Scaling laws for connectivity in random threshold graph models with non-negative fitness variables

Armand M. Makowski* and Osman Yağan†
* Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742. E-mail: armand@isr.umd.edu
† Cybersecurity Laboratory (CyLab), Carnegie Mellon University, Pittsburgh, PA 15213. E-mail: oyagan@andrew.cmu.edu

Abstract—**We explore the scaling properties for graph connectivity in random threshold graphs. In the many node limit, we provide a complete characterization for the existence and type of the underlying zero-one laws, and identify the corresponding critical scalings. These results are consequences of well-known facts in Extreme Value Theory concerning the asymptotic behavior of running maxima on i.i.d. random variables. In the important special case of exponentially distributed fitness, we show that the (essentially unique) critical scaling which ensures a power-law degree distribution, does not result in graph connectivity in the asymptotically almost sure (a.a.s.) sense.**

Keywords: Scale-free networks, Random threshold graphs, Hidden variables, Connectivity, Zero-one laws, Extreme Value Theory.

I. Introduction

Following the work of Barabási and Albert [3], the scale-free nature of complex networks is often explained by means of *growth* models with a preferential attachment mechanism – The so-called "the rich get richer" rule. Although preferential attachment is a reasonable assumption in some contexts, it is predicated on the information about the degree of each vertex being available to newly added nodes, either explicitly or implicitly. There are many situations, including some social networks, where this assumption may be questioned, and where instead the creation of a link between two nodes results from a mutual benefit based on their intrinsic attributes, e.g., authority, friendship, social success, strength of interaction, etc.

A. Random threshold graphs

Hidden variable models form a broad class of random graph models that naturally incorporate this viewpoint by establishing links on the basis of fitness variables associated with individual nodes, e.g., see the papers [7], [10], [40] (and references therein). Interest in such models has been spurred in part by the finding that under certain conditions, hidden variable models do give rise to scale-free networks, and this without resorting to preferential attachment. This issue has been discussed at some length in the more restricted setting of random threshold graphs which we now describe.

The network comprises n nodes, labelled $k = 1, \ldots, n$, and to each node k we assign a *fitness* variable (or weight) ξ_k which measures its importance or rank. The random variables (rvs) $\xi_1, \ldots, \xi_n$ are assumed to form a collection of i.i.d. $\mathbb{R}$-valued rvs, each distributed according to some given probability distribution function $F : \mathbb{R} \to [0, 1]$.[1] For distinct $k, \ell = 1, \ldots, n$, we declare nodes k and ℓ to be adjacent if

$$\xi_k + \xi_\ell > \theta \tag{1}$$

for some threshold θ. We refer to the random graph defined by the adjacency notion (1) as a *random threshold* graph on the set of vertices $\{1, \ldots, n\}$, and hereafter we denote it by $\mathbb{T}(n; \theta)$.

These graphs have recently been the focus of much activity. Their hierarchical structure was elegantly described by Hagberg et al. in terms of so-called creation sequences [26]. The survey by Diaconis et al. [11] presents several equivalent definitions of random threshold graphs. A number of their graph-theoretic properties are developed by Reilly and Scheinerman [36], [37]. Ide et al. [27] explore the spectral properties of the underlying (random) adjacency matrices. Rigorous convergence results are derived in [19], [20], [29] for a number of important network quantities such as the degree distribution, degree correlations, clustering coefficients and the number of triangles. Random threshold graphs are amenable to various generalizations, e.g., see Ide et al [28], and the Ph.D. thesis by Reilly [36]. Random geographical threshold graphs are discussed by Bradonjić et al. [8], [9], and by Masuda et al. [29], [31].

B. Connectivity

Interestingly, most of these earlier papers do not deal with the property of *connectivity* – A notable exception can be found in the papers by Bradonjić et al. [8], [9] concerning the related class of random geographical threshold graphs. See also [36], [37] for results with fixed θ. In social networks (and elsewhere), connectivity is an important property which shapes the characteristics of various dynamical processes on

[1] What we call here a probability distribution function is also called a cumulative distribution function in other literatures.

networks such as the diffusion of cultural fads, beliefs, norms, and innovations [14], [41], [42], the spread of diseases and information [1], [33], [44], and cascading failures [21], [45]. Although analyzing such phenomena in random threshold graphs is beyond the scope of this paper, by analogy with classical results for Erdős-Rényi graphs [34], [41] we expect the following three conjectures to hold when the random threshold network is *connected*: (i) Simple contagion processes such as the spread of information and diseases easily take place, i.e., the network is always in the endemic state for any (positive) transmissibility parameter, and an information (or disease) started from an arbitrary individual reaches out to a positive fraction of the nodes in the large n limit; (ii) However, in the case of a complex contagion process where multiple sources of exposure to the item (e.g., a rumor, a cultural fad, a political view, or a new technological innovation) are required for an individual to adopt a new behavior, network connectivity will create a high resistance to the adoption of the new item by virtue of the high local "stability" of the nodes. In other words, a complex contagion process started from an arbitrary individual will always die-out quickly without reaching a significant proportion of the network; and (iii) A fully connected network is expected to be robust against random failures of the nodes, i.e., even if a positive fraction of the nodes randomly fails, the remaining nodes will still form a giant connected component comprising a positive fraction of the nodes. All three conjectures hold for Erdős-Rényi graphs [6], [34], [41], and their validity in the context of random threshold graphs is a matter currently under investigation.

C. Contributions

Against this backdrop, as a natural first step towards resolving these conjectures, we find it relevant to develop a better understanding of the property of connectivity in the random threshold graphs defined through (1). We do so when the fitness variables are *non-negative* random variables and $\theta > 0$ – See Section II for the mild assumptions enforced on F together with some notation and easy preliminaries. Results for arbitrary distributions F are discussed in [30]; however, as they are technically more involved, we have elected here to restrict our attention to the simpler case of non-negative fitness rvs. Fortunately these more limited results do cover the important situation when F is an exponential distribution; see Section V.

For convenience, we write

$$P(n;\theta) := \mathbb{P}\left[\ \mathbb{T}(n;\theta) \text{ is connected }\right], \quad \begin{matrix} \theta > 0, \\ n = 2,3,\ldots \end{matrix} \qquad (2)$$

We seek to understand how these probabilities behave when the number n of nodes becomes large and the threshold value θ is scaled appropriately. This amounts to making θ depend on n by means of *scaling* functions $\theta : \mathbb{N}_0 \to \mathbb{R}_+ : n \to \theta_n$, and to investigating the limit $\lim_{n\to\infty} P(n;\theta_n)$. We are particularly interested in conditions under which either

$$\lim_{n\to\infty} P(n;\theta_n) = 0 \qquad (3)$$

or

$$\lim_{n\to\infty} P(n;\theta_n) = 1. \qquad (4)$$

We naturally refer to the convergence statements (3) and (4) as a zero law and a one law, respectively.

Such zero-one laws have been discussed extensively in the context of other classes of random graphs, e.g., Erdős-Rényi graphs [6], [13], geometric random graphs [2], [35] and random key graphs [5], [12], [39], [43]. In analogy with results obtained in these earlier studies, we expect a *critical* scaling $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ to define a *boundary* in the space of scalings with the following rough meaning: For n sufficiently large, a threshold value θ_n suitably smaller (resp. larger) than $\theta^\star_n$ ensures $P(n;\theta_n) \simeq 1$ (resp. $P(n;\theta_n) \simeq 0$).

We begin the discussion in Section III-A by adapting the terminology of McColm [32, p. 376] so as to distinguish between *weak* and *strong* critical scalings. We show that the existence and type of a zero-one law, and the form of its critical scaling are completely determined by properties of F. This relies in an essential way on a representation of the probability of graph connectivity. This expression, derived in Section VII-A, highlights the role played by the running minima and maxima of the i.i.d. fitness rvs. In Section III-B we characterize zero-one laws for graph connectivity in terms of the limiting properties of these running minima and maxima. These results are then applied in Section IV and in Section VI. In Section IV we show that the existence of a strong critical scaling is equivalent to the weight distribution F having *infinite support* and being *rapidly varying* (in the sense of Gnedenko [22]). In that case the critical scaling is seen to be asymptotically equivalent to the scaling $\lambda_F : \mathbb{N}_0 \to \mathbb{R}_+$ given by the quantiles

$$\lambda_{F,n} = \left(\frac{1}{1-F}\right)^{\leftarrow}(n), \quad n = 1,2,\ldots$$

where $^{\leftarrow}$ denotes the generalized (left-continuous) inverse operation applied to non-decreasing functions; see a precise definition at (17).

