle.

> **Rectification**
>
> *Rectification means constructing a straight line whose length equals the length of a curved line.*

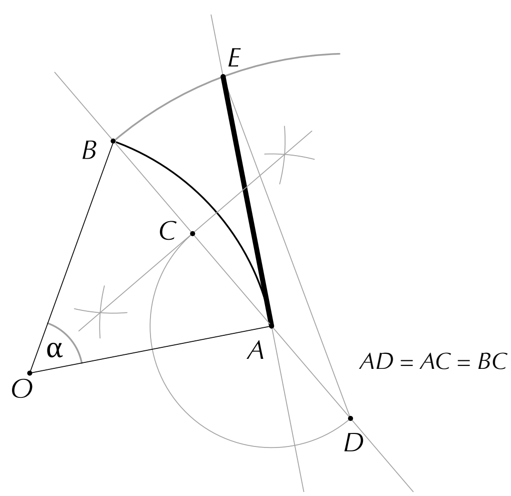
Construction 1-12
The approximate length of a circular arc

Suppose we are given a circular arc AB that subtends a given angle $\angle BOA$ (= α) at center O. We want to determine the length of this circular arc.

1. Draw a tangent to the arc at A (How?).
2. Join A and B by a line and extend it to D with $AD = \frac{1}{2}AB$.
3. Draw the circular arc with center D and radius DB to meet the tangent at E.

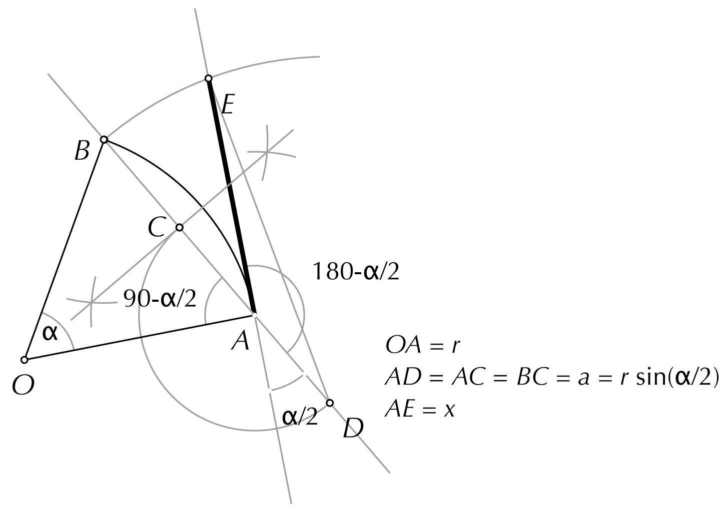
AE is the required length

The construction is shown in Figure 1-44.



1-44
Construction to find the approximate length of a given circular arc
If the angle is less than 30º, the error is less than 1/14,000.

The construction works best for smaller angles. If the angle is less than 30º, the error is less than 1/14,000. When the angle is close to 60º, the error is close 1/900. The following calculation of the error shows this. Figure 1-44 is annotated as follows, where the length *AE* is the unknown variable *x* to be determined.



By the Cosine Law, $DE^2 = AD^2 + AE^2 - 2.AD.AE.\cos(180\text{-}\alpha/2)$

$$9a^2 = a^2 + x^2 - 2ax\cos\left(180 - \frac{\alpha}{2}\right)$$

$$9r^2\sin^2\frac{\alpha}{2} = r^2\sin^2\frac{\alpha}{2} + x^2 + 2x\,r\,\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$x^2 + xr\sin\alpha - 8r^2\sin^2\frac{\alpha}{2} = 0$$

$$\therefore x = -\frac{r}{2}\sin\alpha \pm \sqrt{\frac{r^2}{4}\sin^2\alpha + 8r^2\sin^2\frac{\alpha}{2}} = r\left(-\frac{1}{2}\sin\alpha \pm \sin\frac{\alpha}{2}\sqrt{8+\cos^2\frac{\alpha}{2}}\right)$$

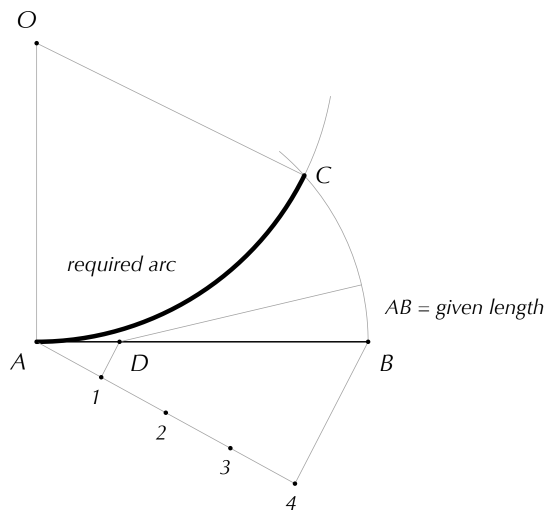
$$x = r\sin\frac{\alpha}{2}\left(\sqrt{\left(3-\sin\frac{\alpha}{2}\right)\left(3+\sin\frac{\alpha}{2}\right)} - \cos\frac{\alpha}{2}\right)$$

α (in º)	radians (arc length α)	sin α/2	cos α/2	*x*	*x*/α	error	precision
10	0.174532925	0.087155743	0.996194698	0.174532775	0.99999914	8.59702E-07	1163194
20	0.34906585	0.173648178	0.984807753	0.34906104	0.99998622	1.378E-05	72569
30	0.523598776	0.258819045	0.965925826	0.52356214	0.999930032	6.99683E-05	14292
40	0.698131701	0.342020143	0.939692621	0.69797669	0.999777963	0.000222037	4504
50	0.872664626	0.422618262	0.906307787	0.872189148	0.999455143	0.000544857	1835
60	1.047197551	0.5	0.866025404	1.046007244	0.99886334	0.00113666	880
70	1.221730476	0.573576436	0.819152044	1.219140067	0.997879721	0.002120279	472
80	1.396263402	0.64278761	0.766044443	1.391174863	0.996355602	0.003644398	274
90	1.570796327	0.707106781	0.707106781	1.561552813	0.994115396	0.005884604	170

Construction 1-13

An approximate circular arc of a given length

This construction is similar to the one above. We assume that we are given a circular arc on which we wish to mark off an arc of a specified length. Let A be a point on the arc. Let AB be the given length on the tangent at A. Mark a point D on the tangent such that $AD = \frac{1}{4}AB$. Draw the circular arc with center D and radius DB to meet the original at C. Arc AC is the required arc. The construction is shown in Figure 1-45.



