On Connectivity and Robustness in Random Intersection Graphs

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***Abstract*—Random intersection graphs have been studied for nearly two decades, and have a wide range of applications ranging from key predistribution in wireless sensor networks to modeling social networks. For these graphs, each node is equipped with a set of objects in a random manner, and two nodes have an undirected edge in between if they have at least one object in common. In this paper, we investigate the strengths of connectivity and robustness in a *general random intersection graph* model. Specifically, we establish sharp asymptotic zero–one laws for k-connectivity and k-robustness, as well as the asymptotically exact probability of k-connectivity, for any positive integer k. The k-connectivity property quantifies how resilient is the connectivity of a graph against node or edge failures. On the other hand, k-robustness measures the effectiveness of *local* diffusion strategies (that do not use global graph topology information) in spreading information over the graph in the presence of misbehaving nodes. In addition to presenting the results under the general random intersection graph model, we consider two special cases of the general model, a *binomial* random intersection graph and a *uniform* random intersection graph, which both have numerous applications as well. For these two specialized graphs, our results on asymptotically exact probabilities of k-connectivity and asymptotic zero–one laws for k-robustness are also novel in the literature.**

***Index Terms*—Connectivity, consensus, random graph, random intersection graph, random key graph, robustness.**

I. Introduction

A. Background

Random intersection graphs have been introduced by Singer-Cohen [18] and different classes of these graphs received considerable attention [2]–[4], [11], [15]–[17], [22] for the past decade. In these graphs, each node is assigned a set of objects selected by some random mechanism. An undirected edge exists between any two nodes that have at least one object in common. Random intersection graphs have proved useful in modeling and analyzing real-world networks in a wide variety of applications. Examples include secure wireless sensor networks [4], [11], [22], classification analysis [12] and social networks [15]–[17]. Several properties such as clustering [3], component evolution [4] and degree distribution [12] have been analyzed for different classes of random intersection graphs.

The graph model considered in this paper, hereafter referred to as a *general random intersection graph*, represents a generalization of the random intersection graphs studied by Bloznelis *et al.* [3], [4], and is defined on a node set $\mathcal{V}_n = \{v_1, v_2, \ldots, v_n\}$ as follows. Each node v_i $(i = 1, 2, \ldots, n)$ is assigned an object set S_i from an object pool $\mathcal{P}_n$ consisting of P_n distinct objects, where P_n is a function of n. Each object set S_i is constructed using the following two-step procedure: First, the size of S_i, $|S_i|$, is determined according to some probability distribution $\mathcal{D}_n : \{1, 2, \ldots, P_n\} \to [0, 1]$. Of course, we have $\sum_{x=1}^{P_n} \mathbb{P}[|S_i| = x] = 1$, with $\mathbb{P}[A]$ denoting the probability that event A occurs. Next, S_i is formed by selecting $|S_i|$ distinct objects uniformly at random from the object pool $\mathcal{P}_n$. In other words, conditioning on $|S_i| = s_i$, set S_i is chosen uniformly among all s_i-size subsets of $\mathcal{P}_n$. This process is repeated independently for all object sets $S_1, \ldots, S_n$. Finally, an undirected edge is assigned between two nodes if and only if their corresponding object sets have at least one object in common; namely, distinct nodes v_i and v_j have an edge in between if and only if $S_i \cap S_j \neq \emptyset$. The graph defined through this adjacency notion is denoted by $G(n, P_n, \mathcal{D}_n)$.

A specific case of the general model $G(n, P_n, \mathcal{D}_n)$, known as the *binomial* random intersection graph, has been widely explored to date [9]–[14]. Under this model, each object set S_i is constructed by a Bernoulli-like mechanism; i.e., by adding each object to S_i independently with probability p_n. Like integer P_n, probability p_n is also a function of n. The term "binomial" accounts for the fact that $|S_i|$ now follows a binomial distribution with P_n as the number of trials and p_n as the success probability in each trial. We denote the binomial random intersection graph by $G_b(n, P_n, p_n)$, where subscript "b" stands for "binomial".

Another well-known special case of the general model $G(n, P_n, \mathcal{D}_n)$ is the *uniform* random intersection graph [2], [11], [15], [22]. Under the uniform model, the probability distribution $\mathcal{D}_n$ concentrates on a single integer K_n, where $1 \leq K_n \leq P_n$; i.e., for each node v_i, the object set size $|S_i|$ equals K_n with probability 1. P_n and K_n are both integer functions of n. We denote by $G_u(n, P_n, K_n)$ the uniform random intersection graph, with "u" meaning "uniform".

B. Applications of Random Intersection Graphs

A concrete example for the application of random intersection graphs can be given in the context of secure wireless sensor networks. As explained in detail in numerous other places [4], [15], [22], the uniform random intersection graph model $G_u(n, P_n, K_n)$ is induced naturally by the Eschenauer–Gligor (EG) random key predistribution scheme [11], which is a typical solution to ensure secure communications in wireless sensor networks. In particular, let the set of n nodes in graph $G_u(n, P_n, K_n)$ stand for the n sensors in the wireless network. Also, let the object pool $\mathcal{P}_n$ (with size P_n) represent the

The material in this paper was presented in part at the 2014 IEEE Conference on Decision and Control (CDC), Los Angeles, CA, USA [28].

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set of cryptographic keys available to the network and let K_n be the number of keys assigned to each sensor (selected uniformly at random from the key pool $\mathcal{P}_n$). Then, the edges in $G_u(n, P_n, K_n)$ represent pairs of sensors that share at least one cryptographic key and thus that can *securely* communicate over existing wireless links in the EG scheme. In the above application, objects that nodes have are cryptographic keys, so uniform random intersection graphs are also referred to as random key graphs [21], [22].

In the secure sensor network area, the general random intersection graph model in this paper captures the differences that may exist among the number of keys, K_n, assigned to each node. by each sensor. These differences appear for a variety of reasons including (a) K_n may vary from sensor to sensor in a heterogeneous sensor network due to differences in the sizes of sensor memories [4]; (b) K_n may decrease due to revocation of compromised nodes and keys [6]; and (c) K_n may increase due to the establishment path keys [11], where new keys are generated and distributed to participating sensors after deployment.

Random intersection graphs can also be used to model social networks, where a node represents an individual, and an object could be an hobby of individuals, a book being read, or a movie being watched, etc. [3], [17]. Then a link between two individuals characterizes a common-interest relation; e.g., two individuals have a connection if they have a common hobby, read the same book, or watched the same movie. In this setting binomial/uniform/general random intersection graphs represent common-interest networks where the sets of interests that individuals have are constructed in different ways. Specifically, in binomial random-intersection graphs, each interest is attached to each person independently with the same probability; in uniform random intersection graphs, all individuals have the same number of interests; and general random intersection graphs provide general possibilities for assigning individuals' interest sets; e.g., without probability or number-of-interest restrictions.

C. Strength of Connectivity and Robustness

We now introduce the graph properties that we are interested in. A graph is connected if there exists at least a path of edges between any two nodes [9]. A graph is said to be k-connected if each pair of nodes has at least k internally node-disjoint path(s) in between [16]; equivalently, a graph is k-connected if it cannot be disconnected by deleting at most $(k-1)$ nodes or edges. Thus, k-connectivity quantifies well-established measures of strength. For instance, it captures the resiliency of graphs against node or edge failures. It also captures the resiliency of consensus protocols in the presence of m adversarial nodes in a large-scale graph (with node size greater than $3m$); i.e., a necessary and sufficient condition is that the graph is $(2m+1)$-connected [8].

Many algorithms rely on graphs with sufficient connectivity; e.g, algorithms to achieve consensus [1], [24], [25] However, these algorithms typically assume that nodes have full knowledge of the graph topology, which is often impractical [24]. To account for lack of full topology knowledge in the general case, Zhang and Sundaram introduced the notion of *graph robustness* [24], which received much attention recently [13], [14], [23], [25], [26]. Formally, a graph with a node set $\mathcal{V}$ is k-robust if at least one of (a) and (b) below hold for any non-empty and strict subset T of $\mathcal{V}$: (a) there exists at least a node $v_a \in T$ such that v_a has no less than k neighbors inside $\mathcal{V} \setminus T$; and (b) there exists at least a node $v_b \in \mathcal{V} \setminus T$ such that v_b has no less than k neighbors inside T. Zhang and Sundaram showed that when nodes have limited (i.e., local) topology knowledge, consensus can still be reached in a sufficiently robust graph in the presence of adversarial/misbehaving nodes, but not in a sufficiently connected and insufficiently robust graph.

Graph robustness provides a different notion of strength than k-connectivity. That is, it quantifies the effectiveness and resiliency of local-information-based consensus algorithms in the presence of adversarial/misbehaving nodes. Robustness also has broad relevance in graph processes beyond consensus; e.g., robustness plays a key role in information cascades [24].

With the above notions of strength in mind, a natural questions to ask is when will random intersection graphs become k-connected or k-robust? Our paper answers this question (viz., Section II) and discusses the relationship between these notions (viz., Section VI-A).

D. Contributions and Organization

We summarize our key contributions as follows:

i) We derive sharp zero–one laws and asymptotically exact probabilities for k-connectivity in general random intersection graphs.
ii) We establish sharp zero–one laws for k-robustness in general random intersection graphs.
iii) For the two specific instances of the general graph model, a binomial random intersection graph and a uniform random intersection graph, we provide the first results on the asymptotically exact probabilities of k-connectivity and zero–one laws for k-robustness.

This paper extends previous work in this area [28] in several significant ways:

i) We strengthen the known results on binomial/uniform/general random intersection graphs. Specifically, Theorems 1–6 in this paper eliminate the condition $|\alpha_n| = o(\ln n)$ appearing in [28, Theorems 1–6].
ii) For k-connectivity of a uniform random intersection graph, we provide a complete proof in Section V. Note that this result serves as the building block for all other results.
iii) We add numerical experiments to better confirm the theoretical results; see Section VIII.
iv) We discuss the parameter conditions of the theorems in detail; see Section IX.
v) We compare our results of binomial/uniform/general random intersection graphs with those of Erdős–Rényi graphs; see the last paragraph of Section II.

The rest of the paper is organized as follows. Section II presents the main results as Theorems 1–6. Then, we introduce some auxiliary lemmas in Section III, before establishing the main results in Sections IV and VI. Section VII details the proofs of the lemmas. We provide numerical experiments in Section VIII and discuss the results in Section IX. We review related work in Section X, and conclude the paper Section XI.

II. The Results

Our main results are presented in Theorems 1–6 below. We defer the proofs of all theorems to Sections IV–VI. Throughout the paper, k is a positive integer and does not scale with n; and e is the base of the natural logarithm function, $\ln$. All limits are understood with $n \to \infty$. We use the standard Landau asymptotic notation $o(\cdot), O(\cdot), \omega(\cdot), \Omega(\cdot), \Theta(\cdot)$ and $\sim$; see [29, Page 2-Footnote 1]. In particular, for two positive sequences f_n and g_n, the relation $f_n \sim g_n$ signifies $\lim_{n\to\infty}(f_n/g_n) = 1$. For a random variable X, the terms $\mathbb{E}[X]$ and Var$[X]$ stand for its expected value and variance, respectively.

As noted in Section I-A, we denote a binomial (resp., uniform) random intersection graph by $G_b(n, P_n, p_n)$ (resp., $G_u(n, P_n, K_n)$). Clearly, $G_b(n, P_n, 0)$ (resp., $G_u(n, P_n, 0)$) is an empty graph, while $G_b(n, P_n, 1)$ (resp., $G_u(n, P_n, P_n)$) being a complete graph is k-connected for $n \geq k+1$ and is k-robust for $n \geq 2k$. Then for each $n \geq 2k$, with P_n fixed and p_n increasing from 0 to 1 (resp., K_n increasing from 0 to P_n), the probabilities of k-connectivity and k-robustness of $G_b(n, P_n, p_n)$ (resp., $G_u(n, P_n, K_n)$) increase from 0 to 1. In addition, for random graphs, results are often obtained in the asymptotic sense since the analysis becomes intractable in the finite regime [9], [10], [18]–[20].

Given the above, it is natural to anticipate that our results are also presented in the form of zero–one laws, where a zero–one law means that the probability of a graph having certain probability asymptotically converges to 0 under some conditions and to 1 under some other conditions. Moreover, it is useful to have a complete picture by obtaining the asymptotically exact probability result [19]. For binomial/uniform/general random intersection graphs, we derive asymptotically exact probabilities for k-connectivity in Theorems 1–3, and zero–one laws for k-robustness in Theorems 4–6. A future work is to establish asymptotically exact probabilities for k-robustness.

Noting that for any graph/network, k-connectivity implies that the minimum node degree is at least k [9], we often present results for the property of minimum node degree being at least k together with k-connectivity results.

A. Asymptotically Exact Probabilities for k-Connectivity and the Property of Minimum Node Degree Being at Least k

1) k-Connectivity and Minimum Node Degree in Binomial Intersection Graphs:

For a binomial random intersection graph, Theorem 1 below shows asymptotically exact probabilities for k-connectivity and the property of minimum node degree being at least k.