When F is *not* rapidly varying but still with *infinite* support, the situation is more complex and we need to appeal to Extreme Value Theory. This is done in Section VI where we leverage the classical characterization of the maximum domain of attraction for each of the three max-stable distributions. In particular we show that a weak (resp. strong) zero-one law holds when F has infinite support and belongs to the maximum domain of attraction of the Fréchet (resp. Gumbel) distribution. The theory developed here needs to be modified in all other cases, including when F belongs to the maximum domain of attraction of the Weibull distribution. We note that connections to Extreme Value Theory were already exploited in the context of random threshold graphs by Ide et al. in [20] (as well as in [19]).

In Section V we discuss the case when F is exponentially distributed with parameter $\lambda > 0$ (a distribution which belongs to the maximum domain of attraction of the Gumbel distribution). This case was used by some authors, e.g., Caldarelli et al. [10], [40], to demonstrate the possibility of generating scale-free networks by proper selection of F. These authors show in the many node limit that degrees follow a power-law distribution only if $\theta^\star_n = \lambda^{-1}\log n$. However, the results obtained here imply

$$\lim_{n\to\infty} P(n;\theta^\star_n) = 1 - e^{-1},$$

and so both (3) and (4) fail under this scaling which acts as the critical scaling in the exponential case! Thus, in the regime where the random threshold graph with exponential fitness is scale-free, neither connectivity nor the lack thereof are typical. This is line with impossibility results recently obtained by Faragó [17], [18] for some homogeneous random networks.

The situation is vastly different with the preferential model of Barabási and Albert [3]: Its degree distribution exhibits power law behavior, and yet, this inhomogeneous network model (as an undirected graph) is connected, in fact tree-like, by construction. In short, while random threshold graphs with exponentially distributed fitness may possibly account for a power law without invoking preferential attachment, the inability to provide a.a.s. connectivity may render such models unsuitable for describing situations where connectivity is an expected network feature or requirement. In a limited sense, this further illustrates the well-known difficulty of building random graph models which are rich enough to exhibit simultaneously *multiple* structural properties.

All the proofs are collected in Section VII, and the main results are illustrated through limited simulation studies for several important special cases.

II. Assumptions, notation and some easy preliminaries

A word on the notation and conventions in use here: All statements involving limits, including asymptotic equivalences, are always understood with n going to infinity. The random variables (rvs) under consideration are all defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. All probabilistic statements are made with respect to the probability measure $\mathbb{P}$, with corresponding expectation operator denoted by $\mathbb{E}$. The notation $\stackrel{P}{\to}_n$ (resp. $\Longrightarrow_n$) is used to signify convergence in probability (resp. convergence in distribution) with n going to infinity.

Let $\{\xi, \xi_k,\ k = 1, 2, \ldots\}$ denote a collection of i.i.d. $\mathbb{R}_+$-valued rvs defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ – We use ξ as a generic representative of this sequence of i.i.d. rvs. Throughout the following assumptions are enforced on their common probability distribution function $F : \mathbb{R} \to [0, 1]$.

Assumption A: The probability distribution function $F : \mathbb{R} \to [0, 1]$

(i) *has support contained in* $\mathbb{R}_+$ *(so* $F(x) = 0$ *if* $x < 0$*);*

(ii) *is continuous on* $\mathbb{R}$*, or equivalently, has no atoms (and so cannot be degenerate at a single point, in particular, the origin);*

Assumption A-(ii) implies $F(0) = 0$, and is most easily satisfied in practice by taking F to be absolutely continuous, say with density function $f : \mathbb{R} \to \mathbb{R}_+$ (so $f(x) = 0$ for $x < 0$ by Assumption A-(i)). In due course additional assumptions will be made on F.

With the probability distribution function F we associate the two quantities

$$\xi^\star_F = \inf\left(x \in \mathbb{R} :\ F(x) > 0\right) \tag{5}$$

and

$$\xi_F = \sup\left(x \in \mathbb{R} :\ F(x) < 1\right). \tag{6}$$

Under the foregoing assumptions, $\xi^\star_F$ is necessarily finite with $\xi^\star_F \geq 0$. It is easy to see that $\xi^\star_F < \xi_F$: While this is clear when $\xi_F = \infty$, it is also the case when ξ_F is finite since the equality $\xi^\star_F = \xi_F$ is precluded by the absence of atoms as per Assumption A-(ii).

Finally, we find it convenient to associate with the rvs $\{\xi_k,\ k = 1, 2, \ldots\}$ their running minima and maxima defined by

$$M^\star_n := \min\left(\xi_1, \ldots, \xi_n\right) \tag{7}$$

and

$$M_n := \max\left(\xi_1, \ldots, \xi_n\right) \tag{8}$$

for each $n = 1, 2, \ldots$. It is a simple matter to check that

$$\lim_{n\to\infty} M^\star_n = \xi^\star_F \quad a.s. \tag{9}$$

and

$$\lim_{n\to\infty} M_n = \xi_F \quad a.s. \tag{10}$$

where these convergence statements are monotone from below and above, respectively. Needless to say, the definitions (5), (6), (7) and (8), and the facts (9) and (10) all hold for any probability distribution $F : \mathbb{R} \to [0, 1]$, not just for those with support in $\mathbb{R}_+$.

III. Zero-one laws and critical thresholds

Recall that a *scaling* is defined as any mapping $\theta : \mathbb{N}_0 \to \mathbb{R}_+$, and that we are interested in finding conditions on such scalings to ensure either $\lim_{n\to\infty} P(n;\theta_n) = 1$ or $\lim_{n\to\infty} P(n;\theta_n) = 0$. Typically there exist scalings, deemed *critical*, which act as boundary in the space of scalings between these two extremes.

A. Strong vs. weak zero-one laws

The terminology, originally developed by McColm for random graphs on a line segment [32, p. 376], is now adapted to the class of random threshold graphs: A *strong* zero-one law is said to hold (for graph connectivity) with critical scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ if for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying

$$\lim_{n\to\infty} \frac{\theta_n}{\theta^\star_n} = c \tag{11}$$

for some $c > 0$, we have

$$\lim_{n\to\infty} P(n;\theta_n) = \begin{cases} 1 & \text{if } 0 < c < 1 \\ & \\ 0 & \text{if } 1 < c. \end{cases} \tag{12}$$

Any scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ appearing in (11)-(12) is called a *strong* critical scaling.

On the other hand, a *weak* zero-one law is said to hold (for graph connectivity) with critical scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ if for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ we have

$$\lim_{n\to\infty} P(n;\theta_n) = \begin{cases} 1 & \text{if } \lim_{n\to\infty} \frac{\theta_n}{\theta^\star_n} = 0 \\ & \\ 0 & \text{if } \lim_{n\to\infty} \frac{\theta_n}{\theta^\star_n} = \infty. \end{cases} \tag{13}$$

Any scaling $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ appearing in (13) is called a *weak* critical scaling.

In its weak form the one law (resp. zero law) emerges when considering scalings $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ which are *at least* an order of magnitude smaller (resp. larger) than $\theta^\star$. On the other hand, under the strong law, for n sufficiently large, a threshold value θ_n suitably smaller (resp. larger) than $\theta^\star_n$ ensures $P(n;\theta_n) \simeq 1$ (resp. $P(n;\theta_n) \simeq 0$) provided $\theta_n \sim c\theta^\star_n$ with $0 < c < 1$ (resp. $c > 1$). This is in sharp contrast with (13) in that the strong one law (resp. zero law) still occurs with scalings $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ which are asymptotically smaller (resp. larger) than $\theta^\star$ but of the *same* order of magnitude as $\theta^\star$.

For each $n = 2, 3, \ldots$, the mapping $\theta \to P(n;\theta)$ is non-increasing on $(0,\infty)$, and an easy monotonicity argument (in θ) shows that a strong zero-one law implies a weak zero-one law, and that a strong critical scaling is also a weak critical scaling.

B. Two basic zero-one laws

In the context of the random threshold graphs discussed in this paper, the asymptotic properties of the running maxima (8) provide useful characterizations of the zero-one laws. The first result along these lines deals with strong zero-one laws.

Theorem 3.1: Assumptions A are enforced on the probability distribution function F. If $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ is a scaling which satisfies

$$\lim_{n\to\infty} \theta^\star_n = \infty, \tag{14}$$

then the strong zero-one law (12) holds for graph connectivity with critical scaling $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ if and only if

$$\frac{M_n}{\theta^\star_n} \xrightarrow{P}_n 1. \tag{15}$$

A requirement weaker than the convergence (15) is shown to suffice for weak zero-one laws.

