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Robustness in clustering-based weighted inter-connected networks

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Abstract. We study the robustness of symmetrically coupled and clustering-based weighted heterogeneous inter-connected networks with respect to load-failure-induced cascades. This is done under the assumption that the flow dynamics are governed by global redistribution of loads based on weighted betweenness centrality. Our results indicate that no weighting bias should be assigned to inter-links when calculating shortest path between node pairs under the clustering-based weighting scheme; i.e., inter-links shall be treated no differently than intra-links. In contrast with *local* load redistribution cases, we show that increasing connectivity is preferred for the robustness against *global* load redistribution-based cascading failures in clustering-based weighted inter-connected networks. Furthermore, comparisons among weighting schemes reveal that, both the clustering-based and degree-based schemes outperform the random one in the sense of requiring lower initial and total investments required to ensure robustness. We also find that clustering-based scheme outperforms degree-based one in terms of requiring lower initial investments. Except in a limited range where weighting is heavily suppressed, clustering-based scheme is shown to outperform degree-based one in terms of total investments. Finally, when there exists a hard investment budget constraint, clustering-based weighting scheme would be a better choice against a two-nodes-induced failure than the degree-based weighting, and the clustering-based scheme is more stable than degree-based scheme against one-or-two-nodes-induced failure. We expect our findings to be significantly useful in designing real-world weighted inter-connected networks that are robust against load-failure-induced cascades.

1 Introduction

Major outages in critical infrastructure networks, such as power grids, telecommunications, and transportation, are a main threat to the society's well-being [1]. To that end, the robustness of network of networks has attracted a great deal of attention recently [2–12]. This line of research was primarily inspired by the observation that many systems are characterized by the inter-connections (or, interdependence) between small sub-networks [13–18]. In particular, the seminal works on the percolation phenomena in interdependent networks [2,19] arrived at a surprising conclusion that the vulnerability of interdependent networks increases as their degree distribution (both intra- [2] and inter-degree [19] distribution) gets broader, which is in contrast with how isolated networks behave [20].

It is therefore of fundamental importance to characterize the robustness of the interdependent and inter-connected networks. Along these lines, a number of important extensions have been made recently. In particular, [6] considered robustness against intentional attacks, whereas [19,21] extended the model from one-to-one correspondence to multiple support-dependence relations. Along this fruitful line of research, the case where the networks are *weighted* [22] has received very little attention although it was pointed out that structural network properties such as the weighting distribution have great influence on the robustness of networks [23]. Therefore, we find it natural and important to investigate the effect of weighting schemes on the robustness of coupled networks against load-failure induced cascades.

Many cascade models have been proposed thus far to study the robustness and load cascading dynamics on isolated networks; e.g., the sand-pile model, the fiber bundle model, local weighted flow redistribution rule [24], global load-based cascading model [25–28] and its application in the problem of the diameter changes by the deletion of a single vertex for various in silico and real-world scale-free networks [29]. Among them, only the sand-pile model has been extended to inter-connected networks [30,31], which utilized a multi-type branching process [31] to analyze robustness and cascading dynamics. However, in the sand-pile model, it is assumed that flow is shed only to neighbor sites, ruling out the possibility of resetting the

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equilibrium via *global* redistribution. This makes it hard for the sand-pile model to capture a real-world situation, raising an immediate need to study the robustness of *weighted* inter-connected networks under global flow redistribution mechanisms.

The *degree-based* weighting scheme was discussed extensively in the literature owing to its applicability to real-world networks [24]; recently, it was also used for interdependent networks [32]. The *clustering-based* weighting is considered as a good alternative to *degree-based* weighting, which captures the relation between the clustering properties and the strength of interactions between constituents of the network [33].

In this paper, we start with an inter-connected network model, where each intra-link (resp. each inter-link) is assigned a weight according to the intra-clustering (resp. inter-clustering) coefficients of the nodes that it is connecting together. We wish to quantify the robustness of this network against single-node failures. We assume that when a node fails, a global (i.e., network-wide) load redistribution takes place leading to some nodes carrying larger loads than before, which possibly triggers further cascading failures if some of the nodes are loaded above their capacity. Our goal is to understand what proactive measures can be implemented to prevent such cascading failures when a single node fails. More specifically, we compute the average load per node in the network when every node sends to (and receives from) every other node a unit package via the *weighted* shortest paths. This quantity will be referred to as the *initial investment* with the understanding that it is the "investment" (in terms of the average node capacity) that needs to be made for the network to function properly in the initial set-up. Then, we consider the situation where a single node has failed and removed from the system. Calculating the new weighted shortest paths, we compute the new average load corresponding to the failure of any one of the nodes. The maximum of these quantities taken over all nodes gives us the *total investment*, defined as the "total investment" one has to make (in terms of average node capacity) to ensure that the network is robust against one-node failures.