Theorem 1 *For a binomial random intersection graph $G_b(n, P_n, p_n)$, with a sequence α_n for all n defined through*

$$p_n{}^2 P_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{1}$$

if $P_n = \omega\big(n(\ln n)^5\big)$,

$$\lim_{n\to\infty} \mathbb{P}\big[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-connected.}\big]$$
$$= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \quad (2a)\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty, \quad (2b)\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty), \quad (2c)\end{cases}$$

and

$$\lim_{n\to\infty} \mathbb{P}\begin{bmatrix}\text{Graph } G_b(n, P_n, p_n) \text{ has a}\\ \text{minimum node degree at least } k.\end{bmatrix}$$
$$= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \quad (3a)\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty, \quad (3b)\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty). \quad (3c)\end{cases}$$

2) k-Connectivity and Minimum Node Degree in Uniform Intersection Graphs:

For a uniform random intersection graph, Theorem 2 below gives asymptotically exact probabilities for k-connectivity and the property of minimum node degree being at least k.

Theorem 2 *For a uniform random intersection graph $G_u(n, P_n, K_n)$, with a sequence α_n for all n defined through*

$$\frac{K_n{}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{4}$$

if $K_n = \Omega\big(\sqrt{\ln n}\big)$, then

$$\lim_{n\to\infty} \mathbb{P}\big[\text{Graph } G_u(n, P_n, K_n) \text{ is } k\text{-connected.}\big]$$
$$= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \quad (5a)\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty, \quad (5b)\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty), \quad (5c)\end{cases}$$

and

$$\lim_{n\to\infty} \mathbb{P}\begin{bmatrix}\text{Graph } G_u(n, P_n, K_n) \text{ has a}\\ \text{minimum node degree at least } k.\end{bmatrix}$$
$$= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \quad (6a)\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty, \quad (6b)\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty). \quad (6c)\end{cases}$$

3) k-Connectivity and Minimum Node Degree in General Random Intersection Graphs:

For a general random intersection graph, Theorem 3 below provides asymptotically exact probabilities for k-connectivity and the property of minimum node degree being at least k.

Theorem 3 *Consider a general random intersection graph $G(n, P_n, \mathcal{D}_n)$. Let X_n be a random variable following probability distribution $\mathcal{D}$. With a sequence α_n for all n defined through*

$$\frac{\big\{\mathbb{E}[X_n]\big\}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{7}$$

if $\mathbb{E}[X_n] = \Omega\big(\sqrt{\ln n}\big)$ and Var$[X_n] = o\Big(\frac{\{\mathbb{E}[X_n]\}^2}{n(\ln n)^2}\Big)$, then

$$\lim_{n\to\infty} \mathbb{P}\big[\text{Graph } G(n, P_n, \mathcal{D}_n) \text{ is } k\text{-connected.}\big]$$
$$= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \quad (8a)\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty, \quad (8b)\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty), \quad (8c)\end{cases}$$

and

$$\lim_{n\to\infty} \mathbb{P}\begin{bmatrix}\text{Graph } G(n, P_n, \mathcal{D}_n) \text{ has a}\\ \text{minimum node degree at least } k.\end{bmatrix}$$
$$= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \quad (9a)\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty, \quad (9b)\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty). \quad (9c)\end{cases}$$

B. Asymptotic Zero–One Laws for k-Robustness

1) k-Robustness in Binomial Random Intersection Graphs: Theorem 4 below gives an asymptotic zero–one law for k-robustness in a binomial random intersection graph.

Theorem 4 *For a binomial random intersection graph $G_b(n, P_n, p_n)$, with a sequence α_n for all n defined through*

$$p_n{}^2 P_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{10}$$

if $P_n = \Omega\big(n(\ln n)^5\big)$, then

$$\lim_{n\to\infty} \mathbb{P}\big[\,\textit{Graph } G_b(n, P_n, p_n) \textit{ is } k\textit{-robust.}\,\big]$$
$$= \begin{cases} 0, & \textit{if } \lim_{n\to\infty} \alpha_n = -\infty, & (11a)\\ 1, & \textit{if } \lim_{n\to\infty} \alpha_n = \infty. & (11b)\end{cases}$$

2) k-Robustness in Uniform Random Intersection Graphs: Theorem 5 below presents an asymptotic zero–one law for k-robustness in a uniform random intersection graph.

Theorem 5 *For a uniform random intersection graph $G_u(n, P_n, K_n)$, with a sequence α_n for all n defined through*

$$\frac{K_n{}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{12}$$

if $K_n = \Omega\big((\ln n)^3\big)$, then

$$\lim_{n\to\infty} \mathbb{P}\big[\,\textit{Graph } G_u(n, P_n, K_n) \textit{ is } k\textit{-robust.}\,\big]$$
$$= \begin{cases} 0, & \textit{if } \lim_{n\to\infty} \alpha_n = -\infty, & (13a)\\ 1, & \textit{if } \lim_{n\to\infty} \alpha_n = \infty. & (13b)\end{cases}$$

3) k-Robustness in General Random Intersection Graphs: Theorem 6 as follows provides an asymptotic zero–one law for k-robustness in a general random intersection graph.

Theorem 6 *Consider a general random intersection graph $G(n, P_n, \mathcal{D}_n)$. Let X_n be a random variable following probability distribution $\mathcal{D}_n$. With a sequence α_n for all n defined through*

$$\frac{\big\{\mathbb{E}[X_n]\big\}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{14}$$

if $\mathbb{E}[X_n] = \Omega\big((\ln n)^3\big)$ and $Var[X_n] = o\Big(\frac{\{\mathbb{E}[X_n]\}^2}{n(\ln n)^2}\Big)$, then

$$\lim_{n\to\infty} \mathbb{P}\big[\,\textit{Graph } G(n, P_n, \mathcal{D}_n) \textit{ is } k\textit{-robust.}\,\big]$$
$$= \begin{cases} 0, & \textit{if } \lim_{n\to\infty} \alpha_n = -\infty,\\ 1, & \textit{if } \lim_{n\to\infty} \alpha_n = \infty.\end{cases}$$

In view of Theorems 1–6, for each binomial/uniform/general random intersection graph, its k-connectivity and k-robustness asymptotically obey the same zero–one laws. Moreover, these zero–one laws are all *sharp* since $|\alpha_n|$ can be much smaller compared to $\ln n$; e.g., even $\alpha_n = \pm \cdot \ln\ln\cdots\ln n$ satisfies $\lim_{n\to\infty} \alpha_n = \pm\infty$.

We compare our results of random intersection graphs with those of Erdős–Rényi graphs below. From [7, Section 1.1], $p_n{}^2 P_n$ in the scaling conditions (1) and (10) of Theorems 1 and 4 is an asymptotics of the edge probability in a binomial random intersection graph $G_b(n, P_n, p_n)$. Also, by [3, Lemma 1], $\frac{K_n{}^2}{P_n}$ in the scaling conditions (4) and (12) of Theorems 2 and 5 (resp., $\frac{\{\mathbb{E}[X_n]\}^2}{P_n}$ in the scaling conditions (7) and (14) of Theorems 3 and 6 is an asymptotics of the edge probability in a general random intersection graph $G(n, P_n, \mathcal{D}_n)$ (resp., a uniform random intersection graph $G_u(n, P_n, K_n)$). Then comparing Theorems 1–3 with Lemma 1, and comparing Theorems 4–6 with Lemma 2, we conclude binomial/uniform/general random intersection graphs under certain parameter conditions[1] exhibit the same behavior with Erdős-Rényi graphs in the sense that for each of (i) k-connectivity, (ii) the property of minimum vertex degree being at least k, and (iii) k-robustness, a common point for the transition from a zero-law to a one-law occurs when the edge probability equals $\frac{\ln n+(k-1)\ln\ln n}{n}$.

III. Auxiliary Lemmas

We present lemmas that are used in proving the theorems.

A. Results of Erdős-Rényi Graphs

Lemma 1 below by Erdős and Rényi [10] investigates k-connectivity and minimum node degree in Erdős-Rényi graphs. An Erdős–Rényi graph $G(n, \hat{p}_n)$ [9] is defined on a set of n nodes such that any two nodes have an edge in between independently with probability $\hat{p}_n$.

Lemma 1 (Erdős and Rényi [10]) *For an Erdős–Rényi graph $G(n, \hat{p}_n)$, with a sequence α_n for all n through*

$$\hat{p}_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n},$$

then it holds that

$$\lim_{n\to\infty} \mathbb{P}\big[G(n, \hat{p}_n) \textit{ is } k\textit{-connected.}\big]$$
$$= \lim_{n\to\infty} \mathbb{P}\big[G(n, \hat{p}_n) \textit{ has a minimum node degree at least } k.\big]$$
$$= \begin{cases} 0, & \textit{if } \lim_{n\to\infty} \alpha_n = -\infty,\\ 1, & \textit{if } \lim_{n\to\infty} \alpha_n = \infty.\end{cases}$$

Lemma 1 is used in the comparison of random intersection graphs and Erdős–Rényi graphs.

Lemma 2 below presents the result on k-robustness in Erdős-Rényi graphs.

Lemma 2 *For an Erdős–Rényi graph $G(n, \hat{p}_n)$, with a sequence α_n for all n through*

$$\hat{p}_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}, \tag{15}$$

then it holds that

$$\lim_{n\to\infty} \mathbb{P}\big[G(n, \hat{p}_n) \textit{ is } k\textit{-robust.}\big] = \begin{cases} 0, & \textit{if } \lim_{n\to\infty} \alpha_n = -\infty,\\ 1, & \textit{if } \lim_{n\to\infty} \alpha_n = \infty.\end{cases} \tag{16}$$

Lemma 2 is applied to Section VI-B for proving Theorem 4. Lemma 2 is also used in the comparison of random intersection graphs and Erdős–Rényi graphs.

To prove Lemma 2, we note the following three facts. (a) The desired result (16) with $|\alpha_n| = o(\ln\ln n)$ is demonstrated in [24, Theorem 3]. (b) By [16, Facts 3 and 7], for any monotone increasing graph property $\mathcal{I}$, the probability that graph $G(n, \hat{p}_n)$ has property $\mathcal{I}$ is non-decreasing as $\hat{p}_n$ increases, where a graph property is called monotone increasing if it holds under the addition of edges. (c) k-robustness is a monotone increasing graph property according to [13, Lemma 3]. In view of (a) (b) and (c) above, we obtain Lemma 2.

[1] Under other parameter conditions, the conclusion may not hold as in the case of binomial random intersection graphs shown by Rybarczyk [16], [17].

B. Lemmas for Graph Coupling

We present several lemmas for graph coupling below. Formally, a coupling [16], [17], [28] of two random graphs G_1 and G_2 means a probability space on which random graphs G_1' and G_2' are defined such that G_1' and G_2' have the same distributions as G_1 and G_2, respectively. If G_1' is a spanning subgraph (resp., spanning supergraph) G_2', we say that under the graph coupling, G_1 is a spanning subgraph (resp., spanning supergraph) G_2.

Following Rybarczyk's notation [16], we write

$$G_1 \preceq G_2 \quad (\text{resp.}, G_1 \preceq_{1-o(1)} G_2) \tag{17}$$

if there exists a coupling under which G_1 is a *spanning subgraph* of G_2 with probability 1 (resp., $1-o(1)$), where a spanning subgraph is a subgraph that has the same node set with the original graph. We write

$$G_2 \succeq G_1 \quad (\text{resp.}, G_2 \succeq_{1-o(1)} G_1) \tag{18}$$

if there exists a coupling under which G_2 is a *spanning supergraph* of G_1 with probability 1 (resp., $1-o(1)$), where a spanning supergraph is a supergraph that has the same node set with the original graph. According to the definitions above, $G_1 \preceq G_2$ and $G_2 \succeq G_1$ are equivalent, while $G_1 \preceq_{1-o(1)} G_2$ and $G_2 \succeq_{1-o(1)} G_1$ are equivalent.

In view that k-connectivity and k-robustness are monotone increasing graph properties, it is natural to obtain that under $G_1 \preceq G_2$ or $G_1 \preceq_{1-o(1)} G_2$, if G_1 is k-connected (resp. k-robust) with high probability, then G_2 is also k-connected (resp. k-robust) with high probability. This result is formally presented in Lemma 3 below given by Rybarczyk [16]. Lemma 3 considers any monotone increasing graph property for generality, where $\mathbb{P}[\cdot]$ means probability.

Lemma 3 (Rybarczyk [16]) *For two random graphs G_1 and G_2, the following results hold for any monotone increasing graph property $\mathcal{I}$.*

- *If $G_1 \preceq G_2$, then $\mathbb{P}\big[G_2 \text{ has } \mathcal{I}.\big] \geq \mathbb{P}\big[G_1 \text{ has } \mathcal{I}.\big]$.*
- *If $G_1 \preceq_{1-o(1)} G_2$, then*
$$\mathbb{P}\big[G_2 \text{ has } \mathcal{I}.\big] \geq \mathbb{P}\big[G_1 \text{ has } \mathcal{I}.\big] - o(1).$$

Lemma 3 is used in many places of this paper: Sections III-B1 and III-B2 next, and Remarks 1–4 after Lemmas 6–9.