Theorem 3.2: Assumptions A are enforced on the probability distribution function F. Assume there exists a scaling $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ satisfying (14) such that

$$\frac{M_n}{\theta^\star_n} \Longrightarrow_n R \tag{16}$$

for some $\mathbb{R}_+$-valued rv R. If R is a non-degenerate rv with $\mathbb{P}[R=0] = 0$, then only the weak zero-one law (13) holds for graph connectivity with critical scaling $\theta^\star : \mathbb{N}_0 \to (0,\infty)$.

Theorem 3.1 and Theorem 3.2 are established in Section VII. Careful inspection of the arguments in Section VII-B shows that the non-negativity of ξ is needed only to ensure $0 \le |\xi^\star_F| < \infty$. Such arguments still work for any rv ξ with *finite* $|\xi^\star_F|$ even if $\xi^\star_F < 0$. The case $\xi^\star_F = -\infty$ is technically more involved, and is addressed in [30].

In order to understand when the conditions (15) and (16) occur, we exploit classical results from Extreme Value Theory in Section IV and Section VI, respectively; e.g., see the monographs [16], [23], [24], [38] for additional details.

IV. Rapid variation and strong zero-one laws

First some notation: With any non-decreasing function $g : \mathbb{R} \to [0,\infty]$, we associate its (left-continuous) generalized inverse $g^\leftarrow : \mathbb{R}_+ \to [-\infty,\infty]$ defined by

$$g^\leftarrow(t) = \inf\left(x \in \mathbb{R} : \ g(x) \ge t\right), \quad t \ge 0 \tag{17}$$

with $g^\leftarrow(t) = \infty$ if the set is empty. Additional information concerning generalized inverses can be found in the monograph by Resnick [38, Section 0.2].

For any probability distribution function $F : \mathbb{R} \to [0,1]$ we introduce the *quantiles*

$$\lambda_{F,n} = \left(\frac{1}{1-F}\right)^\leftarrow (n), \quad n = 1, 2, \ldots \tag{18}$$

If the probability distribution function F has support in $\mathbb{R}_+$ with $F(0) = 0$ (as is the case here under Assumptions A), then somewhat annoyingly, we have $\lambda_{F,1} = -\infty$ according to this definition. As a result, in order to simplify the presentation, we modify the definition by setting $\lambda_{F,1} = 0$ instead. We also note that for each $n = 2, 3, \ldots$, $0 < \lambda_{F,n} < \infty$ with

$$\lim_{n\to\infty} \lambda_{F,n} = \xi_F \tag{19}$$

monotonically from below. With the aforementioned modification, we think of (18) as defining the scaling $\lambda_F : \mathbb{N}_0 \to \mathbb{R}_+$.

A. Rapid variation and relative stability are equivalent

Consider a collection of i.i.d. $\mathbb{R}$-valued rvs $\{\xi, \xi_k, \ k = 1, 2, \ldots\}$, each distributed according to the probability distribution $F : \mathbb{R} \to [0,1]$. Its sequence of maxima $\{M_n, \ n = 1, 2, \ldots\}$ (still given by (8)) is said to be *relatively stable* if there exists a scaling $\lambda : \mathbb{N}_0 \to (0,\infty)$ such that

$$\frac{M_n}{\lambda_n} \xrightarrow{P}_n 1. \tag{20}$$

Note the requirement $\lambda_n > 0$ for all $n = 1, 2, \ldots$, *even* if F does *not* have its support contained in $\mathbb{R}_+$. Scalings appearing at (20) are unique up to asymptotic equivalence.

When $0 < \xi_F < \infty$, the convergence (10) is equivalent to

$$\lim_{n\to\infty} \frac{M_n}{\xi_F} = 1 \quad a.s.,$$

while (19) implies

$$\lambda_{F,n} \sim \xi_F. \tag{21}$$

The validity of (20) (with $\lambda_n = \lambda_{F,n}$) is therefore always guaranteed.

On the other hand, if $\xi_F = \infty$, then (21) is meaningless since $\lim_{n\to\infty} \lambda_{F,n} = \infty$ and a finer analysis is required. When $\xi_F = \infty$, Gnedenko [22] gave a complete characterization of relative stability with the help of the notion of *rapid variation*: The distribution $F : \mathbb{R} \to [0,1]$ is said to be *rapidly varying* if $\xi_F = \infty$ and

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = \begin{cases} 0 & \text{if } x > 1 \\ \infty & \text{if } 0 < x < 1. \end{cases} \tag{22}$$

Combining the remarks above with Theorem 2 in [22, p. 428] leads to the following characterization.

Theorem 4.1: *Consider a collection of i.i.d.* $\mathbb{R}$*-valued rvs* $\{\xi, \xi_k,\ k = 1, 2, \ldots\}$, *each distributed according to the probability distribution* $F : \mathbb{R} \to [0, 1]$. *Its sequence of maxima* $\{M_n,\ n = 1, 2, \ldots\}$ *is relatively stable if and only if either* $0 < \xi_F < \infty$ *or* F *is rapidly varying; in either case the scaling* $\lambda : \mathbb{N}_0 \to (0, \infty)$ *appearing in (20) can be selected through*

$$\lambda_n \sim \lambda_{F,n}, \quad n = 1, 2, \ldots \tag{23}$$

B. A strong zero-one law

We are now in a position to give one of the main results of the paper.

Theorem 4.2: *Assumptions A are enforced on the probability distribution function* F, *and assume* $\xi_F = \infty$. *Consider any scaling* $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ *such that*

$$\theta^\star_n \sim \lambda_{F,n} \tag{24}$$

where the scaling $\lambda_F : \mathbb{N}_0 \to \mathbb{R}_+$ *is given by (18). Then, graph connectivity obeys a strong zero-one law (12) with critical scaling* $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ *if and only if* F *is rapidly varying.*

Proof. By virtue of (19), $\xi_F = \infty$ is equivalent to $\lim_{n\to\infty} \lambda_{F,n} = \infty$, and by Theorem 4.1, the rapid variation of F is equivalent to (20) with (23). The desired result is now immediate from Theorem 3.1. ■

V. Exponential fitness

For some $\lambda > 0$, the rv ξ is said to be exponentially distributed with parameter $\lambda > 0$, written $\xi \sim F_\lambda$, if

$$\mathbb{P}[\xi \le x] = F_\lambda(x) = 1 - e^{-\lambda x}, \quad x \ge 0. \tag{25}$$

This special case was considered already in [10], [40] to show that even non-scale free distributions can generate scale-free networks.

A. Applying Theorem 4.2

With $F = F_\lambda$, we have $\xi^\star_F = 0$ and $\xi_F = \infty$. The exponential distribution is rapidly varying since

$$\frac{1 - F_\lambda(tx)}{1 - F_\lambda(t)} = e^{-\lambda(x-1)t}, \quad x, t > 0 \tag{26}$$

and a strong zero-one law exists by Theorem 4.2. In fact, from (18) and (24) the corresponding critical scaling can be taken as

$$\theta^\star_n = \lambda_{F,n} = \lambda^{-1} \log n, \quad n = 1, 2, \ldots \tag{27}$$

Thus, for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying $\theta_n \sim c\lambda^{-1} \log n$ for some $c > 0$, we have $\lim_{n\to\infty} P(n; \theta_n) = 1$ (resp. $\lim_{n\to\infty} P(n; \theta_n) = 0$) if $0 < c < 1$ (resp. $1 < c$).

B. Sharper results for the exponential distribution

Results sharper than Theorem 4.2 are available when the fitness variables follow the exponential distribution (25). To state such results, write any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ in the form

$$\theta_n = \lambda^{-1} \left(\gamma_n + \log n\right)^+, \quad n = 1, 2, \ldots \tag{28}$$

for some sequence $\gamma : \mathbb{N}_0 \to \mathbb{R}$ – There is no loss of generality in doing so. Here and elsewhere we use the notation $x^+ = \max(0, x)$ for any scalar x in $\mathbb{R}$.

Theorem 5.1: *If the distribution function* $F : \mathbb{R} \to [0, 1]$ *is the exponential distribution (25) with parameter* $\lambda > 0$, *then*

$$\lim_{n\to\infty} P\left(n; \lambda^{-1}\left(\gamma_n + \log n\right)^+\right) = 1 - e^{-e^{-\Gamma}} \tag{29}$$

with sequence $\gamma : \mathbb{N}_0 \to \mathbb{R}$ *satisfying*

$$\lim_{n\to\infty} \gamma_n = \Gamma \tag{30}$$

for some Γ *in* $\mathbb{R}$.