To see additional benefits of clustering-based scheme on inter-connected networks and reveal the cascading dynamics, we also carry on a two-node-removal analysis as an example of multiple node failures. More specifically, we calculate the relative size of the giant component left behind when the two nodes that carry the highest loads fail.

With these in mind, we run an extensive simulation study to understand the robustness of two symmetrically coupled scale-free (SF) networks that are inter-connected. Our study leads to a number of interesting conclusions: first of all, we find that no weighting bias should be assigned to inter-links when calculating the shortest paths between node pairs under the clustering-based weighting scheme. In other words, inter-links and intra-links shall be treated equivalently for robustness costs to be minimum. Second, we show that in contrast with *local* load redistribution cases, increasing connectivity is preferred for robustness under *global* load redistribution based cascades (for clustering-based weighted inter-connected networks). Furthermore, comparisons among weighting schemes reveal that, both the clustering-based and degree-based weighting schemes outperform the random weighting scheme in the sense of requiring *lower* amount of initial *and* total investments. In addition, we show that clustering-based scheme outperforms degree-based one in the sense of requiring smaller *initial* investments. Except in a limited regime where weighting is heavily suppressed, we see that clustering-based scheme outperforms degree-based one in the sense of requiring *lower* total investments. Finally, in the multiple-node-failure analysis, we demonstrate that under hard budget constraints, i.e., when the average capacity per node cannot exceed a certain level, the clustering-based scheme outperforms the degree-based one, in the sense of resulting in a larger giant component size when the two most loaded nodes fail. We also show that clustering-based scheme leads to a more consistent robustness performance across one-or-two-node-induced failure, whereas in the degree-based scheme, the optimal parameter values that leads to highest robustness change significantly between the one-node failure case, and two-node failure case. We expect these findings to be helpful in designing real-world weighted inter-connected networks.

The rest of the paper is organized as follows: in Section 2, the global load-based cascading model on inter-connected networks is briefly introduced. In Section 3, simulation results that reveal the investment costs for robustness under clustering-based weighting scheme are presented. Finally, Section 4 is devoted to concluding remarks.

2 Robustness of inter-connected networks under clustering-based weighting scheme

Our network model is an extension of the model introduced in [25] to inter-connected networks. Namely, we consider two networks, A and B, with the same number of nodes, N. Nodes in each network are connected by intra-links, i.e., links within each network; this is characterized by intra-degree distribution $P_A(k)$ and $P_B(k)$, for network A and network B, respectively. In addition, nodes in network A are connected to nodes in network B, and vice versa, via inter-links; i.e., links between the two networks. The inter-degree distributions are characterized by the distributions [21] $\widetilde{P}_A(k)$ and $\widetilde{P}_B(k)$ for network A and network B, respectively.

We expect this model to capture various real-world applications. In particular, the model applies to cases where two wireless sensor networks that belong to different operators are inter-connected, e.g., see [34] for a study on real-world inter-connected wireless sensor networks.

We assume that each node receives (sends) a unit package from (to) every other node of the network. The weighted path length is given by:

$$d_{i,j} = \sum_{n,m=1}^{2N-1} \delta_{n,m}^{(i,j)} w_{n,m}, \quad (1)$$

where $\delta_{n,m}^{(i,j)} = 1$ when the link $l_{n,m}$ exists and is on the path $p_{(i,j)}$, and $\delta_{n,m}^{(i,j)} = 0$ otherwise. Here, $w_{n,m}$ stands for the *weight* of the link between nodes n and m, and will be specified shortly. The package flows from the origin node i to the destination node j along the weighted shortest path. The load of a particular node n is defined as:

$$L_n = \frac{2}{(2N-1)(2N-2)} \sum_{i,j} L_n^{(i,j)}, \quad (2)$$

where $L_n^{(i,j)}$ is the contribution of the ordered pair (i,j) to the load on node n, and $L_n^{(i,j)} = 1$ when node n is on the shortest path between the ordered pair (i,j). Note that the load L_n is also the weighted relative betweenness centrality [35,36] of node n by definition. The initial investments/cost of the network [25] is defined as:

$$I_0 = \frac{1}{2N} \sum_{n=1}^{2N} L_n, \quad (3)$$

when the investments/cost of a unit package is normalized to one. The quantity I_0 is referred to as the initial investment since it is the *minimum* average capacity (per node) required to ensure that no node is overloaded in the initial set-up and all pairs of nodes can exchange a unit packet of flow.

Under these assumptions, a heterogeneous network structure will lead to a heterogeneous load distribution [25]. In other words, a few nodes in the network will have to carry an exceptionally large load as compared to other nodes. We demonstrate the distribution of loads for the heterogeneous inter-connected networks in Figure 1, where it is seen that a few nodes carry around five times more load than the average. If some of the nodes suffer from failure or targeted attacks, the shortest flow paths will be readjusting, resulting in a global redistribution of load. As a consequence, some nodes may take on a larger load than their capacity and fail, which in turn may trigger new load redistribution and subsequent overload failures if the loads of some nodes exceed their capacities.