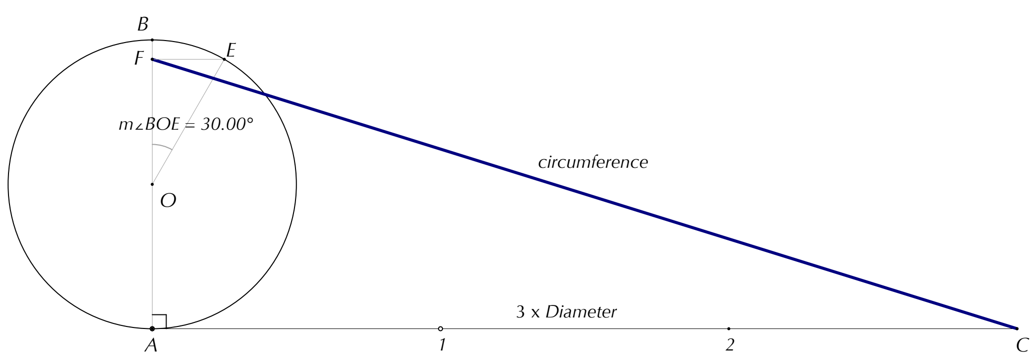
1-45
Construction to mark off a circular arc of given length

Construction 1-14

Rectifying the circumference of circle

Let $O(B)$ be the given circle with diameter AB. Draw a tangent at A and mark off a point C such that $AC = 3AB$. Draw radius OE such that $\angle BOE = 30°$. From E draw $\overline{EF}$ perpendicular to $\overline{AB}$.

CF is the required length.



1-46
Rectifying the circumference of a circle

The error is less than 1/21,700 as the following calculation shows.

$AF = r + \frac{\sqrt{3}}{2}r$, AC= 6r, and CF = l

We have $r^2 + \frac{3}{4}r^2 + \sqrt{3}r^2 + 36r^2 = l^2$

$\therefore\ l = \sqrt{39.4820508}\,r = 6.28347442\ r$

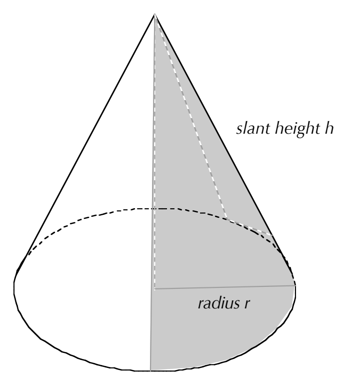
Circumference $c = 2\pi r = 6.28318531\ r$

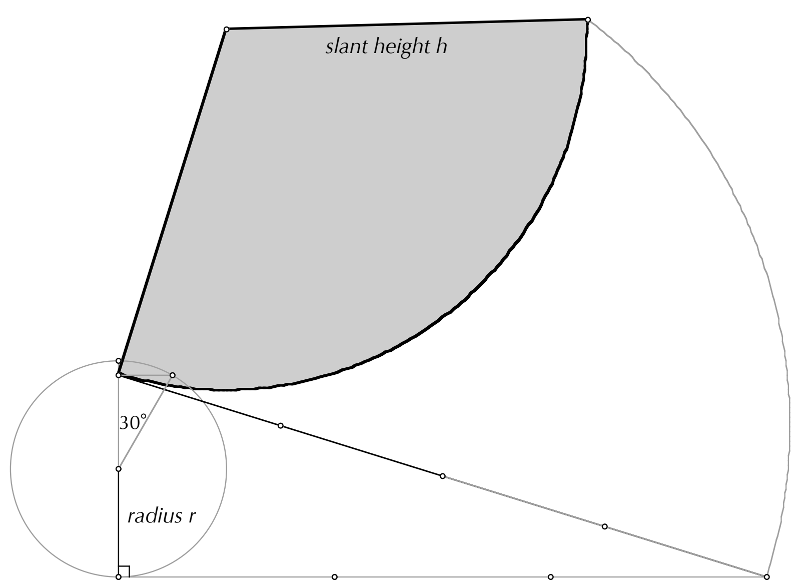
$\frac{l}{c} = 1.00004601$

Thus, error = 0.00004601 and precision = 1/21734

EXAMPLE - Unfolding of a cone

We can use these constructions to produce an unfolding of a cone if we know its slant height and base radius. We first rectify the circumference of the base, and then construct a sector with radius equals the slant height with arc length equaling the circumference.