1) A lemma to confine $|\alpha_n|$ in Theorems 1 and 4 as $o(\ln n)$: We present Lemma 4 below to confine $|\alpha_n|$ in Theorems 1 and 4 as $o(\ln n)$; i.e., if Theorems 1 and 4 hold under an extra condition $|\alpha_n| = o(\ln n)$, then they also hold regardless of this condition.

Lemma 4 ***(a)*** *For graph $G_b(n, P_n, p_n)$ under*

$$ {p_n}^2 P_n = \frac{\ln n + (k-1)\ln\ln n + \beta_n}{n} \tag{19}$$

with $\lim_{n\to\infty} \beta_n = -\infty$, there exists graph $G_b(n, \widetilde{P_n}, \widetilde{p_n})$ under

$$ \widetilde{p_n}^2 \widetilde{P_n} = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\beta_n}}{n} \tag{20}$$

with $\lim_{n\to\infty} \widetilde{\beta_n} = -\infty$ and $\widetilde{\beta_n} = -o(\ln n)$ such that $G_b(n, P_n, p_n) \preceq G_b(n, \widetilde{P_n}, \widetilde{p_n})$.

(b) *For graph $G_b(n, P_n, p_n)$ under*

$$ {p_n}^2 P_n = \frac{\ln n + (k-1)\ln\ln n + \beta_n}{n} \tag{21}$$

with $\lim_{n\to\infty} \beta_n = \infty$, there exists graph $G_b(n, \widehat{P_n}, \widehat{p_n})$ under

$$ \widehat{p_n}^2 \widehat{P_n} = \frac{\ln n + (k-1)\ln\ln n + \widehat{\beta_n}}{n} \tag{22}$$

with $\lim_{n\to\infty} \widehat{\beta_n} = \infty$ and $\widehat{\beta_n} = o(\ln n)$ such that $G_b(n, \widehat{P_n}, \widehat{p_n}) \preceq G_b(n, P_n, p_n)$.

The proof of Lemma 4 is given in Section VII-A.

We now explain that given Lemma 4, if Theorems 1 and Theorem 4 hold under the extra condition $|\alpha_n| = o(\ln n)$, then they also hold regardless of the extra condition. Note that result (2c) (3c) both have a condition $\lim_{n\to\infty} \alpha_n = \alpha^* \in (-\infty, \infty)$, which clearly implies $|\alpha_n| = o(\ln n)$. Hence, we only need to look at results (2a) (2b) (3a) (3b) (11a) (11b). In particular, we will show that

if (2a) (3a) and (11a) hold under condition $\alpha_n = -o(\ln n)$, then they also hold regardless of the condition. (23)

and

if (2b) (3b) and (11b) hold under condition $\alpha_n = o(\ln n)$, then they also hold regardless of the condition. (24)

To see (23), we use Lemma 4-Property (a) and Lemma 3, and note that k-connectivity, the property of minimum node degree being at least k, and k-robustness are all monotone increasing graph properties. Then for graph $G_b(n, P_n, p_n)$ under (19) with $\lim_{n\to\infty} \beta_n = -\infty$, there exists graph $G_b(n, \widetilde{P_n}, \widetilde{p_n})$ under (20) with $\lim_{n\to\infty} \widetilde{\beta_n} = -\infty$ and $\widetilde{\beta_n} = -o(\ln n)$ such that

$$\begin{aligned}&\mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-connected.}\right] \\ &\leq \mathbb{P}\left[G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ is } k\text{-connected.}\right],\end{aligned} \tag{25}$$

$$\begin{aligned}&\mathbb{P}\begin{bmatrix}\text{Graph } G_b(n, P_n, p_n) \text{ has a} \\ \text{minimum node degree at least } k.\end{bmatrix} \\ &\leq \mathbb{P}\begin{bmatrix}\text{Graph } G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ has a} \\ \text{minimum node degree at least } k.\end{bmatrix},\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-robust.}\right] \\ &\leq \mathbb{P}\left[G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ is } k\text{-robust.}\right]\end{aligned} \tag{27}$$

If (2a) and (11a) hold under condition $\alpha_n = -o(\ln n)$, then we use them on graph $G_b(n, \widetilde{P_n}, \widetilde{p_n})$ to obtain

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ is } k\text{-connected.}\right] = 0, \tag{28}$$

$$\lim_{n\to\infty} \mathbb{P}\begin{bmatrix}\text{Graph } G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ has a} \\ \text{minimum node degree at least } k.\end{bmatrix} = 0, \tag{29}$$

and

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ is } k\text{-robust.}\right] = 0. \tag{30}$$

Therefore, (25) and (28) yield

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-connected.}\right] = 0.$$

(26) and (29) lead to

$$\lim_{n\to\infty} \mathbb{P}\begin{bmatrix}\text{Graph } G_b(n, P_n, p_n) \text{ has a} \\ \text{minimum node degree at least } k.\end{bmatrix} = 0.$$

(27) and (30) result in

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-robust.}\right] = 0.$$

In other words, for graph $G_b(n, P_n, p_n)$, with α_n in Theorems 1 and 4 replaced by β_n, (2a) (3a) and (11a) hold. Note that β_n

might or might not have the condition $\beta_n = -o(\ln n)$. Hence, the argument above proves (23).

To see (24), we use Lemma 4-Property (b) and Lemma 3, and note that k-connectivity, the property of minimum node degree being at least k, and k-robustness are all monotone increasing graph properties. Then for graph $G_b(n, P_n, p_n)$ under (21) with $\lim_{n\to\infty} \beta_n = \infty$, there exists graph $G_b(n, \widehat{P_n}, \widehat{p_n})$ under (22) with $\lim_{n\to\infty} \widehat{\beta_n} = -\infty$ and $\widehat{\beta_n} = -o(\ln n)$ such that

$$\begin{aligned}&\mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-connected.}\right] \\ &\quad \geq \mathbb{P}\left[G_b(n, \widehat{P_n}, \widehat{p_n}) \text{ is } k\text{-connected.}\right], \end{aligned} \tag{31}$$

$$\begin{aligned}&\mathbb{P}\left[\begin{array}{l}\text{Graph } G_b(n, P_n, p_n) \text{ has a} \\ \text{minimum node degree at least } k.\end{array}\right] \\ &\quad \geq \mathbb{P}\left[\begin{array}{l}\text{Graph } G_b(n, \widetilde{P_n}, \widetilde{p_n}) \text{ has a} \\ \text{minimum node degree at least } k.\end{array}\right], \end{aligned} \tag{32}$$

and

$$\begin{aligned}&\mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-robust.}\right] \\ &\quad \geq \mathbb{P}\left[G_b(n, \widehat{P_n}, \widehat{p_n}) \text{ is } k\text{-robust.}\right] \end{aligned} \tag{33}$$

If (2b) (3b) and (11b) hold under condition $\alpha_n = o(\ln n)$, then we use them on graph $G_b(n, \widehat{P_n}, \widehat{p_n})$ to obtain

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, \widehat{P_n}, \widehat{p_n}) \text{ is } k\text{-connected.}\right] = 1, \tag{34}$$

$$\lim_{n\to\infty} \mathbb{P}\left[\begin{array}{l}\text{Graph } G_b(n, \widehat{P_n}, \widehat{p_n}) \text{ has a} \\ \text{minimum node degree at least } k.\end{array}\right] = 1, \tag{35}$$

and

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, \widehat{P_n}, \widehat{p_n}) \text{ is } k\text{-robust.}\right] = 1. \tag{36}$$

Therefore, (31) and (34) give rise to

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-connected.}\right] = 1.$$

(32) and (35) yield

$$\lim_{n\to\infty} \mathbb{P}\left[\begin{array}{l}\text{Graph } G_b(n, P_n, p_n) \text{ has a} \\ \text{minimum node degree at least } k.\end{array}\right] = 1.$$

(33) and (36) induce

$$\lim_{n\to\infty} \mathbb{P}\left[G_b(n, P_n, p_n) \text{ is } k\text{-robust.}\right] = 1.$$

In other words, for graph $G_b(n, P_n, p_n)$, with α_n in Theorems 1 and 4 replaced by β_n, (2b) (3b) and (11b) hold. Note that β_n might or might not have the condition $\beta_n = o(\ln n)$. Hence, the argument above proves (24).

2) A lemma to confine $|\alpha_n|$ in Theorems 2 and 5 as $o(\ln n)$: We present Lemma 5 below to confine $|\alpha_n|$ in Theorems 2 and 5 as $o(\ln n)$; i.e., if Theorems 2 and 5 hold under an extra condition $|\alpha_n| = o(\ln n)$, then they also hold regardless of this condition.

Lemma 5 ***(a)*** *For graph $G_u(n, P_n, K_n)$ under $P_n = \Omega(n)$ and*

$$\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \beta_n}{n} \tag{37}$$

with $\lim_{n\to\infty} \beta_n = -\infty$, there exists graph $G_u(n, \widetilde{P_n}, \widetilde{K_n})$ under $\widetilde{P_n} = \Omega(n)$ and

$$\frac{\widetilde{K_n}^2}{\widetilde{P_n}} = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\beta_n}}{n} \tag{38}$$

with $\lim_{n\to\infty} \widetilde{\beta_n} = -\infty$ and $\widetilde{\beta_n} = -o(\ln n)$, such that $G_u(n, P_n, K_n) \preceq G_u(n, \widetilde{P_n}, \widetilde{K_n})$.

(b) *For graph $G_u(n, P_n, K_n)$ under $P_n = \Omega(n)$ and*

$$\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \beta_n}{n} \tag{39}$$

with $\lim_{n\to\infty} \beta_n = \infty$, there exists graph $G_u(n, \widehat{P_n}, \widehat{K_n})$ under $\widehat{P_n} = \Omega(n)$ and

$$\frac{\widehat{K_n}^2}{\widehat{P_n}} = \frac{\ln n + (k-1)\ln\ln n + \widehat{\beta_n}}{n} \tag{40}$$

with $\lim_{n\to\infty} \widehat{\beta_n} = \infty$ and $\widehat{\beta_n} = o(\ln n)$, such that $G_u(n, \widehat{P_n}, \widehat{K_n}) \preceq G_u(n, P_n, K_n)$.

The proof of Lemma 5 is given in Section VII-B.

We now explain that given Lemma 5, if Theorems 2 and Theorem 5 hold under the extra condition $|\alpha_n| = o(\ln n)$, then they also hold regardless of the extra condition. Note that result (5c) (6c) both have a condition $\lim_{n\to\infty} \alpha_n = \alpha^* \in (-\infty, \infty)$, which clearly implies $|\alpha_n| = o(\ln n)$. Hence, we only need to look at results (5a) (5b) (6a) (6b) (13a) (13b). In particular, we need to show that

$$\begin{array}{l}\text{if (5a) (6a) and (13a) hold under condition } \alpha_n = -o(\ln n), \\ \text{then they also hold regardless of the condition.}\end{array} \tag{41}$$

and

$$\begin{array}{l}\text{if (5b) (6b) and (13b) hold under condition } \alpha_n = o(\ln n), \\ \text{then they also hold regardless of the condition.}\end{array} \tag{42}$$

The process of proving (41) and (42) using Lemma 5 is the same as the above process of proving (23) and (24) using Lemma 4. For brevity, we do not repeat the details here.

We present Lemmas 6–9 below. Except Lemma 8 which is from [4, Lemma 4], the proofs of other lemmas are deferred to Section VII.

3) Coupling between general random intersection graphs and uniform random intersection graphs:

Lemma 6 *Let X_n be a random variable with probability distribution $\mathcal{D}_n$. If $Var[X_n] = o\Big(\frac{\{\mathbb{E}[X_n]\}^2}{n(\ln n)^2}\Big)$, then there exists $\epsilon_n = o\big(\frac{1}{\ln n}\big)$ such that*

$$\begin{aligned}&G_u\big(n, P_n, (1-\epsilon_n)\mathbb{E}[X_n]\big) \preceq_{1-o(1)} G(n, P_n, \mathcal{D}_n) \\ &\qquad \preceq_{1-o(1)} G_u\big(n, P_n, (1+\epsilon_n)\mathbb{E}[X_n]\big). \end{aligned} \tag{43}$$

The proof of Lemma 6 is given in Section VII-C. With Lemma 3, Lemma 6 yields Remark 1 below, which is used in Theorems 3 and 6.

Remark 1 *From Lemmas 3 and 6, for any monotone increasing graph property $\mathcal{I}$, we have*

$$\begin{aligned}&\mathbb{P}\big[\text{Graph } G_u(n, P_n, (1-\epsilon_n)\mathbb{E}[X_n]) \text{ has } \mathcal{I}.\big] - o(1) \\ &\leq \mathbb{P}\big[\text{Graph } G(n, P_n, \mathcal{D}_n) \text{ has } \mathcal{I}.\big] \\ &\leq \mathbb{P}\big[\text{Graph } G_u(n, P_n, (1+\epsilon_n)\mathbb{E}[X_n]) \text{ has } \mathcal{I}.\big] + o(1).\end{aligned}$$

4) Coupling between binomial random intersection graphs and Erdős–Rényi graphs:

Lemma 7 *If $p_n = O\left(\frac{1}{n\ln n}\right)$ and ${p_n}^2 P_n = O\left(\frac{1}{\ln n}\right)$, then there exists $\hat{p}_n = {p_n}^2 P_n \cdot \left[1 - O\left(\frac{1}{\ln n}\right)\right]$ such that*

$$G(n, \hat{p}_n) \succeq_{1-o(1)} G_b(n, P_n, p_n). \tag{44}$$

The proof of Lemma 7 is given in Section VII-D. With Lemma 3, Lemma 7 induces Remark 2 below, which is used in Theorem 4.