To quantify the robustness under these assumptions, or more specifically to understand the cost/investment required for robustness against one-node failures, we conduct the following $2N-1$ contingency analysis: begin with the inter-connected network structure introduced earlier. For a fixed node m, $1 \leq m \leq 2N$, suppose that m has failed and removed from the network, and calculate the corresponding weighted shortest paths between the remaining $(2N-1)$ nodes. Then, compute the resulting readjusted loads $L_n(m)$ of the remaining $(2N-1)$ nodes according to (2). Finally, repeat this process from scratch (i.e., start from the case when all nodes are functional) for

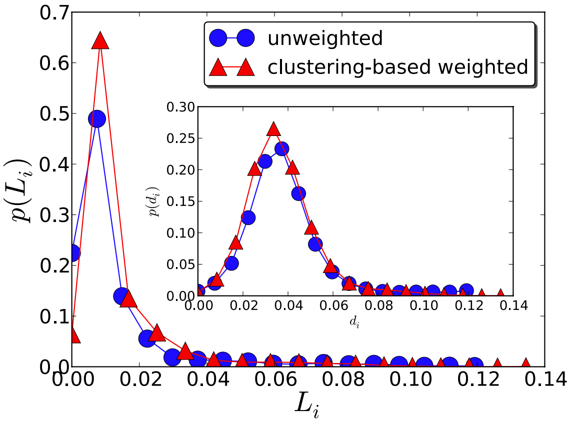


Fig. 1. Distributions of load and (inset) centrality distance following from the unweighted (circle) and the clustering-based weighted (triangle) case. We set $N = 100$, $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 4$, and generate the support degree distribution from a BA network with $\langle \tilde{k} \rangle = \langle \tilde{k}_A \rangle = \langle \tilde{k}_B \rangle = 2$, and with $\alpha = 1$, $\theta = 0.25$ for the clustering-based weighting. The distributions were calculated from 50 independent realizations of the random inter-connected scale-free networks.

each one of the $2N$ nodes in the inter-connected network. We define the *minimum* capacity of a node n as:

$$\Phi_n = \max_{0 \leq m \leq 2N} L_n(m) \quad (4)$$

in which $L_n(0)$ denotes the load from the full network with all $2N$ nodes. Here the understanding is that Φ_n is the minimum capacity required for node n to not get overloaded (hence fail) when any one of the other nodes fail. As a result, if all nodes n $(1 \leq n \leq 2N)$ are assigned a capacity of at least Φ_n, then the corresponding network would be robust against the one-node failures. The "total investment" required for robustness against one-node failure is now given by:

$$I_1 = \frac{1}{2N} \sum_{n=1}^{2N} \Phi_n. \quad (5)$$

Since the investment I_1 is associated with the cost of making the network robust, it is desirable to have smaller I_1.

As mentioned previously, we assume that weighting of the links is done based on the clustering coefficients of the nodes that are connected together. Namely, the weight of a link $i \sim j$ within a network is given by:

$$w_{ij} = (C_i C_j)^\theta, \; j \in \Gamma_i, \quad (6)$$

where Γ_i is the set of neighboring nodes of i in its own isolated sub-network and C_i is the normalized intra-clustering coefficient of node i; i.e.,

$$C_i = \exp\left(\frac{S_i}{T_i}\right), \quad (7)$$

where S_i is the total number of connections between neighbors of node i which are within the same sub-network of node i, and T_i is the total number of pairs of neighbors of node i which are within the same sub-network of node i. Since we are interested in inter-connected networks, we have to specify weights of another type of links as well, namely the inter-links. We assume that the weight of an inter-link $i \sim j$ is given by

$$w_{ij} = \alpha(\widetilde{C}_i\widetilde{C}_j)^\theta,\, j \in \widetilde{\Gamma}_i, \tag{8}$$

where $\widetilde{\Gamma}_i$ is the set of neighboring nodes of i in the other network; e.g., if node i belongs to network A, then $\widetilde{\Gamma}_i$ denotes its neighbors in network B. The inter-clustering coefficient $\widetilde{C}_i$ for node i is defined by

$$\widetilde{C}_i = \exp\left(\frac{\widetilde{S}_i}{\widetilde{T}_i}\right), \tag{9}$$

where $\widetilde{S}_i$ is the total number of connections between neighbors of node i with at least one of them in $\widetilde{\Gamma}_i$, and $\widetilde{T}_i$ is the total number of pairs of neighbors of node i with at least one of them in $\widetilde{\Gamma}_i$. The clustering coefficient is normalized so that the weight of any link is larger than zero, thus no link is virtually removed by the weighting scheme. The multiplier $\alpha \geq 1$ is introduced to capture the possible asymmetric flow dynamics on different types of links in the inter-connected networks. When $\alpha = 1$, there is no weighting bias on inter-links when calculating shortest path between node pairs; they are considered to be equivalent to intra-links. When $\alpha > 1$, inter-links are less preferable than intra-links, meaning that the inter-links are more likely to be avoided when calculating the shortest paths between node pairs. In this work, we do not consider the cases where $\alpha < 1$.