Theorem 5.1, whose proof is given in Section VII-E, is in the spirit of the celebrated double-exponential result for graph connectivity in Erdős-Rényi graphs [6, Thm. 7.3, p. 164], [13, Thm. 3.10, p. 42]. If in (28) we take $\gamma_n \equiv \Gamma$ for all $n = 1, 2, \ldots$, then (29) becomes

$$\lim_{n\to\infty} P\left(n; \lambda^{-1}\left(\Gamma + \log n\right)^+\right) = 1 - e^{-e^{-\Gamma}}. \tag{31}$$

As is the case with Erdős-Rényi graphs and geometric random graphs [25], such a result suggests the possibility of obtaining estimates on the transition width of the phase transition implied by the strong zero-one law; details concerning this estimate are available in [30].

The strong zero-one law of Theorem 4.2 is confirmed through the simulation results displayed in Figure 1 when $\lambda = 1$ so that the critical scaling (27) is now $\theta^\star_n = \log n$ for each $n = 1, 2, \ldots$. Here, and in the other simulation results displayed later in the paper, we estimate $P(n; c\theta^\star_n)$ with $c > 0$

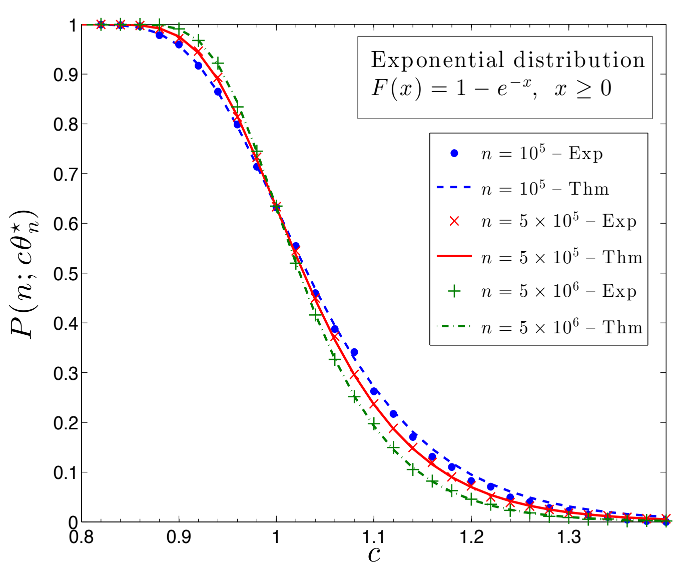

Fig. 1. *Exponential distribution (25) with* $\lambda = 1$.

by the *empirical probability* that the random threshold graph is connected under the scaling $n \to c\theta_n^\star$; this quantity is obtained by averaging over 5000 independent realizations. In Figure 1 these values are indicated by means of marks (Exp.), while the lines give the graphs of the mappings $c \to 1 - e^{-n^{-(c-1)}}$ for the various values of n (Thm). These curves are identified through the following heuristic argument: For each $c > 0$, select the sequence $\gamma : \mathbb{N}_0 \to \mathbb{R}$ appearing in (29) so that

$$\gamma_n + \log n = c \log n, \quad n = 1, 2, \ldots$$

or equivalently, $\gamma_n = (c-1)\log n$. Blind substitution into the sharper result (29) suggests the approximation

$$P(n; c\log n) \simeq 1 - e^{-e^{-(c-1)\log n}} = 1 - e^{-n^{-(c-1)}}$$

for sufficiently large n as we *formally* equate $(c-1)\log n$ with its unbounded limit $\Gamma = \pm\infty$. Yet, in spite of this sleight of hand, the displays show this approximation to be remarkably accurate, expressing the sharp transition alluded to earlier.

C. On degrees

Turning to properties of degree distributions in random threshold graphs, we follow the developments of [19] where additional details can be found. Thus, fix $n = 2, 3, \ldots$ and $\theta > 0$. For each $k = 1, 2, \ldots, n$, the degree of node k in $\mathbb{T}(n;\theta)$ is the rv $D_{n,k}(\theta)$ given by

$$D_{n,k}(\theta) := \sum_{\ell=1,\ \ell\neq k}^{n} \mathbf{1}\left[\xi_k + \xi_\ell > \theta\right]. \tag{32}$$

Under the enforced assumptions on the rvs $\xi_1, \ldots, \xi_n$, the rv $D_{n,k}(\theta)$ is a Binomial rv $\text{Bin}(n-1; 1 - F(\theta - \xi_k))$ conditionally on ξ_k. The rvs $D_{n,1}(\theta), \ldots, D_{n,n}(\theta)$ are obviously equidistributed.

Next, pick $t > 0$. An easy conditioning argument leads to

$$\begin{aligned} &\mathbb{E}\left[e^{-tD_{n,k}(\theta)}\right] \\ &= e^{-\lambda\theta}e^{-(n-1)t} \\ &\quad + \int_0^\theta \left(1 - e^{-\lambda(\theta-x)}(1-e^{-t})\right)^{n-1} \lambda e^{-\lambda x} dx \end{aligned}$$

after some uninteresting calculations. With the critical scaling $\theta^\star : \mathbb{N}_0 \to \mathbb{R}_+$ given here by (27), we find

$$\begin{aligned} &\mathbb{E}\left[e^{-tD_{n,1}(\theta_n^\star)}\right] \\ &= \frac{e^{-(n-1)t}}{n} + \int_0^{\theta_n^\star} \left(1 - \frac{1}{n}(1-e^{-t})e^{\lambda x}\right)^{n-1} \lambda e^{-\lambda x} dx \end{aligned}$$

for each $n = 2, 3, \ldots$. Letting n go to infinity yields

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-tD_{n,1}(\theta_n^\star)}\right] = \int_0^\infty e^{-(1-e^{-t})e^{\lambda x}} \lambda e^{-\lambda x} dx$$

by the Bounded Convergence Theorem since

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}(1-e^{-t})e^{\lambda x}\right)^{n-1} = e^{-(1-e^{-t})e^{\lambda x}}, \quad x \geq 0.$$

Consequently, $D_{n,1}(\theta_n^\star) \Longrightarrow_n D$ where D is a conditionally Poisson rv. It was shown by Fujihara et al. [19, Example 1, p. 366] that

$$\mathbb{P}\left[D = d\right] \sim d^{-2} \quad (\text{as } d \to \infty) \tag{33}$$

Under this critical scaling (27), we also have

$$\lim_{n\to\infty} P(n;\theta_n^\star) = \lim_{n\to\infty} P\left(n; \lambda^{-1}\log n\right) = 1 - e^{-1}$$

upon using (31) with $\Gamma = 0$. Thus, the (essentially unique) scaling which ensures a power-law degree distribution will not result in graph connectivity in the a.a.s sense.

VI. Applying Extreme Value Theory

In Section IV we provided a complete characterization for (15) in terms of the underlying weight distribution F. Characterizing the convergence (16) is more involved, and requires that we delve more deeply into Extreme Value Theory: Recall that the rv ξ, or equivalently, its probability distribution function F, is said to belong to the *maximum domain of attraction* of a probability distribution function $K : \mathbb{R} \to [0,1]$ if there exist sequences of norming constants $\{a_n,\ n = 1, 2, \ldots\}$ and $\{b_n,\ n = 1, 2, \ldots\}$ with $a_n > 0$ for all $n = 1, 2, \ldots$ such that

$$a_n^{-1}\left(M_n - b_n\right) \Longrightarrow_n K. \tag{34}$$

Such a distribution K is called a *max-stable* distribution, and the collection of all distributions $F : \mathbb{R} \to [0,1]$ such that (34) holds constitutes the maximum domain of attraction of K, denoted $\text{MDA}(K)$. According to the Fisher-Tippett Theorem [22, Thm. 3, p. 431] [16, Thm. 3.2.3, p. 121], there are only three max-stable distributions, namely the Fréchet, Weibull and Gumbel distributions (which are introduced next).