Remark 2 *From Lemmas 3 and 7, for any monotone increasing graph property $\mathcal{I}$, we have*

$$\begin{aligned}&\mathbb{P}[\textit{ Graph } G_b(n, P_n, p_n) \textit{ has } \mathcal{I}.]\\&\quad\geq \mathbb{P}[\textit{ Graph } G(n, \hat{p}_n) \textit{ has } \mathcal{I}.] - o(1).\end{aligned}$$

5) Coupling between binomial random intersection graphs and uniform random intersection graphs:

Lemma 8 ([4, Lemma 4]) *If $p_n P_n = \omega(\ln n)$, and for all n sufficiently large,*

$$K_{n,-} \leq p_n P_n - \sqrt{3(p_n P_n + \ln n)\ln n},$$
$$K_{n,+} \geq p_n P_n + \sqrt{3(p_n P_n + \ln n)\ln n},$$

then

$$\begin{aligned}G_u(n, P_n, K_{n,-}) &\preceq_{1-o(1)} G_b(n, P_n, p_n)\\&\preceq_{1-o(1)} G_u(n, P_n, K_{n,+}).\end{aligned}$$

With Lemma 3, Lemma 8 gives rise to Remark 3 below, which is used in Theorem 1.

Remark 3 *From Lemmas 3 and 8, for any monotone increasing graph property $\mathcal{I}$, we have*

$$\begin{aligned}&\mathbb{P}[\textit{ Graph } G_u(n, P_n, K_{n,-}) \textit{ has } \mathcal{I}.] - o(1)\\&\quad\leq \mathbb{P}[\textit{ Graph } G_b(n, P_n, p_n) \textit{ has } \mathcal{I}.]\\&\quad\leq \mathbb{P}[\textit{ Graph } G_u(n, P_n, K_{n,+}) \textit{ has } \mathcal{I}.] + o(1).\end{aligned}$$

Lemma 9 *If $K_n = \omega(\ln n)$ and $p_n = \frac{K_n}{P_n}\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right)$, then*

$$G_u(n, P_n, K_n) \succeq_{1-o(1)} G_b(n, P_n, p_n).$$

The proof of Lemma 9 is given in Section VII-E. With Lemma 3, Lemma 9 yields Remark 4 below, which is used in Theorem 5.

Remark 4 *From Lemmas 3 and 9, for any monotone increasing graph property $\mathcal{I}$, we have*

$$\begin{aligned}&\mathbb{P}[\textit{ Graph } G_u(n, P_n, K_n) \textit{ has } \mathcal{I}.]\\&\quad\geq \mathbb{P}[\textit{ Graph } G_b(n, P_n, p_n) \textit{ has } \mathcal{I}.] - o(1).\end{aligned}$$

IV. Establishing Theorems 1 and 3

Theorems 1–3 describe results on k-connectivity for binomial/uniform/general random intersection graphs. We prove Theorems 1 and 3 in this section, and present the proof of Theorem 2 as a separate section next due to the length of the proof.

We briefly explain the idea of proving Theorems 1 and 3 from Theorem 2 below. First, we demonstrate Theorem 1 from Theorem 2 using the coupling between binomial random intersection graphs and uniform random intersection graphs given by Lemma 8 of Section III-B5. Second, we establish Theorem 3 from Theorem 2 using the coupling between general random intersection graphs and uniform random intersection graphs given by Lemma 6 of Section III-B3.

A. The Proof of Theorem 1

As explained in Section III-B1, we can introduce an extra condition $|\alpha_n| = o(\ln n)$ in proving Theorem 1. Then from Remark 3 after Lemma 8, Theorem 2, and and the fact that both k-connectivity and the property of minimum node degree being at least k are monotone increasing graph properties, the proof of Theorem 1 is completed once we show that with $K_{n,\pm}$ given by

$$K_{n,\pm} = p_n P_n \pm \sqrt{3(p_n P_n + \ln n)\ln n}, \tag{45}$$

under conditions of Theorem 1 and $|\alpha_n| = o(\ln n)$, we have $K_{n,\pm} = \Omega\big(\sqrt{\ln n}\,\big)$ and with $\alpha_{n,\pm}$ defined by

$$\frac{{K_{n,\pm}}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_{n,\pm}}{n}, \tag{46}$$

then

$$\alpha_{n,\pm} = \alpha_n \pm o(1). \tag{47}$$

From conditions (1) and $|\alpha_n| = o(\ln n)$, it is clear that

$${p_n}^2 P_n \sim \frac{\ln n}{n}. \tag{48}$$

Substituting (48) and condition $P_n = \omega\big(n(\ln n)^5\big)$ into (45), we obtain

$$p_n P_n = \sqrt{{p_n}^2 P_n \cdot P_n} = \omega\left(\frac{\ln n}{n} \cdot n(\ln n)^5\right) = \omega\big((\ln n)^3\big),$$
$$K_{n,\pm} = \omega\big((\ln n)^3\big) = \Omega\big(\sqrt{\ln n}\,\big), \tag{49}$$

and

$$\begin{aligned}\frac{{K_{n,\pm}}^2}{P_n} &= {p_n}^2 P_n \cdot \left[1 \pm \sqrt{3\left(1 + \frac{\ln n}{p_n P_n}\right)\frac{\ln n}{p_n P_n}}\right]\\&= {p_n}^2 P_n \cdot \left[1 \pm o\left(\frac{1}{\ln n}\right)\right].\end{aligned} \tag{50}$$

Then from (1) (46) and (50), we obtain (47). As explained before, with (46) (47) and (49), Theorem 1 is proved from Lemma 8 and Theorem 2.

B. The Proof of Theorem 3

Given Remark 1 after Lemma 6 and the fact that both k-connectivity and the property of minimum node degree being at least k are monotone increasing graph properties, we will show Theorem 3 once proving for any $\epsilon_n = o\left(\frac{1}{\ln n}\right)$ that

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\big[G_u(n, P_n, (1 \pm \epsilon_n)\mathbb{E}[X_n]) \text{ is } k\text{-connected.}\big]\\&= \begin{cases}0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty,\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty,\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty),\end{cases}\end{aligned} \tag{51}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\left[\begin{array}{l}\text{Graph } G_u(n, P_n, (1 \pm \epsilon_n)\mathbb{E}[X_n]) \text{ has a}\\ \text{minimum node degree at least } k.\end{array}\right]\\&= \begin{cases}0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty,\\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty,\\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty).\end{cases}\end{aligned} \tag{52}$$

Under $\mathbb{E}[X_n] = \Omega\big(\sqrt{\ln n}\,\big)$ and $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, it follows that $(1 \pm \epsilon_n)\mathbb{E}[X_n] = \Omega\big(\sqrt{\ln n}\,\big)$. From Theorem 2, we will have (51) and (52) once we prove that sequences γ_n^+ and γ_n^- defined through

$$\frac{\big\{(1 \pm \epsilon_n)\mathbb{E}[X_n]\big\}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \gamma_n^{\pm}}{n} \tag{53}$$

satisfy

$$\lim_{n\to\infty}\gamma_n^{\pm} = \begin{cases}-\infty, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty,\\ \infty, & \text{if } \lim_{n\to\infty}\alpha_n = \infty,\\ \alpha^*, & \text{if } \lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty).\end{cases} \tag{54}$$

Now we establish (54). From (7) (53) and $\epsilon_n = o\big(\frac{1}{\ln n}\big)$, it follows that

$$\begin{aligned}\gamma_n^{\pm} &= n \cdot \frac{\big\{(1 \pm \epsilon_n)\mathbb{E}[X_n]\big\}^2}{P_n} - [\ln n + (k-1)\ln\ln n] \\ &= (1 \pm \epsilon_n)^2[\ln n + (k-1)\ln\ln n + \alpha_n] \\ &\quad - [\ln n + (k-1)\ln\ln n] \\ &= \alpha_n + \epsilon_n(\epsilon_n \pm 2)[\ln n + (k-1)\ln\ln n + \alpha_n] \\ &= \alpha_n \pm \left[o\left(\frac{\alpha_n}{\ln n}\right) + o(1)\right], \end{aligned} \tag{55}$$

where the last step uses $\epsilon_n = o\big(\frac{1}{\ln n}\big)$. Then (55) clearly implies (54). Therefore, as mentioned above, we establish (51) (52) and finally Theorem 3.

V. The Proof of Theorem 2

As explained in Section III-B2, we can introduce an extra condition $|\alpha_n| = o(\ln n)$ in proving Theorem 2. Then since a necessary condition for a graph to be k-connected is that the minimum degree is at least k, (6a) implies (5a), and we have

$$\begin{aligned}&\mathbb{P}\big[\text{ Graph } G_u(n, P_n, K_n) \text{ is } k\text{-connected. }\big] \\ &= \mathbb{P}\left[\begin{array}{l}\text{Graph } G_u(n, P_n, K_n) \text{ has a} \\ \text{minimum node degree at least } k.\end{array}\right] \\ &\quad - \mathbb{P}\left[\begin{array}{l} G_u(n, P_n, K_n) \text{ has a minimum degree} \\ \text{at least } k, \text{ but is not } k\text{-connected.}\end{array}\right]. \end{aligned} \tag{56}$$

From (56), we know that (5b) (resp., (5c)) will follow from Lemma 10 below and (6b) (resp., (6c)), where we note that Lemma 10 uses the extra condition $|\alpha_n| = o(\ln n)$ explained above. Also as mentioned before, (6a) implies (5a). Therefore, clearly the proof of Theorem 2 will be completed once we demonstrate (6a) (6b) (6c) and Lemma 10, where we also use the extra condition $|\alpha_n| = o(\ln n)$ in proving (6a) (6b) (6c). We let $e^{-\infty} = 0$ and $e^{\infty} = \infty$, so $e^{-\frac{e^{-\lim_{n\to\infty}\alpha_n}}{(k-1)!}}$ equals 0 if $\lim_{n\to\infty}\alpha_n = -\infty$, 1 if $\lim_{n\to\infty}\alpha_n = \infty$ and $e^{-\frac{e^{-\alpha^*}}{(k-1)!}}$ if $\lim_{n\to\infty}\alpha_n = \alpha^* \in (-\infty, \infty)$. Then (6a) (6b) (6c) under $|\alpha_n| = o(\ln n)$ can be compactly presented by Lemma 11 below. Hence, the proof of Theorem 2 finally reduces to proving Lemmas 10 and 11.

Lemma 10 *For a uniform random intersection graph $G_u(n, P_n, K_n)$ under $K_n = \Omega\big(\sqrt{\ln n}\big)$ and $\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}$, where $\lim_{n\to\infty}\alpha_n$ exists and $|\alpha_n| = o(\ln n)$, it follows that*

$$\lim_{n\to\infty}\mathbb{P}\left[\begin{array}{l} G_u(n, P_n, K_n) \textit{ has a minimum degree} \\ \textit{at least } k, \textit{ but is not } k\textit{-connected.}\end{array}\right] = 0. \tag{57}$$

Lemma 11 *For a uniform random intersection graph $G_u(n, P_n, K_n)$ under $K_n = \Omega\big(\sqrt{\ln n}\big)$ and $\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}$, where $\lim_{n\to\infty}\alpha_n$ exists and $|\alpha_n| = o(\ln n)$, it follows that*

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{P}\left[\, G_u(n, P_n, K_n) \textit{ has a minimum degree at least } k. \right] \\ &= e^{-\frac{e^{-\lim_{n\to\infty}\alpha_n}}{(k-1)!}}.\end{aligned}$$

To prove Lemma 10, we use the following Lemma 12 on $G_u(n, P_n, K_n) \cap G(n, \hat{p}_n)$, where $G(n, \hat{p}_n)$ is an Erdős–Rényi graph with n nodes and edge probability $\hat{p}_n$, and the intersection of two graphs G_A and G_B defined on the same node set is constructed on the node set with the edge set being the intersection of the edge sets of G_A and G_B.

Lemma 12 (Our work [29, Propositions 3 and 4]) *For a uniform random intersection graph $G_u(n, P_n, K_n) \cap G(n, \hat{p}_n)$ under $P_n = \Omega(n)$, $\frac{K_n}{P_n} = o(1)$ and $\frac{{K_n}^2}{P_n} \cdot \hat{p}_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}$, where $\lim_{n\to\infty}\alpha_n$ exists and $|\alpha_n| = o(\ln n)$, it follows that*

$$\lim_{n\to\infty}\mathbb{P}\left[\begin{array}{l} G_u(n, P_n, K_n) \cap G(n, \hat{p}_n) \textit{ has a minimum} \\ \textit{degree at least } k, \textit{ but is not } k\textit{-connected.}\end{array}\right] = 0. \tag{58}$$

Lemma 12 is from our work [29, Propositions 3 and 4]. Setting $\hat{p}_n = 1$, we have $G_u(n, P_n, K_n) \cap G(n, \hat{p}_n) = G_u(n, P_n, K_n)$ and obtain results on $G_u(n, P_n, K_n)$ from Lemma 12:

For $G_u(n, P_n, K_n)$ under $P_n = \Omega(n)$, $\frac{K_n}{P_n} = o(1)$ and $\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n}$, where $\lim_{n\to\infty}\alpha_n$ exists and $|\alpha_n| = o(\ln n)$, result (57 holds.