Throughout, we will present several advantages of the clustering-based weighting scheme as compared to the degree-based and random weighting schemes. But, our starting point in using the clustering-based weighting scheme is to understand the effect of using the information on "neighbors of neighbors" in the robustness of inter-connected networks. Similar considerations have led to more stable systems in the literature before; e.g., see [37] for a work on traffic flow. Furthermore, as compared to the load-based (i.e., betweenness centrality based) weighting scheme used in [25], clustering-based weights are easier to compute as they only require local information about the nodes; clustering-based weight of a link can be computed from only the neighborhood information of the two nodes that the link is connecting together, whereas load-based weight of a link requires the knowledge of the whole network topology.

Finally, we demonstrate that the clustering-based weighting scheme homogenizes the load distribution in the network. Namely, as in [25], we calculated the flow paths based on the hop counts, i.e. when links are *unweighted*, and also the clustering-based weighted path lengths given by (1) with

$$w_{n,m} = \begin{cases} (C_nC_m)^\theta & \text{if } n \sim m \text{ is an intra-link} \\ \alpha(\widetilde{C}_n\widetilde{C}_m)^\theta & \text{if } n \sim m \text{ is an inter-link.} \end{cases} \tag{10}$$

As can be seen from Figure 1, although the distributions of load and centrality distance still contain some heterogeneity, they are more homogeneous under the clustering-based weighting than that under the unweighted situation, i.e., more nodes are centering around the mean load and mean centrality distance under clustering-based scheme. Here, the centrality distance between node i and node f is defined as the sum of betweenness centrality of the nodes on the shortest path between node i and node f, i.e., the centrality distance $d_{i,f} = \sum_{n=1}^{2N} \delta_{i,f}^{(n)} L_n$, in which $\delta_{i,f}^{(n)} = 1$ if node n is on the shortest path between node i and node f, otherwise $\delta_{i,f}^{(n)} = 0$.

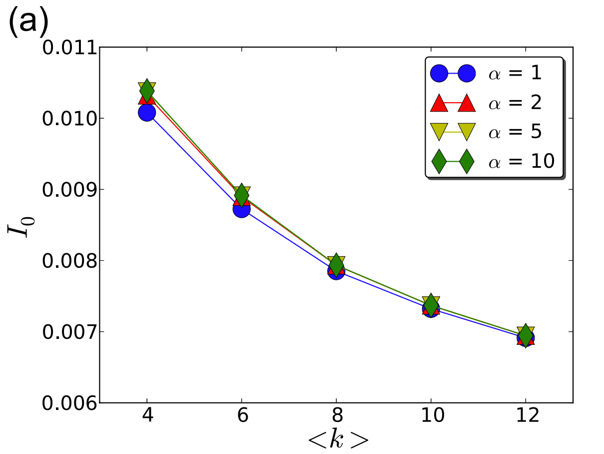
3 Numerical results for the cascading dynamics under the clustering-based weighting scheme

3.1 The role of connectivity and asymmetric flow dynamics

First, we are interested in the effect of possible asymmetric treatment of inter-links (with respect to intra-links) in the robustness of inter-connected networks under the clustering-based weighting scheme. Meanwhile, the variation of the cost/investment (for ensuring robustness against 1-node failures) with the increasing network connectivity is also of interest. Since there are two ways to increase the connectivity of inter-connected networks, i.e., either by increasing intra-connectivity or by increasing inter-connectivity, we will consider the effect of both on the cost/investment.

In our initial set of experiments, the weighting parameter θ is set to $\theta^* = 0.125$, which is optimal as will be shown in the following subsection. As shown in Figures 2a and 2b, when the intra-connectivity was increased, even though inter-connectivity is constant, both the initial cost and the total cost of the network are lower. This means that increasing intra-connectivity on inter-connected networks improves the robustness in the sense that the average node capacity required for initial stability of the network as well as for ensuring the robustness against 1-node failures is smaller. Furthermore, the initial cost and the total cost are also the lowest when $\alpha = 1$, which suggests that no weighting bias should be assigned to the inter-links when calculating shortest paths between node pairs. In other words, in the weighting scheme, inter-links shall be treated in the same manner with intra-links in order for the robustness costs to be the lowest.