At this point we recall that a mapping $L : \mathbb{R}_+ \to \mathbb{R}_+$ is *slowly varying* (at infinity) if

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, \quad x > 0.$$

A. In the domain of attraction of the Fréchet distribution

With $\alpha > 0$, the Fréchet distribution is the probability distribution function $\Phi_\alpha : \mathbb{R} \to [0,1]$ given by

$$\Phi_\alpha(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \\ e^{-x^{-\alpha}} & \text{if } x > 0. \end{cases} \tag{35}$$

Key to the discussion is the following characterization of $\text{MDA}(\Phi_\alpha)$ [16, Thm. 3.3.7, p. 131] [38, Prop. 1.11, p. 54] (adapted to distributions with support on $\mathbb{R}_+$):

Theorem 6.1: The distribution function $F : \mathbb{R} \to [0,1]$ with support contained in $\mathbb{R}_+$ belongs to the maximum domain of attraction of Φ_α with some $\alpha > 0$ if and only if

$$1 - F(x) = x^{-\alpha}L_0(x), \quad x > 0 \tag{36}$$

for some slowly varying function $L_0 : \mathbb{R}_+ \to \mathbb{R}_+$. The norming constants in (34) can be taken as

$$a_n = \lambda_{F,n} \quad \text{and} \quad b_n = 0, \quad n = 1, 2, \ldots \tag{37}$$

Distributions with support contained in $\mathbb{R}_+$ which belong to $\text{MDA}(\Phi_\alpha)$ include the Pareto and other power-law distributions [16, p. 133]. From (36) it follows that $\xi_F = \infty$. The norming constants $\{a_n,\ n=1,2,\ldots\}$ determined by (37) can also be expressed in the form

$$a_n = \lambda_{F,n} = n^{\frac{1}{\alpha}} L_1(n), \quad n=1,2,\ldots \tag{38}$$

for some other slowly varying function $L_1 : \mathbb{R}_+ \to \mathbb{R}_+$; see [38, Prop. 0.8 (v), p. 23] for details.

Corollary 6.2: Assume that the distribution function $F : \mathbb{R} \to [0,1]$ *belongs to* $\mathrm{MDA}(\Phi_\alpha)$ *with some* $\alpha > 0$*. Under Assumptions A, graph connectivity obeys only a weak zero-one law (13) with critical scaling* $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ *given by (24).*

Proof. When F is a member of $\mathrm{MDA}(\Phi_\alpha)$, Theorem 6.1 yields (16) for any scaling $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ asymptotically equivalent to $\lambda_F : \mathbb{N}_0 \to \mathbb{R}_+$. The limiting rv R being distributed according to a Fréchet distribution, we obviously have $\mathbb{P}[R=0] = 0$, and the desired conclusion is now a simple consequence of Theorem 3.2. ■

From these arguments it is plain that any distribution in the maximum domain of attraction of a Fréchet distribution cannot be rapidly varying. The power law distributions discussed next illustrate both this point and Corollary 6.2: For some $\nu > 0$ and $a > 0$, the rv ξ is said to be distributed according to a power law distribution with parameters (a,ν), written $\xi \sim P_{a,\nu}$, if

$$\mathbb{P}[\xi \le x] = P_{a,\nu}(x) = 1 - \left(\frac{a}{a+x}\right)^\nu, \quad x \ge 0. \tag{39}$$

Here, $\xi_F = \infty$ and $\xi_F^\star = 0$. Note that $P_{a,\nu}$ is *not* rapidly varying since

$$\lim_{t\to\infty} \frac{1-P_{a,\nu}(tx)}{1-P_{a,\nu}(t)} = \lim_{t\to\infty}\left(\frac{a+t}{a+tx}\right)^\nu = x^{-\nu}, \quad x>0.$$

We also check that

$$\lambda_{F,n} = a\left(n^{\frac{1}{\nu}}-1\right), \quad n=1,2,\ldots$$

so that $\theta_n^\star \sim \lambda_{F,n} \sim a n^{\frac{1}{\nu}}$. Direct calculations show that

$$\frac{M_n}{a n^{\frac{1}{\nu}}} \Longrightarrow_n R$$

where R is distributed according to the Fréchet distribution Φ_ν given at (35). The corresponding weak zero-one law is illustrated in Figure 2 for the choice of parameters $a=1$ and $\nu=2$, with critical scaling $\theta_n^\star = \sqrt{n}$ for all $n=1,2,\ldots$. The weak nature of the zero-one law (13) is evident from the figure since $P(n; c\,\theta_n^\star) = 1$ only for $0<c<0.3$, becoming close to zero only after $c>8$ – Contrast this with the strong zero-one laws observed in Figure 1 and Figure 3.

B. In the domain of attraction of the Weibull distribution

With $\alpha > 0$, the Weibull distribution is the probability distribution function $\Psi_\alpha : \mathbb{R} \to [0,1]$ given by

$$\Psi_\alpha(x) = \begin{cases} e^{-|x|^\alpha} & \text{if } x \le 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{40}$$

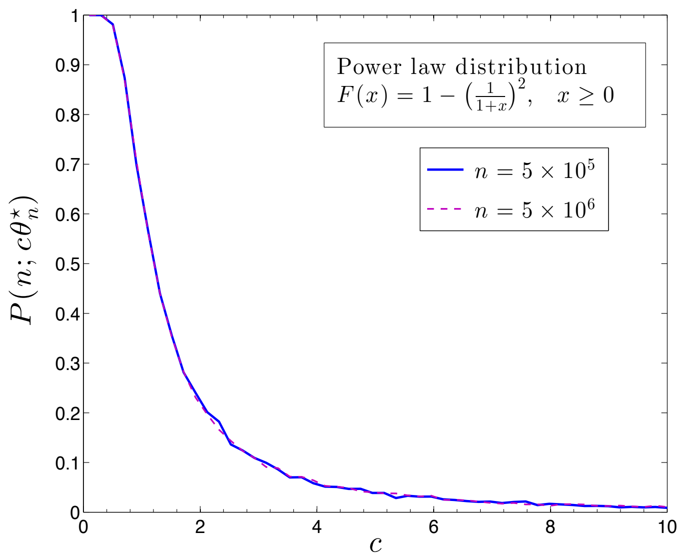

Fig. 2. *Power-law distribution (39) with $a=1$ and $\nu=2$.*

Of particular interest is the following characterization of $\mathrm{MDA}(\Psi_\alpha)$ [16, Thm. 3.3.12, p. 135] (adapted to distributions with support on $\mathbb{R}_+$):

Theorem 6.3: The distribution function $F : \mathbb{R} \to [0,1]$ *with support contained in* $\mathbb{R}_+$ *belongs to the maximum domain of attraction of* Ψ_α *for some* $\alpha > 0$ *if and only if* $0 < \xi_F < \infty$ *and*

$$1 - F(\xi_F - x^{-1}) = x^{-\alpha} L_0(x), \quad x > \xi_F^{-1} \tag{41}$$

for some slowly varying (at infinity) function $L_0 : \mathbb{R} \to \mathbb{R}_+$*. The norming constants in (34) can always be taken as*

$$a_n = \xi_F - \lambda_{F,n} \quad \text{and} \quad b_n = \xi_F, \quad n=1,2,\ldots \tag{42}$$

Distributions with support contained in $\mathbb{R}_+$ which belong to $\mathrm{MDA}(\Psi_\alpha)$ satisfy $0 < \xi_F < \infty$, and include the uniform distribution on $(0,1)$, Beta distributions and distributions with power law behavior at a finite right endpoint [16, p. 137]. The theory in the form developed here does not apply to this class of probability distributions. Indeed, Theorems 3.1 and 3.2 both rely on (14) when taking advantage of the representation result of Lemma 7.1. However, the arguments can be modified to cover both the case $\xi_F < \infty$ under Assumptions A, and probability distributions with support outside $\mathbb{R}_+$; details are available in [30].

C. In the domain of attraction of the Gumbel distribution

The Gumbel distribution is the probability distribution function $\Lambda : \mathbb{R} \to [0,1]$ given by

$$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}. \tag{43}$$

Characterizing $MDA(\Lambda)$ is more involved; see Theorem 2.5.1 in [23, p. 87] as well as the discussion in [38, Section 1.1]. However, the norming constants in (34) can be taken to be

$$a_n = \phi(b_n) \quad \text{and} \quad b_n = \lambda_{F,n}, \quad n=1,2,\ldots \tag{44}$$

with the auxiliary function $\phi : [0, \xi_F) \to \mathbb{R}$ given by

$$\phi(x) = \frac{\int_x^{\xi_F} (1 - F(t))\, dt}{1 - F(x)}, \quad 0 \le x < \xi_F. \qquad (45)$$

Distributions in $\mathrm{MDA}(\Lambda)$ with support contained in $\mathbb{R}_+$ include the exponential, Weibull and log-normal distributions [16, p. 139]. If the distribution $F : \mathbb{R} \to [0,1]$ with support contained in $\mathbb{R}_+$ belongs to $\mathrm{MDA}(\Lambda)$, then both $0 < \xi_F < \infty$ and $\xi_F = \infty$ can occur. In either case, moments of all orders are finite, i.e., $\mathbb{E}[X^r] < \infty$ for all $r \ge 1$ [16, Cor. 3.3.32, p. 148] [38, Ex. 1.1.1, p. 52], and the auxiliary function (45) is indeed well defined.