1-47
Unfolding a cone

1.9.2 Parabola

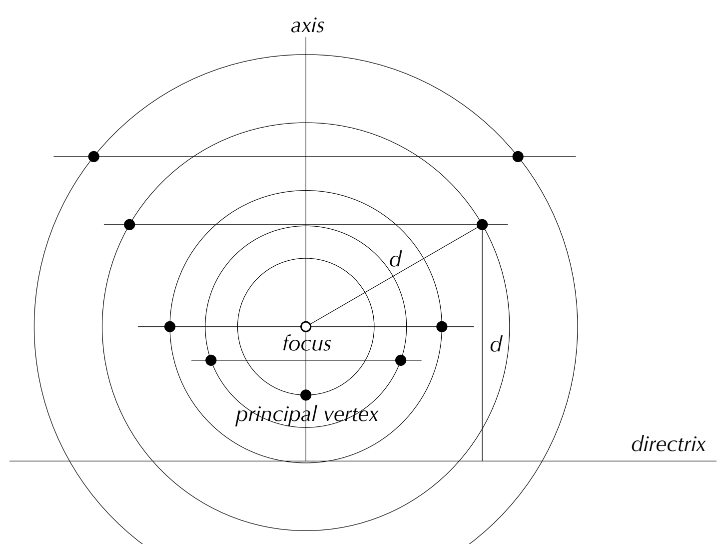
The definition for a parabola suggests the following construction, which can be described in three steps.

Firstly, we specify the focus and directrix. Secondly, we construct the axis of the parabola, which is perpendicular to the directrix and passes through the focus. The principal vertex lies on the axis midway between the focus and the directrix. Lastly, the remainder of the procedure is to construct arbitrary points on the parabola as follows:

1. Draw a line *l* parallel to the directrix at a distance *d*, which can be measured off the axis.
2. Draw a circle with the focus as center and radius *d* to intersect *l* at two points, which lie on the parabola.

> **Parabola**
>
> A *parabola* is a curve on any point at which is *equidistant to both a given fixed point and a given line*.
>
> The fixed point is called the *focus* of the parabola.
>
> The given line is called its *directrix*.
>
> The line through the focus and perpendicular to the directrix is the *axis* of the parabola; the point of inter-section between the axis and the parabola is its *principal vertex*.

We repeat these two steps for different lines parallel to the directrix at different distances from it till we have a sufficient number of points. See Figure 1-48.



1-48
Constructing a parabola given its focus and directrix

The following two constructions suggest practical ways of constructing parabolas with a given height on a base of given width.

Construction 1-15
Constructing a parabola within a rectangle

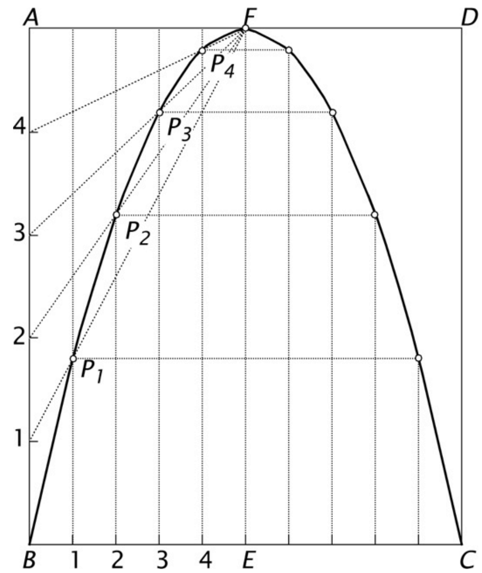
See Figure 1-49.

Bisect the sides, $\overline{BC}$ and $\overline{AD}$, of rectangle *ABCD* and join their midpoints, *E* and *F*, by a line segment.

Divide segments, $\overline{AB}$ and $\overline{BC}$, into the same number of equal parts, say $n = 5$, numbering them as shown.

Join *F* to each of the numbered points on to intersect the lines parallel to through the numbered points on at points P_1, P_2, … $P_{n\text{-}1}$ as shown. These points lie on the required parabola.

The parabola so traced corresponds to the trajectory of a stone thrown into the air at a height equal to *AB* and through a distance equal to *BC*.



1-49
Constructing a parabola inscribed within a rectangle

The next construction is based on the following property of parabolas:

An abscissa is related to any of its double ordinate by the ratio, AB:(PB × BQ), which is always a constant. That is, the abscissa is a scaled multiple of the parts into which it divides the double ordinate.

Construction 1-16
Constructing a parabola by abscissae

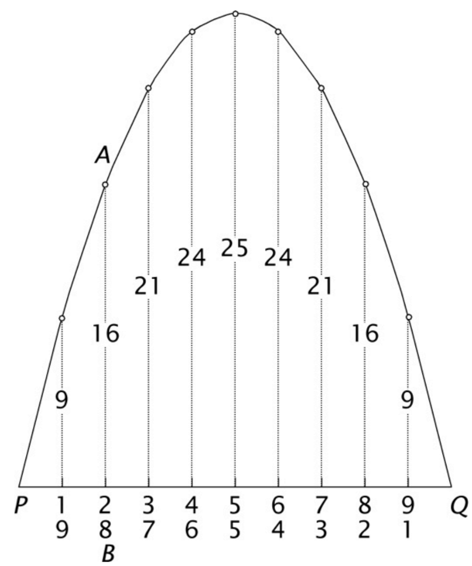
The construction is shown in Figure 1-50.

The base is divided into, say 10, equal parts. Vertical segments are drawn at these points with length proportional to the division of the base. The points lie on the required parabola.

Ordinate and abscissa

The perpendicular from a point *P* on a parabola to its axis is referred to as an *ordinate*; its extension to a second point *Q* on the parabola is referred to as the *double ordinate*.

A line parallel to the axis between a point *A* on the parabola and a point *B* on an ordinate is referred to as an *abscissa*.



1-50
Constructing a parabola by abscissae

1.9.3 Ellipse

Its definition suggests the following construction.

First, we specify the parameters of the ellipse, namely, the foci *A* and *B*, and the distance $r > AB$. Mark off a segment of length *r*.