In Figures 3a and 3b, we observe a similar behavior of the robustness costs as the inter-connectivity is increased. Namely, we see that when the inter-connectivity is increased, even though intra-connectivity is constant, both the initial cost and the total cost of the network are lower. This means that as inter-connectivity increases, the minimum average capacity required to ensure initial stability and robustness against one-node failures will be lower. We also observe that the initial cost and the total cost are lowest when $\alpha = 1$, indicating again that inter-links shall be



$\langle k \rangle$	$I_0(\alpha=1)$	$I_0(\alpha=2)$	$I_0(\alpha=5)$	$I_0(\alpha=10)$
4	0.0100804	0.0103205	0.0103770	0.0103839
6	0.0087239	0.0088960	0.0089135	0.0089152
8	0.0078516	0.0079319	0.0079340	0.0079341
10	0.0073208	0.0073685	0.0073686	0.0073690
12	0.0069124	0.0069433	0.0069433	0.0069433

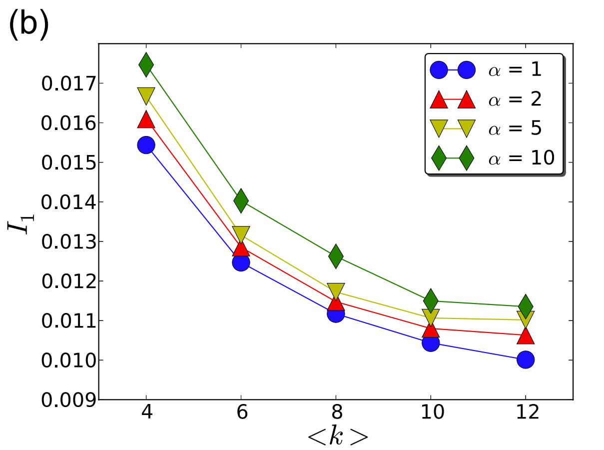
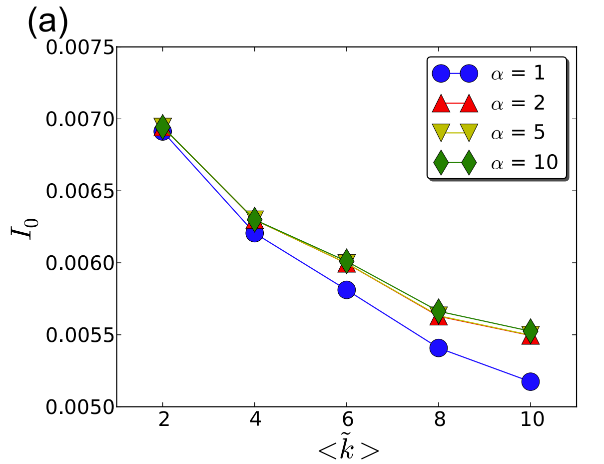


Fig. 2. Simulation results of the global load-based model on symmetrically coupled and clustering-based weighted scale-free networks. (a) The initial investments I_0 vs. $\langle k \rangle$; (b) the total investments I_1 vs. $\langle k \rangle$. The parameters for the coupled BA networks are $N = 100$. The support degree distribution was generated from a BA network with $N = 100$ and $\left\langle \widetilde{k} \right\rangle = \left\langle \widetilde{k}_A \right\rangle = \left\langle \widetilde{k}_B \right\rangle = 2$.



$\langle \tilde{k} \rangle$	$I_0(\alpha=1)$	$I_0(\alpha=2)$	$I_0(\alpha=5)$	$I_0(\alpha=10)$
2	0.0069124	0.0069433	0.0069433	0.0069433
4	0.0062060	0.0062990	0.0062990	0.0062996
6	0.0058119	0.0059962	0.0059965	0.0060106
8	0.0054084	0.0056290	0.0056323	0.0056635
10	0.0051741	0.0054953	0.0054973	0.0055248

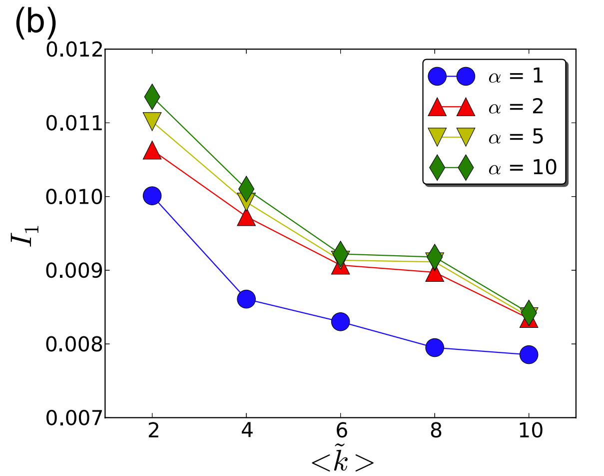


Fig. 3. Simulation results of the global load-based model on symmetrically coupled and clustering-based weighted scale-free networks. (a) The initial investments I_0 vs. $\langle \widetilde{k} \rangle$ and (b) the total investments I_1 vs. $\langle \widetilde{k} \rangle$. The parameters for the coupled BA networks are $N = 100$ and $\langle k \rangle = 12$. The support degree distribution was generated from a BA network with $N = 100$ and $\left\langle \widetilde{k} \right\rangle = \left\langle \widetilde{k}_A \right\rangle = \left\langle \widetilde{k}_B \right\rangle$.

treated equally with the intra-links in order to achieve the lowest possible costs for robustness.