Then clearly, Lemma 10 will proved once we show conditions in Lemma 10 imply $P_n = \Omega(n)$ and $\frac{K_n}{P_n} = o(1)$. From conditions in Lemma 10, we have $K_n = \Omega\big(\sqrt{\ln n}\big)$ and $\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n} \sim \frac{\ln n}{n}$ given $|\alpha_n| = o(\ln n)$. Then we further get $P_n = {K_n}^2 \big/ \frac{{K_n}^2}{P_n} = \Omega\big(\ln n / \frac{\ln n}{n}\big) = \Omega(n)$ and $\frac{K_n}{P_n} = \frac{{K_n}^2}{P_n} \big/ K_n = O\big(\frac{\ln n}{n} \big/ \sqrt{\ln n}\big) = o(1)$. Hence, as mentioned above, Lemma 10 is established.

Now we prove Lemma 11. We let q_n be the edge probability in a uniform random intersection graph $G_u(n, P_n, K_n)$; i.e., two nodes in $G_u(n, P_n, K_n)$ have an edge in between with probability q_n. Under conditions of Lemma 11, given $|\alpha_n| = o(\ln n)$, we have

$$\frac{{K_n}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \alpha_n}{n} \sim \frac{\ln n}{n}. \tag{59}$$

Hence, from [29, Lemma 8-Property (a)], it follows that

$$q_n = \frac{{K_n}^2}{P_n}\left[1 \pm O\left(\frac{{K_n}^2}{P_n}\right)\right] \sim \frac{\ln n}{n}. \tag{60}$$

Then, by [27, Section 3], Lemma 11 will follow once we show Lemma 13 below, where $\mathcal{V}_n = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes in graph $G_u(n, P_n, K_n)$.

Lemma 13 *For a uniform random intersection graph $G_u(n, P_n, K_n)$ under $K_n = \Omega\big(\sqrt{\ln n}\big)$ and $q_n \sim \frac{\ln n}{n}$, it follows for integers $m \geq 1$ and $h \geq 0$ that*

$$\begin{aligned}&\mathbb{P}[\textit{Nodes } v_1, v_2, \ldots, v_m \textit{ have degree } h] \\ &\sim (h!)^{-m}(nq_n)^{hm}e^{-mnq_n}.\end{aligned} \tag{61}$$

The rest of this section is devoted to proving Lemma 13.

In a uniform random intersection graph $G_u(n, P_n, K_n)$, recalling that $\mathcal{V}_n = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes, we let S_i be the set of K_n distinct objects assigned to node $v_i \in \mathcal{V}_n$. We further define $\mathcal{V}_m$ as $\{v_1, v_2, \ldots, v_m\}$ and $\overline{\mathcal{V}_m}$ as $\mathcal{V}_n \setminus \mathcal{V}_m$. Among nodes in $\overline{\mathcal{V}_m}$, we denote by N_i the set of nodes neighboring to v_i for $i = 1, 2, \ldots, m$. We denote $N_i \cap N_j$ by N_{ij}, and $S_i \cap S_j$ by S_{ij}.

We have the following two observations:

i) If node v_i has degree h, then $|N_i| \leq h$, where the equal sign holds if and only if v_i is directly connected to none of nodes in $V_m \setminus \{v_i\}$; i.e., if and only if event $\bigcap_{j\in\{1,2,\ldots,m\}\setminus\{i\}}(S_{ij} = \emptyset)$ happens.

ii) If $|N_i| \le h$ for any $i = 1, 2, \ldots, m$, then

$$\Big|\bigcup_{1\le i\le m} N_i\Big| \le \sum_{1\le i\le m} N_i \le hm, \tag{62}$$

where the two equal signs in (62) *both* hold if and only if

$$\Big(\bigcap_{1\le i<j\le m}(N_{ij}=\emptyset)\Big)\cap\Big(\bigcap_{1\le i\le m}(|N_i|=h)\Big). \tag{63}$$

From i) and ii) above, if nodes $v_1, v_2, \ldots, v_m$ have degree h, we have either of the following two cases:

(a) Any two of $v_1, v_2, \ldots, v_m$ have no edge in between (namely, $\bigcap_{1\le i<j\le m}(S_{ij}=\emptyset)$); and event (63) happens.

(b) $\big|\bigcup_{1\le i\le m} N_i\big| \le hm-1$.

In addition, if case (a) happens, then nodes $v_1, v_2, \ldots, v_m$ have degree h. However, if case (b) occurs, there is no such conclusion. With P_a (resp., P_b) denoting the probability of case (a) (resp., case (b)), we obtain

$$P_a \le \mathbb{P}[\text{Nodes } v_1, v_2, \ldots, v_m \text{ have degree } h] \le P_a + P_b,$$

where

$$P_a = \mathbb{P}\Big[\Big(\bigcap_{1\le i<j\le m}(S_{ij}=\emptyset)\Big)\cap\Big(\bigcap_{1\le i<j\le m}(N_{ij}=\emptyset)\Big) \cap\Big(\bigcap_{1\le i\le m}(|N_i|=h)\Big)\Big],$$

and

$$P_b = \mathbb{P}\Big[\Big|\bigcup_{1\le i\le m} N_i\Big| \le hm-1\Big].$$

Hence, (61) holds after we prove the following (64) and (65):

$$P_b = o\left((nq_n)^{hm}e^{-mnq_n}\right). \tag{64}$$

and

$$P_a \sim (h!)^{-m}(nq_n)^{hm}e^{-mnq_n}\cdot[1+o(1)], \tag{65}$$

We will prove (64) and (65) below. We let $\mathbb{S}_m$ denote the tuple $(S_1, S_2, \ldots, S_m)$. The expression "$\,|\,\mathbb{S}_m = \mathbb{S}_m^*$" means "given $S_1 = S_1^*, S_2 = S_2^*, \ldots, S_m = S_m^*$", where $\mathbb{S}_m^* = (S_1^*, S_2^*, \ldots, S_m^*)$ with $S_1^*, S_2^*, \ldots, S_m^*$ being arbitrary K_n-size subsets of the object pool. Note that $S_{ij}^* := S_i^* \cap S_j^*$. For two different nodes v and w in the graph $G_u(n, P_n, K_n)$, we use $v \leftrightarrow w$ to denote the event that there is an edge between v and w; i.e., the symbol "$\leftrightarrow$" means "is directly connected to".

A. The Proof of (64)

Let w be an arbitrary node in $\overline{V_m}$. We have

$$\begin{aligned}
&\mathbb{P}\Big[\Big|\bigcup_{1\le i\le m} N_i\Big| = t\,\Big|\,\mathbb{S}_m = \mathbb{S}_m^*\Big] \qquad (66)\\
&= \frac{(n-m)!}{t!(n-m-t)!}\\
&\times\left\{\mathbb{P}[w\leftrightarrow \text{ at least one of nodes in } V_m \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*]\right\}^t\\
&\times\left\{\mathbb{P}[w\leftrightarrow \text{none of nodes in } V_m \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*]\right\}^{n-m-t}. \qquad (67)
\end{aligned}$$

By the union bound, it holds that

$$\begin{aligned}
&\mathbb{P}[w\leftrightarrow \text{at least one of nodes in } V_m \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*]\\
&\le \sum_{1\le i\le m}\mathbb{P}[w\leftrightarrow v_i \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*] = mq_n, \qquad (68)
\end{aligned}$$

which yields

$$\mathbb{P}[w\leftrightarrow \text{none of nodes in } V_m \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*] \ge 1-mq_n. \tag{69}$$

In addition,

$$\begin{aligned}
&\mathbb{P}[w\leftrightarrow \text{none of nodes in } V_m \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*]\\
&= \frac{\binom{P_n - |\bigcup_{1\le i\le m} S_i^*|}{K_n}}{\binom{P_n}{K_n}}\\
&\le (1-q_n)^{K_n{}^{-1}|\bigcup_{1\le i\le m} S_i^*|} \quad \text{(by [20, Lemma 5.1])}\\
&\le e^{-K_n{}^{-1}q_n|\bigcup_{1\le i\le m} S_i^*|} \quad (\text{by } 1+x\le e^x \text{ for any real } x).
\end{aligned} \tag{70}$$

We will prove

$$\begin{aligned}
&\sum_{\mathbb{S}_m^*}\Big\{\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\\
&\quad\times\left\{\mathbb{P}[w\leftrightarrow \text{none of nodes in } V_m \,\big|\, \mathbb{S}_m=\mathbb{S}_m^*]\right\}^{n-m-hm}\Big\} \qquad (71)\\
&\le e^{-mnq_n}\cdot[1+o(1)]. \qquad (72)
\end{aligned}$$

From (67) (68) and (72), we derive

$$\begin{aligned}
P_b &= \mathbb{P}\Big[\Big|\bigcup_{1\le i\le m} N_i\Big| \le hm-1\Big]\\
&= \sum_{t=0}^{hm-1}\sum_{\mathbb{S}_m^*}\Big\{\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\cdot(66)\Big\}\\
&\le \sum_{t=0}^{hm-1}\Big[n^t\cdot(mq_n)^t\cdot(71)\Big]\\
&\le (nq_n)^{hm}e^{-mnq_n}\cdot[1+o(1)]\cdot m^{hm}\sum_{t=0}^{hm-1}(mnq_n)^{t-hm}.
\end{aligned} \tag{73}$$

Applying (60) to (73), we obtain (64). Hence, we complete the proof of (64) once showing (72), whose proof is detailed below.

From (69) (70) and (60), we have

$$\begin{aligned}
(71) &\le (1-mq_n)^{-m-hm}\\
&\quad\times\sum_{\mathbb{S}_m^*}\Big\{\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\cdot e^{-K_n{}^{-1}nq_n|\bigcup_{1\le i\le m} S_i^*|}\Big\}\\
&\le [1+o(1)]\cdot\sum_{\mathbb{S}_m^*}\Big\{\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\cdot e^{-K_n{}^{-1}nq_n|\bigcup_{1\le i\le m} S_i^*|}\Big\},
\end{aligned} \tag{74}$$

so (72) holds once we demonstrate

$$\begin{aligned}
&\sum_{\mathbb{S}_m^*}\Big\{\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\cdot e^{-K_n{}^{-1}nq_n|\bigcup_{1\le i\le m} S_i^*|}\Big\}\\
&\quad\le e^{-mnq_n}\cdot[1+o(1)].
\end{aligned} \tag{75}$$

We denote the left hand side of (75) by $Z_{m,n}$. Dividing $\mathbb{S}_m^*$

into two parts $\mathbb{S}_{m-1}^*$ and S_m^*, we derive

$$Z_{m,n} = \sum_{\substack{\mathbb{S}_{m-1}^* \\ S_m^*}} \Big\{\mathbb{P}[(\mathbb{S}_{m-1} = \mathbb{S}_{m-1}^*) \cap (S_m = S_m^*)] \times e^{-K_n{}^{-1} n q_n |\bigcup_{1\leq i\leq m} S_i^*|}\Big\}$$
$$= \sum_{\mathbb{S}_{m-1}^*} \mathbb{P}[\mathbb{S}_{m-1} = \mathbb{S}_{m-1}^*]\Big\{e^{-K_n{}^{-1} n q_n |\bigcup_{1\leq i\leq m-1} S_i^*|} \times \sum_{S_m^*} \mathbb{P}[S_m = S_m^*] e^{-K_n{}^{-1} n q_n |S_m^* \setminus \bigcup_{1\leq i\leq m-1} S_i^*|}\Big\}, \tag{76}$$

where

$$\sum_{S_m^*} \mathbb{P}[S_m = S_m^*] e^{-K_n{}^{-1} n q_n |S_m^* \setminus \bigcup_{1\leq i\leq m-1} S_i^*|}$$
$$\leq e^{-nq_n} \sum_{S_m^*} \mathbb{P}[S_m = S_m^*] e^{K_n{}^{-1} n q_n \left|S_m^* \cap \left(\bigcup_{i=1}^{m-1} S_i^*\right)\right|}$$
$$= e^{-nq_n} \sum_{r=0}^{K_n} \mathbb{P}\left[\left|S_m \cap \left(\bigcup_{i=1}^{m-1} S_i^*\right)\right| = r\right] e^{K_n{}^{-1} n q_n r}. \tag{77}$$