Corollary 6.4: Assume that the distribution function $F : \mathbb{R} \to [0,1]$ *belongs to* $\mathrm{MDA}(\Lambda)$ *with* $\xi_F = \infty$. *Under Assumptions A, graph connectivity obeys a strong zero-one law (12) with critical scaling* $\theta^\star : \mathbb{N}_0 \to (0,\infty)$ *given by (24).*

Proof. Again, $\xi_F = \infty$ implies $\lim_{n\to\infty} \lambda_{F,n} = \infty$, hence $\lim_{n\to\infty} b_n = \infty$. Gnedenko [22, Lemma 5, p. 445] has shown that the norming constants $\{a_n,\ n = 1, 2, \ldots\}$ and $\{b_n,\ n = 1, 2, \ldots\}$ at (34) with a Gumbel limit necessarily satisfy

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Using this fact we get $\frac{M_n}{b_n} \xrightarrow{P}_n 1$, and distributions in $\mathrm{MDA}(\Lambda)$ are necessarily of rapid variation. The conclusion now follows by appealing to Theorem 4.2. ∎

We close by showing how the theory developed thus far applies to Weibull distributions: For some $\lambda > 0$ and $\beta > 0$, the rv ξ is said to be distributed according to a Weibull distribution with parameters (λ, β), written $\xi \sim F_{\lambda,\beta}$, if[2]

$$\mathbb{P}[\xi \le x] = F_{\lambda,\beta}(x) = 1 - e^{-\lambda x^\beta}, \quad x \ge 0. \qquad (46)$$

The case $\beta = 1$ corresponds to the exponential distribution. Again we have $\xi_F^\star = 0$ and $\xi_F = \infty$. It is well known that $F_{\lambda,\beta}$ is in $\mathrm{MDA}(\Lambda)$. A direct argument (based on the appropriate version of (26)) shows that $F_{\lambda,\beta}$ is of rapid variation. By virtue of either Theorem 4.2 or Corollary 6.4, strong critical scalings exist and they are of the form

$$\theta_n^\star = \lambda_{F,n} = \left(\lambda^{-1} \log n\right)^{\frac{1}{\beta}}, \quad n = 1, 2, \ldots$$

Therefore, for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ such that $\theta_n \sim c\left(\lambda^{-1} \log n\right)^{1/\beta}$ for some $c > 0$, we have $\lim_{n\to\infty} P(n; \theta_n) = 1$ (resp. $\lim_{n\to\infty} P(n; \theta_n) = 0$) if $0 < c < 1$ (resp. $1 < c$). The corresponding zero-one law is illustrated in Figure 3 for a Weibull distribution with parameters $\lambda = 1$ and $\beta = 2$, and critical scaling $\theta_n^\star = \sqrt{\log n}$ for all $n = 1, 2, \ldots$.

[2]The distributions given by (40) and (46), are both called Weibull distributions in the literature – After all, although they have non-intersecting supports, they both display the same exponential decay. This accepted terminology, while perhaps unfortunate, should not cause any confusion in this paper.

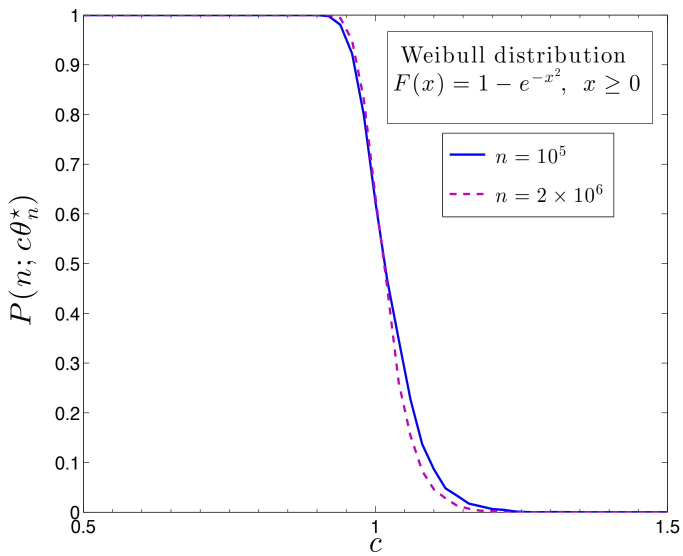

Fig. 3. *Weibull distribution (46) with* $\lambda = 1$ *and* $\beta = 2$.

VII. Proofs

A. Representing the probability of graph connectivity

The scaling laws for graph connectivity in random threshold graphs all flow from the following observation.

Lemma 7.1: Under Assumptions A on the probability distribution function F, *we have*

$$P(n; \theta) = \mathbb{P}[M_n^\star + M_n > \theta], \quad \begin{array}{c} \theta > 0, \\ n = 2, 3, \ldots \end{array} \qquad (47)$$

Proof. Fix $\theta > 0$ and $n = 2, 3, \ldots$. Recall that for distinct $i, j = 1, \ldots, n$, the nodes i and j are adjacent in the random threshold graph $\mathbb{T}(n; \theta)$ if and only if $\xi_i + \xi_j > \theta$.

Let $\xi_{n,1}, \ldots, \xi_{n,n}$ denote the values of $\xi_1, \ldots, \xi_n$ arranged in increasing order, i.e., $\xi_{n,1} \le \xi_{n,2} \le \ldots \le \xi_{n,n-1} \le \xi_{n,n}$, so that $\xi_{n,1} = M_n^\star$ and $\xi_{n,n} = M_n$. The rvs $\xi_{n,1}, \ldots, \xi_{n,n}$ are the so-called order statistics associated with the rvs $\xi_1, \ldots, \xi_n$.

If $\xi_{n,1} + \xi_{n,n} > \theta$, then for each $j = 1, \ldots, n-1$, we must necessarily have $\theta < \xi_{n,j} + \xi_{n,n}$, and any node whose weight coincides with $\xi_{n,n}$ is indeed connected to each of the other $n - 1$ nodes. In other words, the graph $\mathbb{T}(n; \theta)$ is connected if $\xi_{n,1} + \xi_{n,n} > \theta$, whence

$$P(n; \theta) \ge \mathbb{P}[\xi_{n,1} + \xi_{n,n} > \theta] = \mathbb{P}[M_n^\star + M_n > \theta]. \qquad (48)$$

Assume now that the random threshold graph $\mathbb{T}(n; \theta)$ is connected. We claim that the inequality $\xi_{n,1} + \xi_{n,n} \ge \theta$ necessarily holds. We proceed by contradiction: Assume instead that $\xi_{n,1} + \xi_{n,n} < \theta$, so that $\xi_{n,1} + \xi_{n,k} < \theta$ for $k = 2, \ldots, n$. Then, any node whose weight coincides with $\xi_{n,1}$ will not be adjacent to each of the other $n - 1$ nodes – This node is therefore isolated and the random threshold graph $\mathbb{T}(n; \theta)$ cannot be connected, contrary to assumption! These arguments show that

$$P(n; \theta) \le \mathbb{P}[\xi_{n,1} + \xi_{n,n} \ge \theta] = \mathbb{P}[M_n^\star + M_n > \theta] \qquad (49)$$

since $\mathbb{P}[\xi_{n,1} + \xi_{n,n} = \theta] = 0$ under the foregoing Assumptions A-(ii) that the weight distribution F has no atom. We get (47) by combining (48) and (49). ∎

B. Exploiting Lemma 7.1

Lemma 7.1 highlights the role to be played by the running minima (7) and maxima (8), and their limiting theory. Indeed, let $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ be a scaling under consideration as a possible critical scaling for graph connectivity, and write

$$R_n^\star = \frac{M_n}{\theta_n^\star} + \frac{M_n^\star}{\theta_n^\star}, \quad n = 1, 2, \ldots \tag{50}$$

For any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$, the expression (47) becomes

$$P(n; \theta_n) = \mathbb{P}[R_n^\star > r_n], \quad n = 1, 2, \ldots \tag{51}$$

with

$$r_n = \frac{\theta_n}{\theta_n^\star}. \tag{52}$$

Now, let n go to infinity under the following assumptions: Suppose there exists some $\mathbb{R}_+$-valued rv $R^\star$ such that

$$R_n^\star \Longrightarrow_n R^\star \tag{53}$$

and assume that

$$\lim_{n\to\infty} r_n = r \tag{54}$$

for some scalar r in $\mathbb{R}$. Then, standard facts concerning distributional convergence [4] readily yield

$$\lim_{n\to\infty} \mathbb{P}[R_n^\star > r_n] = \mathbb{P}[R^\star > r] \tag{55}$$

whenever the limit r is a *point of continuity* for $R^\star$, i.e., $\mathbb{P}[R^\star = r] = 0$.