3.2 Comparisons of the weighting schemes

To further see the benefits of clustering-based weighting scheme on the robustness of inter-connected networks, in the following simulation, we compare it with two other weighting schemes.

First, we consider the degree-based weighting scheme in which the weight of an intra-link $i \sim j$ is taken to be

$$w_{ij} = (k_i k_j)^{\theta},\ j \in \Gamma_i, \tag{11}$$

where k_i and k_j are the intra-degrees of node i and node j, respectively. Similarly, the weight of an inter-link $i \sim j$ is assumed to be given by:

$$w_{ij} = \alpha(\widetilde{k}_i \widetilde{k}_j)^{\theta},\ j \in \widetilde{\Gamma}_i, \tag{12}$$

where $\widetilde{k}_i$ and $\widetilde{k}_j$ are the inter-degree of node i and node j, respectively.

Second, we consider the random weighting scheme, in which we first compute the degree-based weights distribution and then reshuffle these weights randomly in the network.

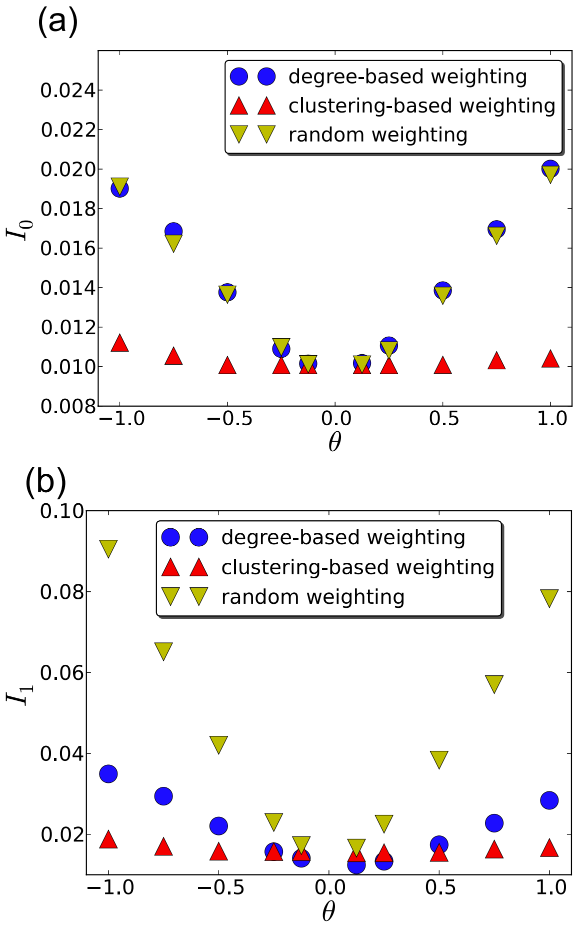


Fig. 4. Comparison of three weighting schemes. (a) The initial investments I_0 vs. weighting parameter θ and (b) the total investments I_1 vs. weighting parameter θ for random, degree-based and clustering-based weighting schemes, respectively.

We show the results of Monte-Carlo simulations for the model on symmetrically coupled scale-free networks generated by the Barabási-Albert (BA) algorithm [38] with $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 4$, and the support degree distribution was generated from a BA network with $\left\langle \widetilde{k} \right\rangle = \left\langle \widetilde{k}_A \right\rangle = \left\langle \widetilde{k}_B \right\rangle = 2$ in Figure 4. The asymmetric flow dynamics parameter is set to $\alpha = 1$.

As seen from Figures 4a and 4b, it is evident that both clustering-based and degree-based weighting schemes outperform random weighting scheme in the sense of requiring *lower* initial and/or total investments.

As shown from Figure 4a, the initial investments I_0 is lower under clustering-based weighting scheme as compared to the degree-based one, meaning that clustering-based weighting scheme leads to a smaller average capacity requirement in the initial configuration of network for ensuring that no node fails. Furthermore, from Figure 4b, it is seen that the total investments I_1 is larger under clustering-based weighting scheme than that under degree-based one only in a small range of θ values. Outside that range, the clustering-based weighting scheme is better than the degree-based one from the total cost point of view. A possible explanation for this phenomenon might be that extra protections were introduced by considering the impact of neighbors of neighbors which is embedded in the clustering coefficient. Except for a limited range where the weighting was suppressed heavily, the benefits from the consideration of neighbors of neighbors outweigh those from the pure consideration of neighbors, which is witnessed in other disciplines like traffic flow [37,39] as well. Another interesting observation is that the optimal configuration of the inter-connected networks under degree-based clustering is achieved at $\theta^* = 0.125$, not at $\theta = 0.4$ which is the optimal weighting for isolated networks as shown in [40]. The shift of the optimal weighting parameter might be attributed to the break of symmetry in inter-connected networks between intra-degree distribution and inter-degree distribution.