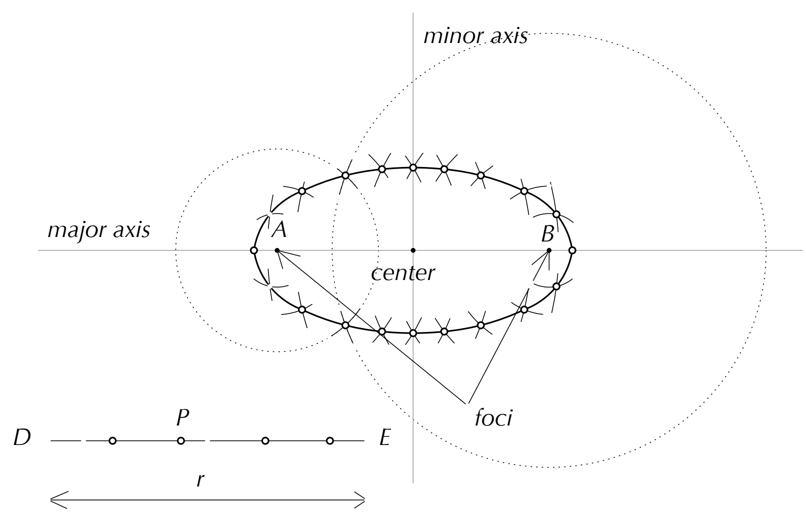
Let *P* be an arbitrary point between *D* and *E*. Construct circles $A(DP)$ and $B(EP)$. The circles intersect at two points that lie on the ellipse. Repeating construction with the radii reversed gives another two points on the ellipse. The construction is repeated for different choices of *P* until enough points have been generated. The construction for a few points on the ellipse is illustrated in Figure 1-51.

Ellipse

An *ellipse* is a curve on any point of which the *sum of the distances to two fixed points equals a constant*.

The fixed points are called the *foci* of the ellipse.

The mid-point of the segment joining the foci is called the *center* of the ellipse.



1-51
Constructing the points on the ellipse

Trammel method of constructing ellipses

If the axes of the ellipse are known, the most convenient method of construction is by means of a ***trammel***, a strip of material on which designated points have been marked.

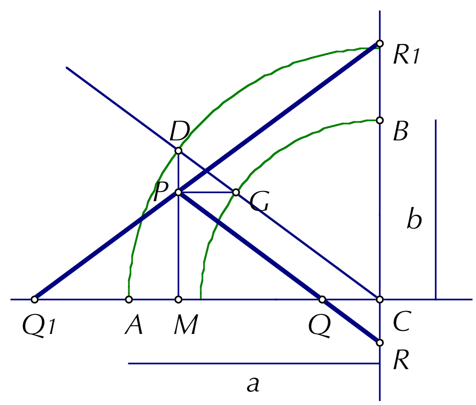
Refer to Figure 1-52 where C is the center of the ellipse. We can draw two circles, centered at C with radii a, the semi-major axis, and b, the semi-minor axis. These are known as the ***major*** and ***minor circles*** respectively. Consider the common radius DC passing through point G. Let P be a point on the ellipse. Let PM be its ordinate. Draw a line through P parallel to DC to meet the axes at Q and R. Because $\overline{DM}$ is parallel to BC, and PG is parallel to $\overline{MC}$, it follows that $PR = DC = a$, and $PQ = GC = b$. Then, the line $–P–Q–R–$can serve as a trammel by means of which points on the curve can be located.

Alternatively, we can draw a line through P to meet the axes at points Q_1 and R_1 such that $\angle MQ_1P = \angle PQM$. From congruent triangles, $Q_1P = PQ = b$ and $PR_1 = PR = a$, and we can use the line $–Q_1–P–R_1–$ as a trammel. Both trammel methods are illustrated in Figure 1-52.

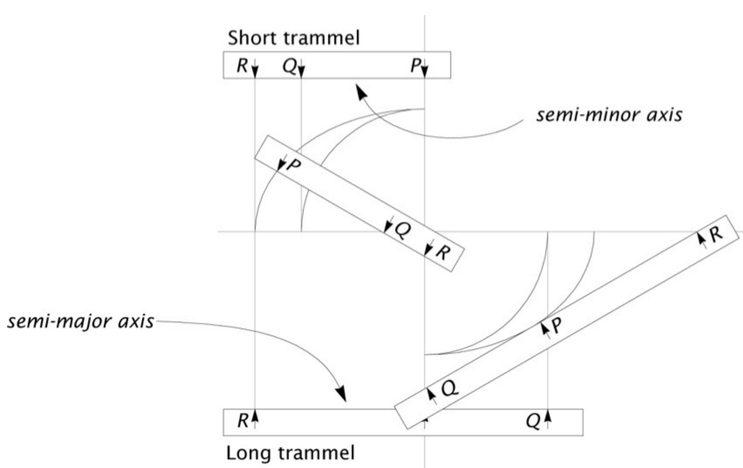
The Trammel Method

Draw the axes and mark off along a straight strip of card-board the distances *PQ* and *PR*. Apply the trammel so that *Q* lines up with the major axis and R lines up with the minor axis; *P* is a point on the ellipse. More points *P* can be plotted, by moving the trammel so that *Q* and *R* slide along their respective axes.

W Abbott
Practical Geometry and Engineering Graphics
Blackie & Son Ltd, Glasgow, 1971.



1-52
Principle of the Trammel



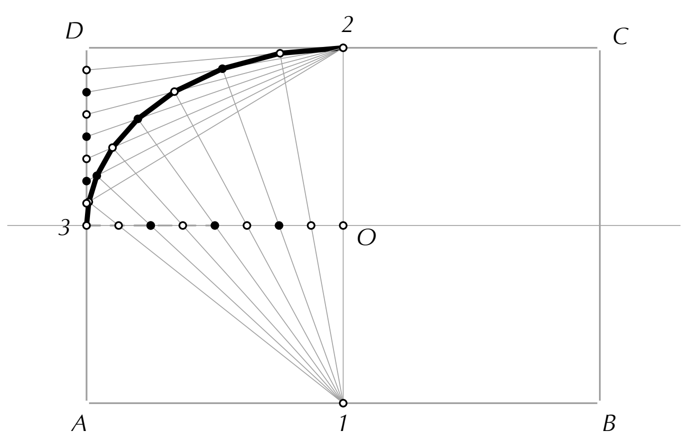
1-53
The trammel method of constructing ellipses

Construction 1-17
Constructing an ellipse within a rectangle

This is a convenient way to quickly construct an ellipse.