If the scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ satisfies (14), then (9) yields

$$\lim_{n\to\infty} \frac{M_n^\star}{\theta_n^\star} = 0 \quad a.s. \tag{56}$$

since here we have $0 \le \xi_F^\star < \infty$ under Assumptions A. As a result,

$$\lim_{n\to\infty} \left(R_n^\star - \frac{M_n}{\theta_n^\star} \right) = 0 \quad a.s., \tag{57}$$

and the sequence $\{R_n^\star,\ n = 2, 3, \ldots\}$ converges in probability (resp. in distribution) if and only if the sequence $\{\frac{M_n}{\theta_n^\star},\ n = 2, 3, \ldots\}$ converges in probability (resp. in distribution), in which case their limits coincide. These observations pave the way to proving Theorem 3.1 and Theorem 3.2.

C. A proof of Theorem 3.1

In view of (14), (56) and (57) we note that (15) reduces to $R_n^\star \Longrightarrow_n R^\star$ with $R^\star = 1$ a.s. Thus, pick any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ which satisfies (11) for some $c \neq 1$ in $(0, \infty)$. Since $c \neq 1$ is a point of continuity for the degenerate rv $R = 1$, we conclude from (55) (with the notation (52)) that

$$\lim_{n\to\infty} \mathbb{P}[R_n^\star > r_n] = \mathbb{P}[R^\star > c] = \begin{cases} 1 & \text{if } 0 < c < 1 \\ \\ 0 & \text{if } 1 < c. \end{cases}$$

Letting n going to infinity in (51) yields the strong zero-one law (12).

Conversely, assume that the strong zero-one law (12) holds with critical scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ satisfying (14). For any $c \neq 1$ in $(0, \infty)$, consider the scaling $\theta_c : \mathbb{N}_0 \to \mathbb{R}_+ : n \to c\theta_n^\star$. Using (50), (51) and (52), we find

$$P(n; c\theta_n^\star) = \mathbb{P}[R_n^\star > c], \quad n = 2, 3, \ldots$$

and the zero-one law (12) for θ_c can now be rewritten as

$$\lim_{n\to\infty} \mathbb{P}[R_n^\star > c] = \begin{cases} 1 & \text{if } 0 < c < 1 \\ \\ 0 & \text{if } 1 < c. \end{cases}$$

This amounts to $R_n^\star \Longrightarrow_n 1$, or equivalently, $R_n^\star \xrightarrow{P}_n 1$, whence $\frac{M_n}{\theta_n^\star} \xrightarrow{P}_n 1$ by virtue of (14), (56) and (57) as noted earlier. ∎

D. A proof of Theorem 3.2

Under (14) we conclude from (16), (56) and (57) that $R_n^\star \Longrightarrow_n R$. To establish (13), consider scalings $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ which satisfy (11), this time with either $c = 0$ or $c = \infty$ – The notation (50)-(52) is in use in what follows.

Assume first that $c = 0$ so that $\lim_{n\to\infty} r_n = 0$. The condition $\mathbb{P}[R = 0] = 0$ amounts to the origin $r = 0$ being a point of continuity of the non-negative rv R. Hence, the convergence $R_n^\star \Longrightarrow_n R$ implies $\lim_{n\to\infty} \mathbb{P}[R_n^\star > r_n] = \mathbb{P}[R > 0] = 1$, and upon letting n go to infinity in (51), we get $\lim_{n\to\infty} P(n; \theta_n) = 1$ when $c = 0$ as desired.

Next, take $c = \infty$ in (11): For every $M > 0$, there exists a finite integer $n^\star(M)$ such that $r_n > M$ whenever $n \ge n^\star(M)$, and $\mathbb{P}[R_n^\star > r_n] \le \mathbb{P}[R_n^\star > M]$ on that range. Letting n go to infinity we conclude

$$\limsup_{n\to\infty} \mathbb{P}[R_n^\star > r_n] \le \limsup_{n\to\infty} \mathbb{P}[R_n^\star > M]. \tag{58}$$

In particular, if M is a point of continuity for R, then the convergence $R_n^\star \Longrightarrow_n R$ implies $\lim_{n\to\infty} \mathbb{P}[R_n^\star > M] = \mathbb{P}[R > M]$, and the inequality (58) becomes

$$\limsup_{n\to\infty} \mathbb{P}[R_n^\star > r_n] \le \mathbb{P}[R > M] \tag{59}$$

with the left handside being independent of M.

In order to conclude, write (59) for a sequence $\{M_j,\ j = 1, 2, \ldots\}$ of points of continuity for R such $\lim_{j\to\infty} M_j = \infty$ – It is always possible to find such a sequence. The rv R being honest, we get $\lim_{j\to\infty} \mathbb{P}[R > M_j] = 0$ and the conclusion $\limsup_{n\to\infty} \mathbb{P}[R_n^\star > r_n] = 0$ follows, whence $\lim_{n\to\infty} \mathbb{P}[R_n^\star > r_n] = 0$. Making use of (51) again we conclude that $\lim_{n\to\infty} P(n; \theta_n) = 0$ when $c = \infty$. This completes the proof of (13).

Finally, Theorem 3.1 readily implies that only the weak zero-one law (13) can hold under the assumption (16) for some non-degenerate $\mathbb{R}_+$-valued rv R. ∎

E. A Proof of Theorem 5.1

The rvs $\xi_1, \ldots, \xi_n$ are exponentially distributed with parameter $\lambda > 0$; see (25). With the help of (44)-(45) the convergence (34) becomes

$$\lambda M_n - \log n \Longrightarrow_n \Lambda \tag{60}$$

as we note that

$$a_n = \lambda^{-1} \quad \text{and} \quad b_n = \lambda_{F,n} = \lambda^{-1} \log n, \quad n = 1, 2, \ldots$$

Since $\lim_{n\to\infty} M_n^\star = 0$ a.s., we have $\lambda M_n^\star \xrightarrow{P}_n 0$, and (60) implies

$$(\lambda M_n - \log n) + \lambda M_n^\star \Longrightarrow_n \Lambda. \tag{61}$$

Now, consider a scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ written in the form (28) for some sequence $\gamma : \mathbb{N}_0 \to \mathbb{R}$. With this representation, for n sufficiently large, we get from (47) that

$$\begin{aligned} P(n; \theta_n) &= \mathbb{P}\left[M_n^\star + M_n > \theta_n\right] \\ &= \mathbb{P}\left[(\lambda M_n - \log n) + \lambda M_n^\star > \lambda\theta_n - \log n\right] \\ &= \mathbb{P}\left[(\lambda M_n - \log n) + \lambda M_n^\star > \gamma_n\right] \\ &= \mathbb{P}\left[(\lambda M_n - \log n) + \lambda M_n^\star - \gamma_n > 0\right]. \end{aligned}$$

Let n go to infinity: The convergence (61) together with (30) yields $(\lambda M_n - \log n) + \lambda M_n^\star - \gamma_n \Longrightarrow_n \Lambda - \Gamma$, whence $\lim_{n\to\infty} P(n; \theta_n) = \mathbb{P}[\Lambda - \Gamma > 0]$ since the Gumbel distribution has only points of continuity. The desired result (29) follows from (43). ■

ACKNOWLEDGMENT

We thank the reviewers for their careful reading of the manuscript in its earlier incarnation, and for their constructive comments. We also thank Professor Y. Ide for pointing out errors in the bibliography. This work was supported by NSF Grant CCF-0830702 and by the Laboratory for Telecommunications Sciences, College Park (MD). The first author also acknowledges the hospitality and support of the Basque Center for Applied Mathematics (BCAM) in Bilbao (Spain) for month-long stays in June 2010 and June 2011 when this work was started and partially carried out.

REFERENCES

[1] R. M. Anderson and R. M. May, *Infectious Diseases of Humans*, Oxford University Press, Oxford (UK), 1991.

[2] M.J.B. Appel and R.P. Russo, "The connectivity of a graph on uniform points on $[0,1]^d$," *Statistics & Probability Letters* **60** (2002), pp. 351-357.

[3] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science* **286** (1999), pp. 509-512.

[4] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York (NY), 1968.

[5] S.R. Blackburn and S. Gerke, "Connectivity of the uniform random intersection graph," *Discrete Mathematics* **309** (2009), pp. 5130-5140.

[6] B. Bollobás, *Random Graphs*, Second Edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (UK), 2001.

[7] M. Boguñá and R. Pastor-Storras, "Class of correlated random networks with hidden variables," *Physical Review* **E 68** (2003), 036112.