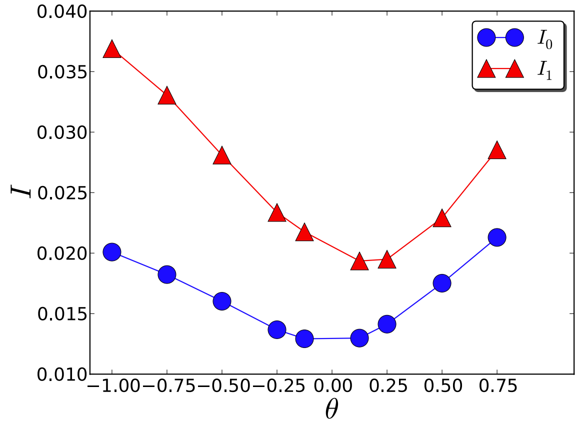


Fig. 5. Seemingly symmetric case under the degree-based weighting scheme. With $N = 100$, $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 2$, and the support degree distribution was generated from a BA network with $\left\langle \widetilde{k} \right\rangle = \left\langle \widetilde{k}_A \right\rangle = \left\langle \widetilde{k}_B \right\rangle = 2$, and $\alpha = 1$.

To clear the possible uncertainty, we investigate the case of $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 2$, and the support degree distribution was generated from a BA network with $\left\langle \widetilde{k} \right\rangle = \left\langle \widetilde{k}_A \right\rangle = \left\langle \widetilde{k}_B \right\rangle = 2$ for the degree-based weighting scheme, which is seemingly a symmetric case. As shown in Figure 5, the optimal weighting parameter θ^* was still 0.125. We also tested the case of $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 4$, and the support degree distribution was generated from a BA network with $\left\langle \widetilde{k} \right\rangle = \left\langle \widetilde{k}_A \right\rangle = \left\langle \widetilde{k}_B \right\rangle = 4$. The result is the same.

The reason for observing different optimal θ values in inter-connected networks (as compared to isolated networks) even in the seemingly symmetric case is explained as follows. Even when the inter-degree distribution is set to be equivalent to the intra-degree distribution, an inter-connected network is never equivalent to an isolated network. This is so even if one generates an isolated network whose degree distribution is equal to the *total* degree distribution of the inter-connected network; total degree distribution of an inter-connected network can be computed by taking the convolution of the inter-degree distribution and the intra-degree distribution. This is already pointed out in the literature [41] and is related to the *multiplex* nature of inter-connected networks. In particular, consider the construction of an inter-connected network via the configuration model, where each node is given a random number of intra-stubs drawn from the intra-degree distribution, and another random number of inter-stubs drawn from the inter-degree distribution. Note that intra-stubs can only be combined with other intra-stubs to form intra-links with nodes from the same sub-network, whereas inter-stubs can only be connected to other inter-stubs to form inter-links with the nodes in the other sub-network. Since no such constraint exists in an isolated network with the same total degree distribution, i.e., any stub can be combined with any other stub, the resulting network would be significantly different than the inter-connected network case; e.g., in [41] the percolation of thresholds of the two cases are shown to differ significantly. Therefore, even in the case where inter-links are treated the same with intra-links in the weighting scheme, the sole fact that inter-links are distinguished from intra-links in the network construction leads to a significant difference in network behavior which manifests from a significant shift of the optimal weighting parameter in our case.

3.3 Multiple nodes failure

Similar to [25], we want to explore the effect of clustering-based weighting on the robustness against two-node-failures. Since $2N-1$ contingency analysis does not guarantee robustness against two-or-more-nodes-induced failure, we assume that every node takes on the same extra protection [25] and has a capacity given by:

$$C_n(\gamma) = (1+\gamma)C_n(0). \tag{13}$$

The larger γ is, the larger robustness will be against a failure of two or more nodes. For two-node-induced failure, the investment cost is given by:

$$I_2 = (1+\gamma)I_0. \tag{14}$$

When I_2 is not large enough, the failure of two nodes may induce a cascading failure, leaving behind a network not necessarily connected. As shown in Figures 6 and 7, the relative size N_{gc}/N of giant component is increased with tolerance parameter γ after removal of two most-loaded nodes.