1. Bisect the sides of the rectangle *ABCD* and join their midpoints of opposite sides by line segments to meet at the center *O*.
2. We consider the upper left quadrant *O2D3*. The construction is similar for each quadrant.
3. Divide segments $\overline{O3}$ and $\overline{D3}$ into the same number of equal parts, say $n = 8$, as shown.
4. Join *2* to each of the points on *D3*. Join 1 to each of the points on *O3* to intersect the corresponding lines from 2 as shown.

These points lie on the required ellipse. See Figure 1-54.



1-54
Constructing an ellipse inscribed within a rectangle

☛ *QUESTION*: *What do you construct if ABCD is a square?*

☛ *ANSWER*

Left as an exercise

1.9.4 Hyperbola

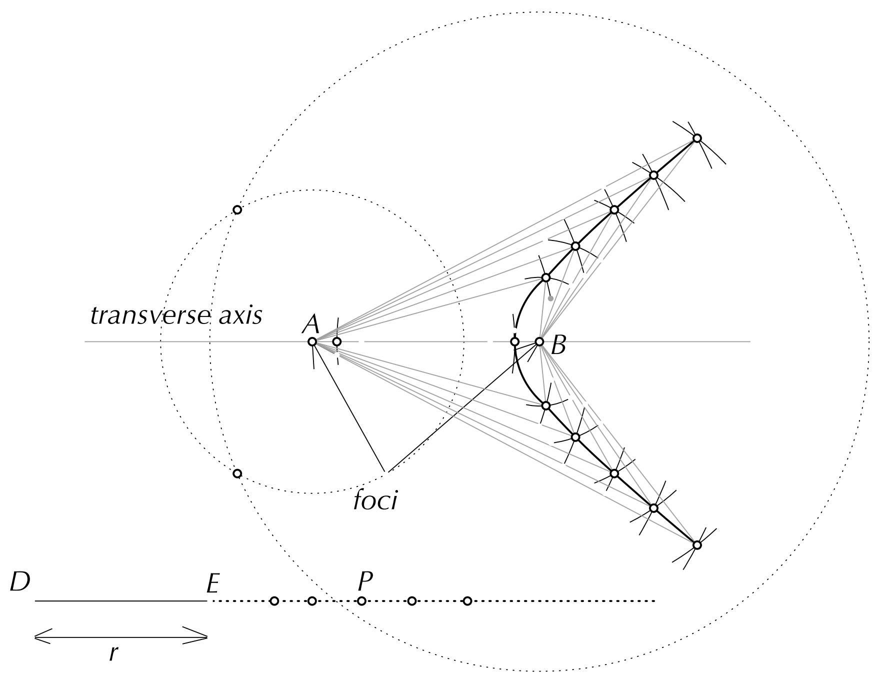
As before, we consider a technique suggested by the definition to find the points of a hyperbola. Like the ellipse, the hyperbola is specified by two focal points and a real number *r*, the distance between two arbitrary points, say *D* and *E*. The construction is illustrated in Figure 1-55. Given these points, we repeatedly apply the following procedure to generate arbitrary points that are on the hyperbola. Select a point *P* so that *P* is not between *D* and *E*. Construct circles *A*(*DP*) and *B*(*EP*). The two circles meet at two points of the hyperbola. Repeating the two steps with the radii reversed gives two additional points.

Hyperbola

A *hyperbola* is a curve on any point at which the *absolute difference of the distances to two fixed points equals a constant*.

The fixed points are called the *foci* of the hyperbola.

The mid-point of the segment joining the foci is called the *center* of the hypebola.



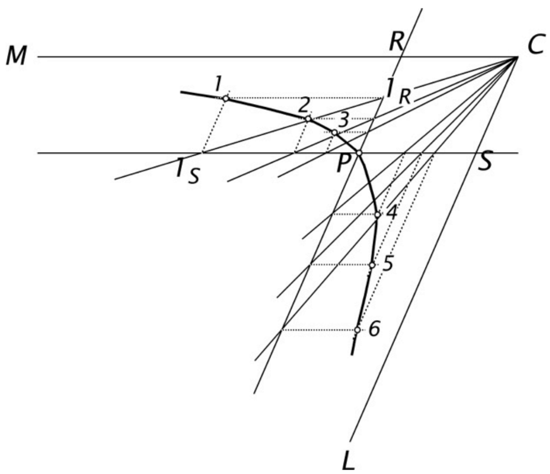
1-55
Constructing points on the hyperbola

The constructions below give practical techniques for producing hyperbolas. The first relies on being given the asymptotes; the second requires the semi-transverse axis. For both, we need to know a point on the curve.

Construction 1-18

Constructing a hyperbola given its asymptotes and a point on the curve

Let *CL*– and *CM*– be the asymptotes. Construct lines –*P*–*R*– and – *P*–*S*– parallel to them. Construct any radial line from *C* cutting –*P*–*R*– and –*P*–*S*– at points, 1_R and 1_S. Through these points construct lines parallel to the asymptotes to intersect at *1*, which is on the curve. Similarly construct points 2, 3, … as shown in Figure 1-56.