Denoting $\left|\bigcup_{i=1}^{m-1} S_i^*\right|$ by v, then for r satisfying $0 \leq r \leq |S_m^*| = K_n$ and $S_m^* \cup \left(\bigcup_{i=1}^{m-1} S_i^*\right) = K_n + v - r \leq P_n$ (i.e., for $r \in [\max\{0, K_n + v - P_n\}, K_n]$), we obtain

$$\mathbb{P}\left[\left|S_m \cap \left(\bigcup_{i=1}^{m-1} S_i^*\right)\right| = r\right] = \binom{v}{r}\binom{P_n - v}{K_n - r} \Big/ \binom{P_n}{K_n}, \tag{78}$$

which together with $K_n \leq v \leq mK_n$ yields

$$\text{L.H.S. of (78)} \leq \frac{(mK_n)^r}{r!} \cdot \frac{(P_n - K_n)^{K_n - r}}{(K_n - r)!} \cdot \frac{K_n!}{(P_n - K_n)^{K_n}}$$
$$\leq \frac{1}{r!}\left(\frac{mK_n{}^2}{P_n - K_n}\right)^r \text{ for } r \in [\max\{0, K_n + v - P_n\}, K_n]. \tag{79}$$

Also, it is clear that

$$\text{L.H.S. of (78)} = 0 \text{ for } r \notin [\max\{0, K_n + v - P_n\}, K_n]. \tag{80}$$

Applying (79) and (80) to (77), we establish

$$\sum_{S_m^*} \mathbb{P}[S_m = S_m^*] e^{-K_n{}^{-1} n q_n |S_m^* \setminus \bigcup_{1\leq i\leq m-1} S_i^*|}$$
$$\leq e^{-nq_n} \sum_{r=0}^{K_n} \frac{1}{r!}\left(\frac{mK_n{}^2}{P_n - K_n}\right)^r \cdot e^{K_n{}^{-1} n q_n r}$$
$$\leq e^{-nq_n} \cdot e^{\frac{mK_n{}^2}{P_n - K_n} \cdot e^{K_n{}^{-1} n q_n}}. \tag{81}$$

From and (59) (i.e., $\frac{K_n{}^2}{P_n} \sim \frac{\ln n}{n}$), we have $P_n = \omega(K_n)$ and further

$$\frac{mK_n{}^2}{P_n - K_n} \sim \frac{mK_n{}^2}{P_n} \sim \frac{m \ln n}{n}. \tag{82}$$

For an arbitrary $\epsilon > 0$, from (60), we obtain $q_n \leq (1+\epsilon)\frac{\ln n}{n}$ for all n sufficiently large, which with condition $K_n \geq 2$ yields that for all n sufficiently large,

$$e^{K_n{}^{-1} n q_n} \leq e^{\frac{1}{2}(1+\epsilon)\ln n} = n^{\frac{1}{2}(1+\epsilon)}. \tag{83}$$

From (82) and (83), we get

$$\frac{mK_n{}^2}{P_n - K_n} \cdot e^{K_n{}^{-1} n q_n} \leq \frac{m \ln n}{n} \cdot [1 + o(1)] \cdot n^{\frac{1}{2}(1+\epsilon)}$$
$$\leq m \ln n \cdot n^{\frac{1}{2}(\epsilon - 1)} \cdot [1 + o(1)]. \tag{84}$$

Since $\epsilon > 0$ is arbitrary, it follows from (84) that for arbitrary $0 < c < \frac{1}{2}$, then for all n sufficiently large, it is clear that

$$\frac{mK_n{}^2}{P_n - K_n} \cdot e^{K_n{}^{-1} n q_n} \leq n^{-c}. \tag{85}$$

Using (85) in (81), for all n sufficiently large, it follows that

$$\sum_{S_m^*} \mathbb{P}[S_m = S_m^*] e^{-K_n{}^{-1} n q_n |S_m^* \setminus \bigcup_{1\leq i\leq m-1} S_i^*|} \leq e^{-nq_n} \cdot e^{n^{-c}}. \tag{86}$$

Substituting (86) into (76), for all n sufficiently large, we obtain

$$Z_{m,n} \leq e^{-nq_n} \cdot e^{n^{-c}} \cdot \sum_{\mathbb{S}_{m-1}^*} \mathbb{P}[\mathbb{S}_{m-1} = \mathbb{S}_{m-1}^*] e^{-K_n{}^{-1} n q_n |\bigcup_{1\leq i\leq m-1} S_i^*|}$$
$$\leq e^{-nq_n} \cdot e^{n^{-c}} \cdot Z_{m-1,n}. \tag{87}$$

We then evaluate $Z_{2,n}$. By (75), it holds that

$$Z_{2,n} = \sum_{S_1^*} \sum_{S_2^*} \Big\{\mathbb{P}[(S_1 = S_1^*) \cap (S_2 = S_2^*)] \cdot e^{-K_n{}^{-1} n q_n |S_1^* \cup S_2^*|}\Big\}$$
$$= \sum_{S_1^*} \mathbb{P}[S_1 = S_1^*] \sum_{S_2^*} \mathbb{P}[S_2 = S_2^*] e^{-K_n{}^{-1} n q_n |S_1^* \cup S_2^*|}. \tag{88}$$

Setting $m = 2$ in (86), for all n sufficiently large, we derive

$$\sum_{S_2^*} \mathbb{P}[S_2 = S_2^*] e^{-K_n{}^{-1} n q_n |S_2^* \setminus S_1^*|} \leq e^{-nq_n} \cdot e^{n^{-c}}.$$

Then for all n sufficiently large, it follows that

$$\sum_{S_2^*} \mathbb{P}[S_2 = S_2^*] e^{-K_n{}^{-1} n q_n |S_1^* \cup S_2^*|}$$
$$= e^{-nq_n} \sum_{S_2^*} \mathbb{P}[S_2 = S_2^*] e^{-K_n{}^{-1} n q_n |S_2^* \setminus S_1^*|}$$
$$\leq e^{-2nq_n} \cdot e^{n^{-c}}. \tag{89}$$

From (88) and (89), for all n sufficiently large, we obtain

$$Z_{m,n} \leq \left(e^{-nq_n} \cdot e^{n^{-c}}\right)^{m-2} \cdot Z_{2,n}$$
$$\leq \left(e^{-nq_n} \cdot e^{n^{-c}}\right)^{m-2} \cdot e^{-2nq_n} \cdot e^{n^{-c}}$$
$$\leq e^{-mnq_n} \cdot e^{(m-1)n^{-c}}. \tag{90}$$

Letting $n \to \infty$, we finally establish

$$Z_{m,n} \leq e^{-mnq_n} \cdot [1 + o(1)];$$

i.e., (75) is proved. Then as explained above, (72) holds; and then (64) follows.

B. The Proof of (65)

Again let w be an arbitrary node in $\overline{V_m}$. We have

$$\mathbb{P}\bigg[\bigg(\bigcap_{1\leq i<j\leq m}(N_{ij}=\emptyset)\bigg)\cap\bigg(\bigcap_{1\leq i\leq m}(|N_i|=h)\bigg)\,\bigg|\,\mathbb{S}_m=\mathbb{S}_m^*\bigg] \tag{91}$$

$$\begin{aligned}&=\frac{(n-m)!}{(h!)^m(n-m-hm)!}\\&\quad\times\prod_{1\leq i\leq m}\left(\left\{\mathbb{P}\left[\begin{array}{l}w\leftrightarrow v_i,\\ \text{but } w\leftrightarrow \text{none of}\\ \text{nodes in } V_m\setminus\{v_i\}\end{array}\,\middle|\,\mathbb{S}_m=\mathbb{S}_m^*\right]\right\}^h\right)\\&\quad\times\left\{\mathbb{P}[w\leftrightarrow\text{none of nodes in }V_m\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\right\}^{n-m-hm}\end{aligned} \tag{92}$$

and

$$P_a=\sum_{\mathbb{S}_m^*:\,\bigcap_{1\leq i<j\leq m}(S_{ij}^*=\emptyset)}\Big\{\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\cdot(91)\Big\}, \tag{93}$$

where $S_{ij}^*:=S_i^*\cap S_j^*$.

For $i=1,2,\ldots,m$, under $\mathbb{S}_m^*:\bigcap_{1\leq i<j\leq m}(S_{ij}^*=\emptyset)$, it follows that

$$\begin{aligned}&\mathbb{P}[w\leftrightarrow v_i,\text{ but none of nodes in }V_m\setminus\{v_i\}\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&\geq\mathbb{P}[w\leftrightarrow v_i\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&\quad-\sum_{\substack{1\leq j\leq m\\ j\neq i}}\mathbb{P}[w\leftrightarrow\text{both }v_i\text{ and }v_j\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\end{aligned} \tag{94}$$

where we note

$$\mathbb{P}[w\leftrightarrow v_i\,|\,\mathbb{S}_m=\mathbb{S}_m^*]=q_n, \tag{95}$$

and

$$\begin{aligned}&\mathbb{P}[w\leftrightarrow\text{both }v_i\text{ and }v_j\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&=\mathbb{P}[w\leftrightarrow v_i\,|\,\mathbb{S}_m=\mathbb{S}_m^*]+\mathbb{P}[w\leftrightarrow v_j\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&\quad-\mathbb{P}[(w\leftrightarrow v_i)\cup(w\leftrightarrow v_j)\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&=q_n+q_n-\binom{P_n-2K_n}{K_n}\Big/\binom{P_n}{K_n}\end{aligned} \tag{96}$$

given $\mathbb{S}_m^*:\bigcap_{1\leq i<j\leq m}(S_{ij}^*=\emptyset)$. From [20, Lemma 5.1], we get $\binom{P_n-2K_n}{K_n}\big/\binom{P_n}{K_n}\leq(1-q_n)^2$, which with (95) and (96) are used in (94) to derive

$$\begin{aligned}&\mathbb{P}[w\leftrightarrow v_i,\text{ but none of nodes in }V_m\setminus\{v_i\}\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&\quad\geq q_n-(m-1)\cdot 2{q_n}^2.\end{aligned} \tag{97}$$

Substituting (69) and (97) to (92), and then from (93), we obtain

$$\begin{aligned}P_a&\geq\frac{(n-m-hm)^{hm}}{(h!)^m}\cdot[q_n-2(m-1)q_n^2]^{hm}\\&\quad\times(1-mq_n)^{n-m-hm}\sum_{\mathbb{S}_m^*:\,\bigcap_{1\leq i<j\leq m}(S_{ij}^*=\emptyset)}\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*].\end{aligned}$$

Then from (60), it further hold that

$$\begin{aligned}P_a&\geq\frac{n^{hm}}{(h!)^m}\cdot(q_n)^{hm}\cdot e^{-mnq_n}\\&\quad\times[1-o(1)]\cdot\mathbb{P}\bigg[\bigcap_{1\leq i<j\leq m}(S_{ij}=\emptyset)\bigg].\end{aligned} \tag{98}$$

From (70), under $\mathbb{S}_m^*:\bigcap_{1\leq i<j\leq m}(S_{ij}^*=\emptyset)$, it holds that

$$\mathbb{P}[w\leftrightarrow\text{none of nodes in }V_m\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\leq e^{-mq_n}. \tag{99}$$

For each $i=1,2,\ldots,m$, we have

$$\begin{aligned}&\mathbb{P}[w\leftrightarrow v_i,\text{ but }w\leftrightarrow\text{none of nodes in }V_m\setminus\{v_i\}\,|\,\mathbb{S}_m=\mathbb{S}_m^*]\\&\leq\mathbb{P}[w\leftrightarrow v_i\,|\,\mathbb{S}_m=\mathbb{S}_m^*]=q_n.\end{aligned} \tag{100}$$

Substituting (100) and (99) to (92), and then from (93), we obtain

$$\begin{aligned}P_a&\leq\frac{n^{hm}}{(h!)^m}\cdot(q_n)^{hm}\cdot e^{-mnq_n}\cdot\sum_{\mathbb{S}_m^*:\,\bigcap_{1\leq i<j\leq m}(S_{ij}^*=\emptyset)}\mathbb{P}[\mathbb{S}_m=\mathbb{S}_m^*]\\&=\frac{n^{hm}}{(h!)^m}\cdot(q_n)^{hm}\cdot e^{-mnq_n}\cdot\mathbb{P}\bigg[\bigcap_{1\leq i<j\leq m}(S_{ij}=\emptyset)\bigg].\end{aligned} \tag{101}$$

From (98) and (101), we obtain

$$P_a\sim\frac{n^{hm}}{(h!)^m}\cdot(q_n)^{hm}\cdot e^{-mnq_n}\cdot\mathbb{P}\bigg[\bigcap_{1\leq i<j\leq m}(S_{ij}=\emptyset)\bigg]. \tag{102}$$

By the union bound, it is clear that

$$\begin{aligned}&\mathbb{P}\bigg[\bigcap_{1\leq i<j\leq m}(S_{ij}=\emptyset)\bigg]\\&=1-\mathbb{P}\bigg[\bigcup_{1\leq i<j\leq m}(S_{ij}\neq\emptyset)\bigg]\\&\geq1-\sum_{1\leq i<j\leq m}\mathbb{P}[S_{ij}\neq\emptyset]=1-\binom{m}{2}q_n.\end{aligned} \tag{103}$$

From (60) and (103), since a probability is at most 1, we get

$$\lim_{n\to\infty}\mathbb{P}\bigg[\bigcap_{1\leq i<j\leq m}(S_{ij}=\emptyset)\bigg]=1. \tag{104}$$

Using (104) in (102), we establish (65).

VI. Establishing Theorems 4–6

Before proving Theorems 4–6, we first discuss the relationship between k-connectivity and k-robustness.