From Figure 6, it is evident that when the tolerance parameter γ is relatively small, say, $\gamma < 0.2$, i.e., when there is a hard investment budget constraint, clustering-based weighting scheme would be a better choice against a two-node-induced failures even though the degree-based weighting is shown to perform better through $2N-1$ contingency analysis with the same parameters.

From Figure 7, we can see that the best degree-based weighting through $2N-1$ contingency analysis is no longer the best one against a two-nodes-induced failure, indicating that degree-based weighting is not so stable a scheme under multiple nodes' failure. In other words, it

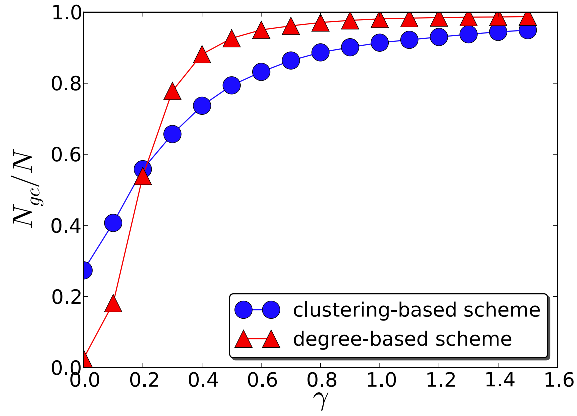


Fig. 6. Relative size of the giant component which is left behind a cascading failure after removal of the two most loaded nodes, as a function of the investment parameter γ. With $N = 100$, $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 4$, and the support degree distribution was generated from a BA network with $\langle \tilde{k} \rangle = \langle \tilde{k}_A \rangle = \langle \tilde{k}_B \rangle = 2$, and $\alpha = 1$, $\theta = 0.125$. Each curve is averaged over 50 independent realizations of random inter-connected scale-free networks.

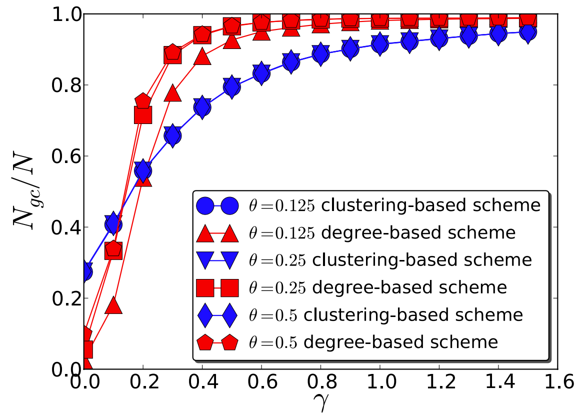


Fig. 7. Relative size of the giant component which is left behind a cascading failure after removal of the two most loaded nodes, as a function of the investment parameter γ. With $N = 100$, $\langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 4$, and the support degree distribution was generated from a BA network with $\langle \tilde{k} \rangle = \langle \tilde{k}_A \rangle = \langle \tilde{k}_B \rangle = 2$, and $\alpha = 1$.

would be hard if not impossible to optimize the degree-based weighting scheme simultaneously against the failure of one, two, and more nodes. On the other hand, the clustering-based scheme is seen to yield a relatively stable robustness performance against one-or-two-node-induced failures.

4 Conclusions

In conclusion, we have investigated the robustness of inter-connected networks under a clustering based weighting scheme through a global load redistribution based contingency analysis. In particular, our contributions include

(i) extending the clustering based weighting scheme to inter-connected networks, (ii) analyzing the cost of initial stability and robustness against one-or-two-nodes failures in inter-connected networks, and (iii) comparing the clustering based scheme with the random and degree-based weighting schemes in terms of initial and total costs. Our main findings are that, no weighting bias should be assigned to inter-links when calculating shortest path between node pairs under the clustering-based weighting scheme. Furthermore, we show that increasing connectivity reduces robustness costs in inter-connected networks, which is in contrast with local flow redistribution cases [42]. Also, we show that except for a limited range where weighting is heavily suppressed, clustering-based weighting scheme outperforms both the random and degree-based ones in terms of requiring a lower initial and total cost in the symmetrically inter-connected scale-free networks. Last but not least, we show that, when there exists a hard investment budget constraint, clustering-based weighting scheme would be a better choice for resisting against two-node-failures than the degree-based weighting, and the clustering-based scheme has a more stable robustness performance than degree-based scheme against one-or-two-node-induced failures.

All results presented so far are restricted to symmetrically coupled scale free networks. Other types of networks, like symmetrically coupled Erdõs-Rényi (ER) networks or small-world networks as well as asymmetrically coupled networks could be also studied. More interesting findings are to be expected from follow-up extensions including the consideration of the transient dynamics. In case of the existence of relaxation time for the node to overload, transient oscillations or overshooting could be expected. The role of intra- and inter- connectivity might be carefully reviewed under that circumstance.

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