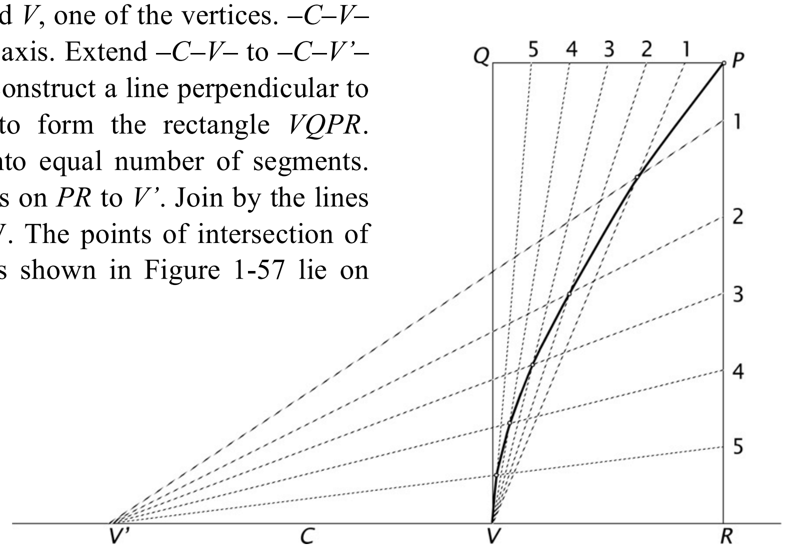
1-56

Constructing a hyperbola given its asymptotes and a point on the curve

Construction 1-19

Constructing a hyperbola given the semi-transverse axis and a point on the curve

Let *C* be the center and *V*, one of the vertices. –*C*–*V*– is the semi-transverse axis. Extend –*C*–*V*– to –*C*–*V’*– such that *CV’* = *CV*. Construct a line perpendicular to the axes through *P* to form the rectangle *VQPR*. Divide *PQ* and *PR* into equal number of segments. Join by lines the points on *PR* to *V’*. Join by the lines the points on *PQ* to V. The points of intersection of corresponding lines as shown in Figure 1-57 lie on the required curve.



1-57

Constructing a hyperbola given the semi-transverse axis and a point on the curve

1.10 THE "DIVINE PROPORTION" OR GOLDEN SECTION

The proportion called the *golden section* has played an important role in many architectural proportional systems through the centuries.

> *Let AB be a segment and C a point so that A-C-B (that is, C belongs to the segment and it is between A and B). C divides AB in the* **golden ratio** *if AB:AC = AC:CB*

That is, C divides the segment so that the ratio between the length of the segment and its larger part is equal to the ratio between the larger and the smaller part.

> *Any division that satisfies the golden ratio is called a* **golden section**

If we define $AB = l$, $AC = a$ and $CB = b$, this ratio simplifies to $l{:}a = a{:}b$. Or, $a+b{:}a= a{:}b$. That is, $a^2–ab – b^2 = 0$.

This ratio is independent of l, a, or b. In calculating this ratio, it is convenient to set b =1. That is, $a^2– a – 1 = 0$.

The positive solution to this equation is which is the ratio ϕ we are looking for. ϕ is called the *golden ratio*.

That is, $\phi = ½ (1+\sqrt{5})$

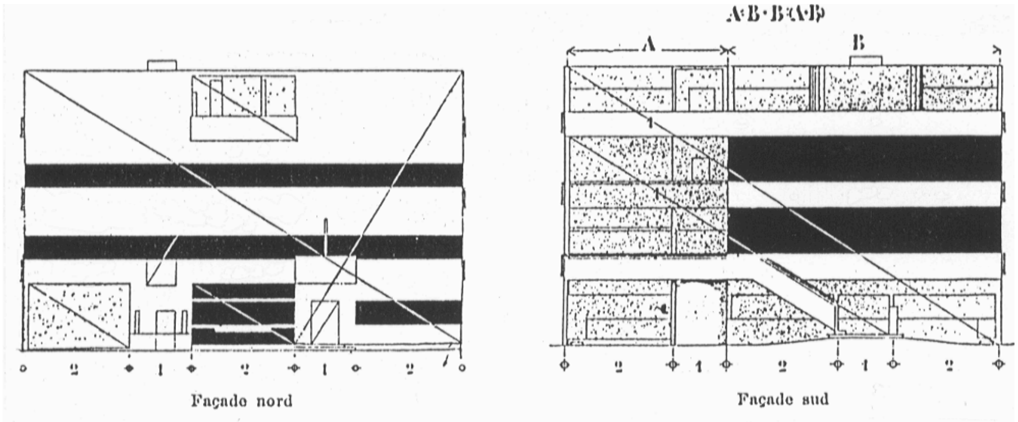
Alternatively, $\phi \times (\phi –1) = 1$. Or, $\phi = 2/(\sqrt{5}–1)$.

1.10.1 Golden rectangles

A rectangle *ABCD* is *golden* if $AB = \phi\ BC$.

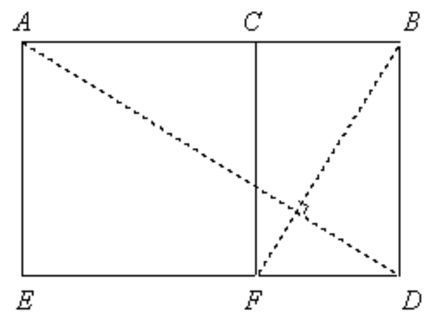
We can always construct a golden rectangle with a given longer side, by extending it in the golden ratio using a construction to divide a segment in the golden ratio. Golden rectangles are used as regulating lines frequently in building designs. An example is the facades of Le Corbusier's Villa Stein at Garches, which is shown below.