A. The Relationships of k-Robustness with k-Connectivity and Minimum Node Degree

The references [13], [23], [24] all claim that for any graph/network, k-robustness implies k-connectivity. However, we show that the above claim does not hold, using a simple example graph illustrated in Figure 1. Recalling the definition of k-robustness in Section I-C, it is straightforward to show that the graph in Figure 1 is 2-robust yet only 1-connected (i.e., connected), not 2-connected. Hence, k-robustness actually does not imply k-connectivity for general k. We also note that 1-robustness (i.e., robustness) is clearly equivalent to 1-connectivity (i.e., connectivity). Thus, with "$\Longleftrightarrow$" denoting equivalence and "$\not\Longrightarrow$" denoting the relation of not implying, we can write

1-robustness $\Longleftrightarrow$ 1-connectivity, and

k-robustness $\not\Longrightarrow$ k-connectivity for $k\geq2$.

On the other hand, as shown in [24, Figure 1], it holds that

k-connectivity $\not\Longrightarrow$ k-robustness for $k\geq2$.

In addition, k-robustness implies that the minimum node degree is at least k as given by [24, Lemma 1], which we put as Lemma 14 below for clarity. Note that k-connectivity also implies that the minimum node degree is at least k.

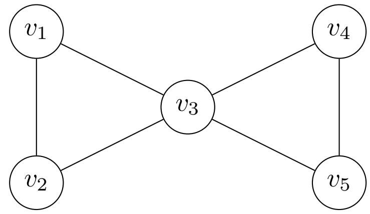

Fig. 1: A graph that is 2-robust yet only 1-connected (i.e., connected), not 2-connected.

Lemma 14 ([24, Lemma 1]) *For any graph/network, k-robustness implies that the minimum node degree is at least k.*

We now explain the idea of proving Theorems 4–6, which present zero–one laws on k-robustness for binomial/uniform/general random intersection graphs. First, the zero-law of Theorem 4 is established from the zero-law of Theorem 1 since k-robustness implies the property of minimum node degree being at least k from Lemma 14 above, while the one-law of Theorem 4 is proven from the coupling between binomial random intersection graphs and Erdős–Rényi graphs given by Lemma 7 of Section III-B4. Second, the zero-law of Theorem 5 is demonstrated from the zero-law of Theorem 2 because k-robustness implies the property of minimum node degree being at least k from Lemma 14 above, while the one-law of Theorem 5 is established from the coupling between binomial random intersection graphs and uniform random intersection graphs given by Lemma 8 of Section III-B5. Finally, both the zero-law and one-law of Theorem 6 are proved from the coupling between general random intersection graphs and uniform random intersection graphs given by Lemma 6 of Section III-B3.

B. The Proof of Theorem 4

Since k-robustness implies the property of minimum node degree being at least k from Lemma 14, the zero-law of Theorem 4 is clear from (6a) of Theorem 1 in view that under conditions of Theorem 4, if $\lim_{n\to\infty}\alpha_n = -\infty$,

$$\begin{aligned}&\mathbb{P}\big[\, G_b(n,P_n,p_n) \text{ is } k\text{-robust.}\,\big]\\ &\leq \mathbb{P}\left[\begin{array}{l} G_b(n,P_n,p_n) \text{ has a} \\ \text{minimum node degree at least } k. \end{array}\right] \to 0, \text{ as } n\to\infty.\end{aligned} \tag{105}$$

Below we prove the one-law of Theorem 4. As explained in Section III-B1, we can introduce an extra condition $|\alpha_n| = o(\ln n)$ in proving Theorem 4. Given (10) and $|\alpha_n| = o(\ln n)$, we have

$$p_n{}^2 P_n \sim \frac{\ln n}{n},$$

which together with condition $P_n = \Omega\big(n(\ln n)^5\big)$ leads to

$$p_n \sim \sqrt{\frac{\ln n}{nP_n}} = O\left(\sqrt{\frac{\ln n}{n^2(\ln n)^5}}\right) = O\left(\frac{1}{n(\ln n)^2}\right). \tag{106}$$

Noting that (106) implies condition $p_n = O\left(\frac{1}{n\ln n}\right)$ in Lemma 7, we apply Lemma 2, Remark 2 after Lemma 7, and condition (10) to derive the following: there exists $\hat{p}_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n - O(1)}{n}$ such that if $\lim_{n\to\infty}\alpha_n = \infty$,

$$\begin{aligned}&\mathbb{P}[\,\text{Graph } G_b(n,P_n,p_n) \text{ is } k\text{-robust.}\,]\\ &\geq \mathbb{P}[\,\text{Graph } G(n,\hat{p}_n) \text{ is } k\text{-robust.}\,] - o(1) \to 1, \text{ as } n\to\infty.\end{aligned} \tag{107}$$

The proof of Theorem 4 is completed via (105) and (107).

C. The Proof of Theorem 5

As explained in Section III-B2, we can introduce an extra condition $|\alpha_n| = o(\ln n)$ in proving Theorem 5. Since k-robustness implies that the minimum node degree is at least k from Lemma 14, the zero-law of Theorem 5 is clear from Lemma 11 in view that under conditions of Theorem 5 with the extra condition $|\alpha_n| = o(\ln n)$, if $\lim_{n\to\infty}\alpha_n = -\infty$,

$$\begin{aligned}&\mathbb{P}\big[\, G_u(n,P_n,K_n) \text{ is } k\text{-robust.}\,\big]\\ &\leq \mathbb{P}\left[\begin{array}{l} G_u(n,P_n,K_n) \text{ has a} \\ \text{minimum node degree at least } k. \end{array}\right] \to 0, \text{ as } n\to\infty.\end{aligned} \tag{108}$$

Below we establish the one-law of Theorem 5 with the help of Theorem 4. Given $K_n = \Omega\big((\ln n)^3\big) = \omega(\ln n)$, we use Lemma 9 to obtain that with p_n set by

$$p_n = \frac{K_n}{P_n}\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right), \tag{109}$$

it holds that

$$\begin{aligned}&\mathbb{P}[\,\text{Graph } G_u(n,P_n,K_n) \text{ is } k\text{-robust.}\,]\\ &\quad\geq \mathbb{P}[\,\text{Graph } G_b(n,P_n,p_n) \text{ is } k\text{-robust.}\,] - o(1).\end{aligned} \tag{110}$$

From (12) and $|\alpha_n| = o(\ln n)$, we obtain $\frac{K_n{}^2}{P_n} \sim \frac{\ln n}{n}$, which together with $K_n = \Omega\big((\ln n)^3\big)$ results in

$$P_n \sim \frac{nK_n{}^2}{\ln n} = \Omega\big(n(\ln n)^5\big), \tag{111}$$

From $K_n = \Omega\big((\ln n)^3\big)$ and (109), it follows that

$$\begin{aligned} p_n{}^2 P_n &= \left[\frac{K_n}{P_n}\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right)\right]^2 \cdot P_n \\ &= \frac{K_n{}^2}{P_n}\cdot\left[1 - O\left(\frac{1}{\ln n}\right)\right]. \end{aligned} \tag{112}$$

By (12) and (112), it is clear that

$$p_n{}^2 P_n = \frac{\ln n + (k-1)\ln\ln n + \alpha_n - O(1)}{n}. \tag{113}$$

Given (111) (113) and $|\alpha_n| = o(\ln n)$, we use Theorem 4 and (110) to get that if $\lim_{n\to\infty}\alpha_n = \infty$,

$$\mathbb{P}\big[\, G_u(n,P_n,K_n) \text{ is } k\text{-robust.}\,\big] \to 1, \text{ as } n\to\infty. \tag{114}$$

The proof of Theorem 5 is completed via (108) and (114).

D. The Proof of Theorem 6

Similar to the process of proving Theorem 3 with the help of Theorem 2, we demonstrate Theorem 6 using Theorem 5, the proof of which is given in Section VI-C.

Given Remark 1 after Lemma 6 and the fact that k-robustness is a monotone increasing graph property, we will show Theorem 6 once proving for any $\epsilon_n = o\left(\frac{1}{\ln n}\right)$ that

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{P}\big[G_u(n,P_n,(1\pm\epsilon_n)\mathbb{E}[X_n]) \text{ is } k\text{-robust.}\big]\\ &= \begin{cases} 0, & \text{if } \lim_{n\to\infty}\alpha_n = -\infty, \\ 1, & \text{if } \lim_{n\to\infty}\alpha_n = \infty. \end{cases}\end{aligned} \tag{115}$$

Under $\mathbb{E}[X_n] = \Omega\big(\sqrt{\ln n}\,\big)$ and $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, it follows that $(1 \pm \epsilon_n)\mathbb{E}[X_n] = \Omega\big(\sqrt{\ln n}\,\big)$. From Theorem 5, we will have (51) once we prove that sequences γ_n^+ and γ_n^- defined through

$$\frac{\big\{(1 \pm \epsilon_n)\mathbb{E}[X_n]\big\}^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \gamma_n^{\pm}}{n} \tag{116}$$

satisfy

$$\lim_{n\to\infty} \gamma_n^{\pm} = \begin{cases} -\infty, & \text{if } \lim_{n\to\infty} \alpha_n = -\infty, \\ \infty, & \text{if } \lim_{n\to\infty} \alpha_n = \infty. \end{cases} \tag{117}$$

Note that (14) and (116) are exactly the same as (7) and (53), while (117) is a subset of (54). Since (55) follows from (7) (53) and $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, we use (14) (116) and $\epsilon_n = o\left(\frac{1}{\ln n}\right)$ to obtain (55), which further yields (117). Therefore, as mentioned above, we establish (115) and finally Theorem 6.

VII. Establishing Lemmas in Section III

Lemmas 2 and 8 are clear in Section III. Below we prove Lemmas 4, 5, 6, 7 and 9.

A. Proof of Lemma 4

(a) We set

$$\widetilde{P_n} = P_n, \tag{118}$$

and

$$\widetilde{\beta_n} = \max\{\beta_n, -\ln\ln n\}. \tag{119}$$

Given (119) and $\lim_{n\to\infty} \beta_n = -\infty$, we obtain $\lim_{n\to\infty} \widetilde{\beta_n} = -\infty$ and $\widetilde{\beta_n} = -o(\ln n)$. We use $\widetilde{\beta_n} = -o(\ln n)$ and (20) to have $\widetilde{p_n}^2 \widetilde{P_n} \sim \frac{\ln n}{n}$, so it is clear for all n sufficiently large that $\widetilde{p_n}$ is less than 1 and can be used as a probability. Under $p_n \leq \widetilde{p_n}$ and $\widetilde{P_n} = P_n$, by [16, Section 3], there exists a graph coupling under which $G_b(n, P_n, p_n)$ is a spanning subgraph of $G_b(n, \widetilde{P_n}, \widetilde{p_n})$; i.e., $G_b(n, P_n, p_n) \preceq G_b(n, \widetilde{P_n}, \widetilde{p_n})$.

(b) We set

$$\widehat{P_n} = P_n, \tag{120}$$

and

$$\widehat{\beta_n} = \min\{\beta_n, \ln\ln n\}. \tag{121}$$

Given (121) and $\lim_{n\to\infty} \beta_n = \infty$, we clearly obtain $\lim_{n\to\infty} \widehat{\beta_n} = \infty$ and $\widehat{\beta_n} = o(\ln n)$.

It holds from (121) that $\widehat{\beta_n} \leq \beta_n$, which along with (21) (22) and (120) yields $p_n \geq \widehat{p_n}$. Under $p_n \geq \widehat{p_n}$ and $\widehat{P_n} = P_n$, by [16, Section 3], there exists a graph coupling under which $G_b(n, P_n, p_n)$ is a spanning supergraph of $G_b(n, \widehat{P_n}, \widehat{p_n})$; i.e., $G_b(n, \widehat{P_n}, \widehat{p_n}) \preceq G_b(n, P_n, p_n)$.

B. The Proof of Lemma 5

a) Proving property (a): We define $\widetilde{\beta_n}^*$ by

$$\widetilde{\beta_n}^* = \max\{\beta_n, -\ln\ln n\}, \tag{122}$$

and define $\widetilde{K_n}^*$ such that

$$\frac{(\widetilde{K_n}^*)^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \widetilde{\beta_n}^*}{n}. \tag{123}$$

Note that $\widetilde{K_n}^*$ might or might not be an integer. We set

$$\widetilde{K_n} := \lfloor \widetilde{K_n}^* \rfloor, \tag{124}$$

and

$$\widetilde{P_n} := P_n, \tag{125}$$

where the floor function $\lfloor x \rfloor$ means the largest integer not greater than x.

From (39) (122) and (123), it holds that

$$K_n \leq \widetilde{K_n}^*. \tag{126}$$

Then by (124) (126) and the fact that K_n and $\widetilde{K_n}$ are both integers, it follows that

$$K_n \leq \widetilde{K_n}. \tag{127}$$

From (125) and (127), by [4, Lemma 3], there exists a graph coupling under which $G_u(n, P_n, K_n)$ is a spanning subgraph of $G_u(n, \widetilde{P_n}, \widetilde{K_n})$; i.e., $G_u(n, P_n, K_n) \preceq G_u(n, \widetilde{P_n}, \widetilde{K_n})$. Therefore, the proof of property (a) is completed once we show $\widetilde{\beta_n}$ defined in (38) satisfies

$$\lim_{n\to\infty} \widetilde{\beta_n} = -\infty, \tag{128}$$

$$\widetilde{\beta_n} = -o(\ln n). \tag{129}$$

We first prove (128). From (38) (123) and (124), it holds that

$$\widetilde{\beta_n} \leq \widetilde{\beta_n}^*, \tag{130}$$

which together with (122) and $\lim_{n\to\infty} \beta_n = -\infty$ yields (128).