[8] M. Bradonjić, A. Hagberg and A.G. Percus, "Giant component and connectivity in geographical threshold graphs," in Proceedings of WAW 2007, San Diego (CA), December 2007. Lecture Notes in Computer Science LCNS **4863**. Edited by A. Bonato and F.R.K Chung, pp. 209-216, New York Springer 2007.

[9] M. Bradonjić, A. Hagberg and A.G. Percus, "The structure of geographical threshold graphs," *Internet Mathematics* **5** (2008), pp. 113-139.

[10] G. Caldarelli, A. Capocci, P. De Los Rios and M.A. Muñoz, "Scale-free networks from varying vertex intrinsic fitness," *Physical Review Letters* **89** (2002), 258702.

[11] P. Diaconis, S. Holmes and S. Janson, "Threshold graph limits and random threshold graphs," *Internet Mathematics* **5** (2008), pp. 267-320

[12] R. Di Pietro, L.V. Mancini, A. Mei, A. Panconesi and J. Radhakrishnan, "Redoubtable sensor networks," *ACM Transactions on Information Systems Security* **TISSEC 11** (2008), pp. 1-22.

[13] M. Draief and L. Massoulié, *Epidemics and Rumours in Complex Networks*, London Mathematical Society lecture Note series **369**, Cambridge University Press, Cambridge (UK), 2010.

[14] P. Dodds and D. J. Watts, "Universal behavior in a generalized model of contagion," *Physical Review Letters* **92** (2004), 218701.

[15] P. Erdős and A. Rényi, "On the evolution of random graphs," *Publ. Math. Inst. Hung. Acad. Sci.* **5** (1960), pp. 17-61.

[16] P. Embrechts, C. Klüppelberg and Th. Mikosch, *Modelling Extremal Events*, Applications of Mathematics: Stochastic Modelling and Applied Probability **33**, Springer-Verlag, Berlin Heidelberg, 1997.

[17] A. Faragó, "Scalability of node degrees in random wireless network topologies," *IEEE Journal on Selected Areas in Communications* **JSAC-27** (2009), Special issue on Stochastic Geometry and Random Graphs for Wireless Networks, pp. 12381244.

[18] A. Faragó, "Asymptotically optimal trade-off between local and global connectivity in wireless networks," *Performance Evaluation* **68** (2011), pp. 142-156.

[19] A. Fujihara, Y. Ide, N. Konno, N. Masuda, H. Miwa and M. Uchida, "Limit theorems for the average distance and the degree distribution of the threshold network model," *Interdisciplinary Information Sciences* **15** (2009), pp. 361-366.

[20] A. Fujihara, M. Uchida and H. Miwa, "Universal power laws in threshold network model: Theoretical analysis based on Extreme Value Theory," *Physica A* **389** (2010), pp. 1124-1130.

[21] J. Gao, S. V. Buldyrev, H. E. Stanley and S. Havlin, "Networks formed from interdependent networks," *Nature Physics* **8** (2012), pp. 40–48.

[22] B. Gnedenko, "Sur la distribution limite du terme maximum d'une série aléatoire," *The Annals of Mathematics* Second Series **44** (1943), pp. 423-453.

[23] L. de Haan, *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*, Mathematical Center Tract **32**, Mathematics Center, Amsterdam (Holland), 1970.

[24] L. de Haan and A. Ferreira, *Extreme Value Theory: An Introduction*, Springer Series in Operations Research and Financial Engineering, Springer 2006.

[25] G. Han and A.M. Makowski, "Sensitivity of critical transmission ranges to node placement distributions," *IEEE Journal on Selected Areas in Communications* **JSAC–27** (2009), Special Issue on Stochastic Geometry and Random Graphs for Wireless Networks, pp. 1066-1078.

[26] A. Hagberg, P.J. Swart and D.A. Schult, "Designing threshold networks with given structural and dynamical properties," *Physical Review E* **74** (2006), 056116.

[27] Y. Ide, N. Konno and N. Obata, "Spectral properties of the threshold network model," *Internet Mathematics* **6** (2009), pp. 173-187.

[28] Y. Ide, N. Konno and N. Masuda, "Statistical properties of a generalized threshold network model," *Methodology and Computing in Applied Probability* **12** (2010), pp. 361-377.

[29] N. Konno, N. Masuda, R. Roy and A. Sarkar, "Rigorous results on the threshold network model," *Journal of Physics A: Mathematical and General* **38** (2005), pp. 6277-6291.

[30] A.M. Makowski and O. Yağan, "Zero-one laws for connectivity in random threshold graphs," In preparation. To be submitted to *Internet Mathematics.*

[31] N. Masuda, H. Miwa and N. Konno, "Geographical threshold graphs with small-world and scale-free properties," *Physical Review E* **71** (2005), 036108.

[32] G.L. McColm, "Threshold functions for random graphs on a line segment," *Combinatorics, Probability and Computing* **13** (2004), pp. 373-387.

[33] J. D. Murray, *Mathematical Biology*, Third Edition, Springer, New York (NY), 2002.

[34] M. E. J. Newman, "Spread of epidemic disease on networks," *Physical Review* E **66** (2002), 16128.

[35] M.D. Penrose, *Random Geometric Graphs*, Oxford Studies in Probability **5**, Oxford University Press, New York (NY), 2003.

[36] E.P. Reilly, *Random Threshold Graphs and Related Topics*, Ph.D. Thesis, Department of Applied Mathematics and Statistics, Johns Hopkins University, Baltimore (MD), 2010.

[37] E.P. Reilly and E.R. Scheinerman, "Random threshold graphs," *Electronic Journal of Combinatorics* **16** (2009).

[38] S.I. Resnick, *Extreme Values, Regular Variation, and Point Processes*, Springer-Verlag, 1987.

[39] K. Rybarczyk, "Diameter of the uniform random intersection graph with a note on the connectivity and the phase transition," *Discrete Mathematics* **311** (2011), pp. 1998-2019.

[40] V.D.P. Servedio and G. Caldarelli, "Vertex intrinsic fitness: How to produce arbitrary scale-free networks," *Physical Review E* **70** (2004), 056126.

[41] D. J. Watts, "A simple model of global cascades on random networks," *Proceedings of the National Academy of Sciences* **99** (2002), pp. 5766–5771.

[42] O. Yağan and V. Gligor, "Analysis of complex contagions in random multiplex networks," *Physical Review E* **86**, 036103, September 2012.

[43] O. Yağan and A.M. Makowski, "Zero-one laws for connectivity in random key graphs." *IEEE Transactions on Information Theory* **IT-58** (2012), pp. 2983-2999.

[44] O. Yağan, D. Qian, J. Zhang and D. Cochran, "Conjoining speeds up information diffusion in overlaying social-physical networks." Available online at arXiv:1112.4002[cs.SI].

[45] O. Yağan, D. Qian, J. Zhang and D. Cochran, "Optimal allocation of interconnecting links in cyber-physical systems: Interdependence, cascading failures and robustness." *IEEE Transactions on Parallel and Distributed Systems* **TPDS-23** (2012), Special issue on Cyber-Physical Systems, pp. 1708-1720.

Armand M. Makowski (M'83-SM'94-F'06) received the Licence en Sciences Mathématiques from the Université Libre de Bruxelles in 1975, the M.S. degree in Engineering-Systems Science from U.C.L.A. in 1976 and the Ph.D. degree in Applied Mathematics from the University of Kentucky in 1981. In August 1981, he joined the faculty of the Electrical Engineering Department at the University of Maryland College Park, where he is Professor of Electrical and Computer Engineering. He has held a joint appointment with the Institute for Systems Research since its establishment in 1985.

Armand Makowski was a C.R.B. Fellow of the Belgian-American Educational Foundation (BAEF) for the academic year 1975-76; he is also a 1984 recipient of the NSF Presidential Young Investigator Award and became an IEEE Fellow in 2006.

His research interests lie in applying advanced methods from the theory of stochastic processes to the modeling, design and performance evaluation of engineering systems, with particular emphasis on communication systems and networks.

Osman Yağan (S'07) received the B.S. degree in Electrical and Electronics Engineering from the Middle East Technical University, Ankara (Turkey) in 2007, and the Ph.D. degree in Electrical and Computer Engineering from the University of Maryland at College Park in 2011.

He was a visiting Postdoctoral Scholar at Arizona State University during Fall 2011. Since December 2011, he has been a Postdoctoral Research Fellow in CyLab at Carnegie Mellon University. His research interests include wireless network security, dynamical processes in complex networks, percolation theory, random graphs and their applications.