1-58
Le Corbusier's Villa Stein at Garches

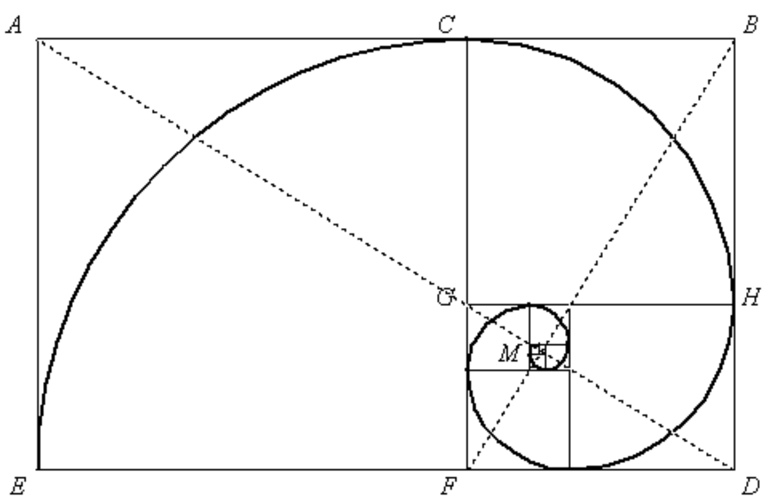


It is easy to see that a golden rectangle, *ABDE* (with longer side), can be divided into a square and a (smaller) golden rectangle, *CBDF*, as shown on the right. If *C* divides in the golden ratio with longer side, the segment through the rectangle with *C* as one of its end points accomplishes the desired division. This can be seen if one again sets $BD = 1$. Then $AB = \phi$ and $CB = \phi - 1$. Thus, $BD{:}CB = \phi{:}1$, and *CBDF* is a golden rectangle.

The division of a golden rectangle into a square and a smaller golden rectangle is used prominently in the south (garden) elevation of the Villa Stein and less prominently in its street elevation.



We can use this division to construct a golden spiral as shown below.



1-59
Golden spiral

The spiral is composed of quarter circles inside the squares that successively divide the golden rectangles *ABDE*, *CBDF*, *HDFG* … constructed according to the above construction. The major diagonals of the first two rectangles are perpendicular and cross at *M*.

Construction 1-20
Constructing a golden rectangle given one of its sides

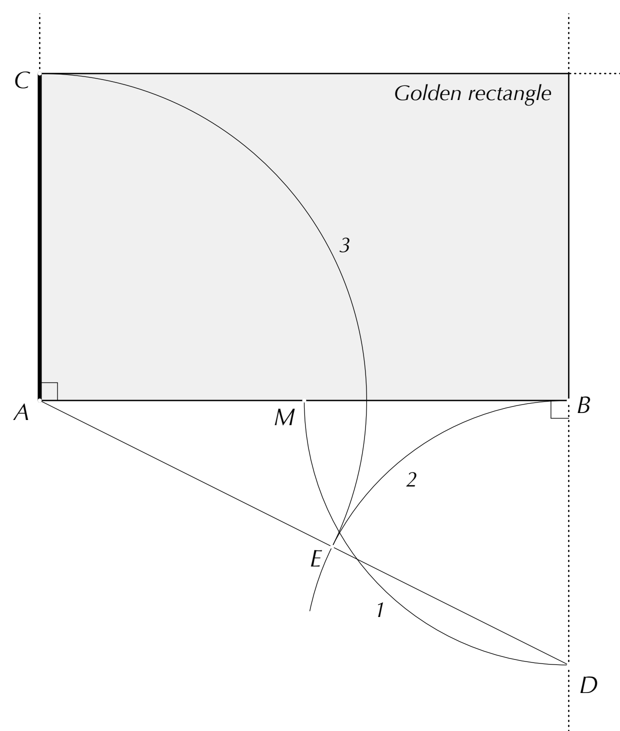
We adapt Construction 1-10 (see page 18) and the fact that $\phi = ½ (1+\sqrt{5})$. There are two possibilities.

Suppose we are given the *longer side*, say, *AB*. Then, we have to construct *C* such that $AB:AC = \phi$

There are three steps:

1. Let *M* be the mid-point of *AB*. Draw a perpendicular at *B* and arc *B*(*M*) to meet it at *D*.
2. Draw arc *D*(*B*) to meet *AD* at *E*.
3. Draw a perpendicular at *A* and arc *A*(*E*) to meet it at *C*

AC is required shorter side. See Figure 1-60.



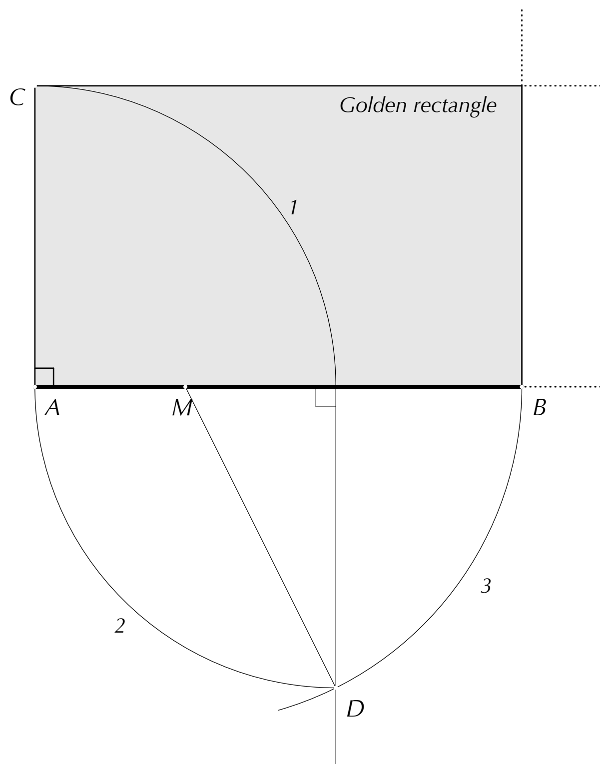
1-60
Constructing the golden rectangle given longer side

Suppose we are given the *shorter side*, say, AC. Then, we have to construct C such that AB:$AC = \phi$

Again, there are three steps:

1. Draw perpendicular to AC at A. Draw arc $A(C)$ to meet it at E.
2. Draw perpendicular to AE at E. Draw arc $E(A)$ to meet it at D.
3. Let M be the mid-point of AE. Draw arc $M(D)$ to meet $-AE-$ at B.

AB is required longer side See Figure 1-61.



1-61
Constructing a golden rectangle given one its shorter side

1.10.2 Golden series

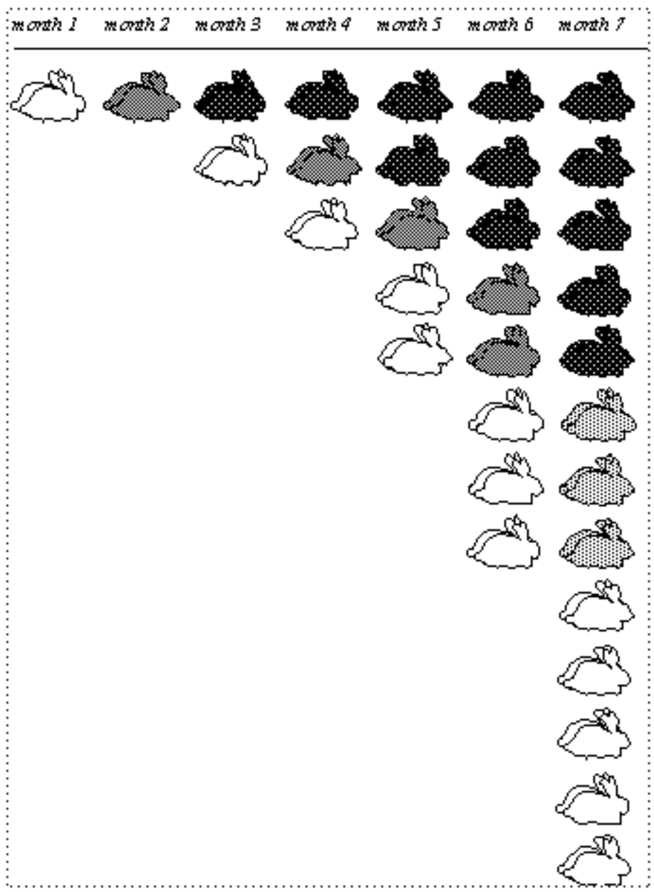
A Tale of Rabbits

In 1202, Leonardo da Pisa, also known as Fibonacci, constructed a simplified model of the breeding habits of rabbits as follows: he assumed that rabbits live forever; every pair of rabbits produces a new pair of baby rabbits every month, and the new pair starts to breed on its own after two months. Starting with a single, new born pair, we have one pair in the first and second months; two pairs in month 3; 3 pairs in month 4; 5 pairs in month 5 etc. Figure 1-62 illustrates this for the first 7 months.

Table 1-1 The first ten Fibonacci numbers

n	1	2	3	4	5	6	7	8	9	10
u_n	1	1	2	3	5	8	13	21	34	55
u_{n+1}/u_n	1	2	1.5	1.667	1.6	1.625	1.6154	1.619	1.6176	1.6182

Table 1-1 above gives these numbers for the first 10 months, together with the ratios u_n+1/u_n, where u_n is the number of pairs in month n. These numbers suggest that the ratios u_n+1/u_n approach ϕ more and more closely as n increases. This is true although we do not prove this here.



1-62
The first seven months

The Golden Series

The sequence of numbers 1, ϕ, ϕ^2, ϕ^3, ϕ^4, … , ϕ^n, … is the *golden series*.

The golden series is a *geometric progression*: If the terms of the progression are denoted by u_n, u_n/u_{n-1} is constant for every n; in the present case, it is obviously ϕ.

It is easy to prove that the golden series has the following additive property: $u_n = u_{n-1} + u_{n-2}$. That is, each term is the sum of its two preceding terms.

The Fibonacci numbers enter the picture when we compute the first terms of the series:

$\phi^2 = \phi + 1$

$\phi^3 = 2\phi + 1$

$\phi^4 = 3\phi + 2$

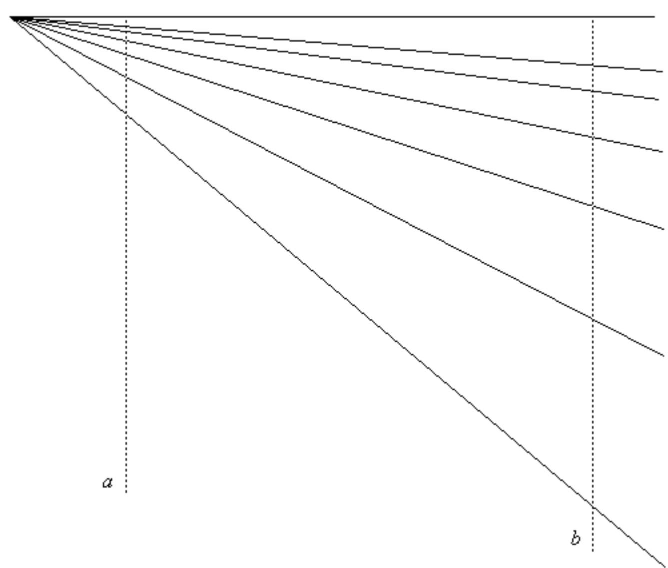
$\phi^5 = 5\phi + 3$

…

That is, we can express the members of the series as first order expressions in ϕ using the Fibonacci numbers as coefficients.

The importance of the golden series for proportional systems in architecture results from the combination of additive and multiplicative properties that establish a flexible system of dimensions related to each other by the golden ratio.

It has been suggested by Scholfield (*Theory of Proportions in Architecture*, Cambridge University Press, 1958) that the golden series can be incorporated into a universal golden scale that can be used to find a golden series for any base unit of measurement as shown below in Figure 1-63.



1-63
A universal golden scale