Now we establish (129). From (124), we have $\widetilde{K_n} > \widetilde{K_n}^* - 1$. Then from (38) and (125), it holds that

$$\begin{aligned} \widetilde{\beta_n} &= n \cdot \frac{\widetilde{K_n}^2}{P_n} - [\ln n + (k-1)\ln\ln n] \\ &> n \cdot \frac{(\widetilde{K_n}^* - 1)^2}{P_n} - [\ln n + (k-1)\ln\ln n] \\ &> n \cdot \frac{(\widetilde{K_n}^*)^2 - 2\widetilde{K_n}^*}{P_n} - [\ln n + (k-1)\ln\ln n]. \end{aligned} \tag{131}$$

By $\lim_{n\to\infty} \beta_n = -\infty$, it holds that $\beta_n \leq 0$ for all n sufficiently large. Then from (122), it follows that

$$\widetilde{\beta_n}^* = -O(\ln\ln n), \tag{132}$$

which along with (123) yields

$$\frac{\widetilde{K_n}^*}{P_n} \sim \sqrt{\frac{\ln n}{nP_n}} = O\bigg(\frac{\sqrt{\ln n}}{n}\bigg). \tag{133}$$

Applying (123) (133) and $P_n = \Omega(n)$ to (131), we obtain

$$\begin{aligned} \widetilde{\beta_n} &> \left\{ n \cdot \frac{(\widetilde{K_n}^*)^2}{P_n} - [\ln n + (k-1)\ln\ln n] \right\} - 2n \cdot \frac{\widetilde{K_n}^*}{P_n} \\ &= \widetilde{\beta_n}^* - O\big(\sqrt{\ln n}\,\big). \end{aligned} \tag{134}$$

Thus, from (130) (132) and (134), clearly $\widetilde{\beta_n}$ can be written as $-O\big(\sqrt{\ln n}\,\big)$ and further $-o(\ln n)$; i.e., (129) is proved. Then as explained above, since we have shown (128) and (129), property (a) of Lemma 5 is established.

b) Proving property (b): We define $\widehat{\beta_n}^*$ by

$$\widehat{\beta_n}^* = \min\{\beta_n, \ln\ln n\}, \tag{135}$$

and define $\widehat{K_n}^*$ such that

$$\frac{(\widehat{K_n}^*)^2}{P_n} = \frac{\ln n + (k-1)\ln\ln n + \widehat{\beta_n}^*}{n}. \tag{136}$$

We set

$$\widehat{K_n} := \lceil \widehat{K_n}^* \rceil, \tag{137}$$

and

$$\widehat{P_n} := P_n. \tag{138}$$

From (39) (135) and (136), it holds that

$$K_n \geq \widehat{K_n}^*. \tag{139}$$

Then by (137) (139) and the fact that K_n and $\widehat{K_n}$ are both integers, it follows that

$$K_n \geq \widehat{K_n}. \tag{140}$$

From (138) and (140), by [4, Lemma 3], there exists a graph coupling under which $G_u(n, P_n, K_n)$ is a spanning supergraph of $G_u(n, \widehat{P_n}, \widehat{K_n})$; i.e., $G_u(n, \widehat{P_n}, \widehat{K_n}) \preceq G_u(n, P_n, K_n)$. Therefore, the proof of property (b) is completed once we show $\widehat{\beta_n}$ defined in (40) satisfies

$$\lim_{n\to\infty} \widehat{\beta_n} = \infty, \tag{141}$$

$$\widehat{\beta_n} = o(\ln n). \tag{142}$$

We first prove (141). From (40) (136) and (137), it holds that

$$\widehat{\beta_n} \geq \widehat{\beta_n}^*, \tag{143}$$

which together with (135) and $\lim_{n\to\infty} \beta_n = \infty$ yields (141).

Now we establish (142). From (137), we have $\widehat{K_n} < \widehat{K_n}^* + 1$. Then from (40) and (138), it holds that

$$\begin{aligned}
\widehat{\beta_n} &= n \cdot \frac{\widehat{K_n}^2}{P_n} - [\ln n + (k-1)\ln\ln n] \\
&< n \cdot \frac{(\widehat{K_n}^* + 1)^2}{P_n} - [\ln n + (k-1)\ln\ln n] \\
&\leq n \cdot \frac{(\widehat{K_n}^*)^2 + 3\widehat{K_n}^*}{P_n} - [\ln n + (k-1)\ln\ln n]. \quad (144)
\end{aligned}$$

By $\lim_{n\to\infty} \beta_n = \infty$, it holds that $\beta_n \geq 0$ for all n sufficiently large. Then from (135), it follows that

$$\widehat{\beta_n}^* = O(\ln\ln n), \tag{145}$$

which along with (136) and condition $P_n = \Omega(n)$ induces

$$\frac{\widehat{K_n}^*}{P_n} \sim \sqrt{\frac{\ln n}{nP_n}} = O\bigg(\frac{\sqrt{\ln n}}{n}\bigg). \tag{146}$$

Hence, we have $\lim_{n\to\infty} \widehat{K_n}^* = \infty$ and it further holds for all n sufficient large that

$$(\widehat{K_n}^* + 1)^2 < (\widehat{K_n}^*)^2 + 3\widehat{K_n}^*. \tag{147}$$

Applying (136) (146) and $P_n = \Omega(n)$ to (144), we obtain

$$\begin{aligned}
\widehat{\beta_n} &< \left\{ n \cdot \frac{(\widehat{K_n}^*)^2}{P_n} - [\ln n + (k-1)\ln\ln n] \right\} + 3n \cdot \frac{\widehat{K_n}^*}{P_n} \\
&= \widehat{\beta_n}^* + O\big(\sqrt{\ln n}\,\big). \quad (148)
\end{aligned}$$

Thus, from (143) (145) and (148), clearly $\widehat{\beta_n}$ can be written as $O\big(\sqrt{\ln n}\,\big)$ and further $o(\ln n)$; i.e., (142) is proved. Then as explained above, since we have shown (141) and (142), property (b) of Lemma 5 is established.

C. The Proof of Lemma 6

According to [4, Lemma 3], for any monotone increasing graph property $\mathcal{I}$ and any $|\epsilon_n| < 1$,

$$\begin{aligned}
&\mathbb{P}\big[G(n, P_n, \mathcal{D}_n) \text{ has } \mathcal{I}.\big] - \mathbb{P}\big[G_u(n,P_n,(1-\epsilon_n)\mathbb{E}[X_n]) \text{ has } \mathcal{I}.\big] \\
&\geq \big\{1 - \mathbb{P}[X_n < (1-\epsilon_n)\mathbb{E}[X_n]]\big\}^n - 1, \quad (149)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\big[G(n, P_n, \mathcal{D}_n) \text{ has } \mathcal{I}.\big] - \mathbb{P}\big[G_u(n,P_n,(1+\epsilon_n)\mathbb{E}[X_n]) \text{ has } \mathcal{I}.\big] \\
&\leq 1 - \big\{1 - \mathbb{P}[X_n > (1+\epsilon_n)\mathbb{E}[X_n]]\big\}^n. \quad (150)
\end{aligned}$$

By (149) (150) and the fact that $\lim_{n\to\infty}(1-m_n)^n = 1$ for $m_n = o\big(\frac{1}{n}\big)$ (this can be proved by a simple Taylor series expansion as in [29, Fact 2]), the proof of Lemma 6 is completed once we demonstrate that with $\text{Var}[X_n] = o\Big(\frac{\{\mathbb{E}[X_n]\}^2}{n(\ln n)^2}\Big)$, there exists $\epsilon_n = o\big(\frac{1}{\ln n}\big)$ such that

$$\mathbb{P}\left[X_n < (1-\epsilon_n)\mathbb{E}[X_n]\right] = o\bigg(\frac{1}{n}\bigg), \tag{151}$$

and

$$\mathbb{P}\left[X_n > (1+\epsilon_n)\mathbb{E}[X_n]\right] = o\bigg(\frac{1}{n}\bigg). \tag{152}$$

To prove (151) and (152), Chebyshev's inequality yields

$$\mathbb{P}\big[\, |X_n - \mathbb{E}[X_n]| > \epsilon_n \mathbb{E}[X_n]\big] \leq \frac{\text{Var}[X_n]}{\big\{\epsilon_n \mathbb{E}[X_n]\big\}^2}. \tag{153}$$

We set ϵ_n by $\epsilon_n = \sqrt[4]{\frac{n\text{Var}[X_n]}{\{\mathbb{E}[X_n]\}^2}} \cdot \frac{1}{\sqrt{\ln n}}$. Then given condition $\text{Var}[X_n] = o\Big(\frac{\{\mathbb{E}[X_n]\}^2}{n(\ln n)^2}\Big)$, we obtain

$$\epsilon_n = o\Bigg(\sqrt[4]{\frac{1}{(\ln n)^2}}\Bigg) \cdot \frac{1}{\sqrt{\ln n}} = o\bigg(\frac{1}{\ln n}\bigg), \tag{154}$$

and

$$\frac{\text{Var}[X_n]}{\big\{\epsilon_n \mathbb{E}[X_n]\big\}^2} = \sqrt{\frac{\text{Var}[X_n]}{n\big\{\mathbb{E}[X_n]\big\}^2}} \cdot \ln n = o\bigg(\frac{1}{n}\bigg). \tag{155}$$

By (153) (154) and (155), it is straightforward to see that (151) and (152) hold with $\epsilon_n = o\big(\frac{1}{\ln n}\big)$. Therefore, we have completed the proof of Lemma 6.

D. The Proof of Lemma 7

In view of [16, Lemma 3], if ${p_n}^2 P_n < 1$ and $p_n = o\left(\frac{1}{n}\right)$, with $\hat{p}_n := {p_n}^2 P_n \cdot \Big(1 - np_n + 2p_n - \frac{{p_n}^2 P_n}{2}\Big)$, then (44) follows. Given conditions $p_n = O\left(\frac{1}{n\ln n}\right)$ and ${p_n}^2 P_n = O\left(\frac{1}{\ln n}\right)$ in Lemma 7, ${p_n}^2 P_n < 1$ and $p_n = o\left(\frac{1}{n}\right)$ clearly hold. Then Lemma 7 is proved once we show $\hat{p}_n$ satisfies $\hat{p}_n = {p_n}^2 P_n \cdot \left[1 - O\left(\frac{1}{\ln n}\right)\right]$, which is easy to see via

$$\begin{aligned}
&-np_n + 2p_n - \frac{{p_n}^2 P_n}{2} \\
&= (-n+2) \cdot O\left(\frac{1}{n\ln n}\right) - \frac{1}{2} \cdot O\left(\frac{1}{\ln n}\right) = -O\left(\frac{1}{\ln n}\right).
\end{aligned}$$

Hence, the proof of Lemma 7 is completed.

E. The Proof of Lemma 9

We use Lemma 8 to prove Lemma 9. From conditions $K_n = \omega\left(\ln n\right)$ and $p_n = \frac{K_n}{P_n}\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right)$, we first obtain $p_n P_n = \omega\left(\ln n\right)$ and then for all n sufficiently large,

$$\begin{aligned}
&K_n - \Big[p_n P_n + \sqrt{3(p_n P_n + \ln n)\ln n}\,\Big] \\
&= K_n\sqrt{\frac{3\ln n}{K_n}} - \sqrt{3\left[K_n\left(1 - \sqrt{\frac{3\ln n}{K_n}}\right) + \ln n\right]\ln n} \\
&= \sqrt{3K_n \ln n} - \sqrt{3\left[K_n + \sqrt{\ln n}\left(\sqrt{\ln n} - \sqrt{3K_n}\right)\right]\ln n} \\
&\geq 0.
\end{aligned}$$

Then by Lemma 8, Lemma 9 is now established.

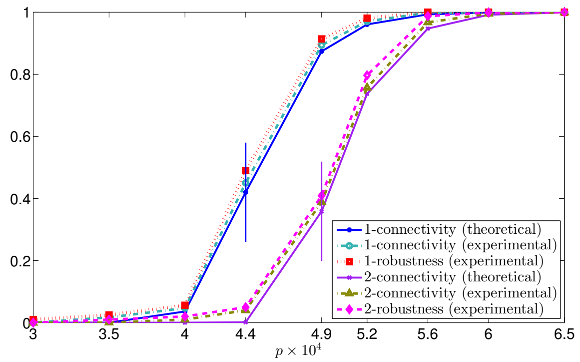

Fig. 2: A plot of the empirical probabilities that binomial random intersection graph $G_b(n, P, p)$ has k-connectivity or k-robustness as a function of p, with $n = 2000$, $P = 20000$ and $k = 2, 6$.

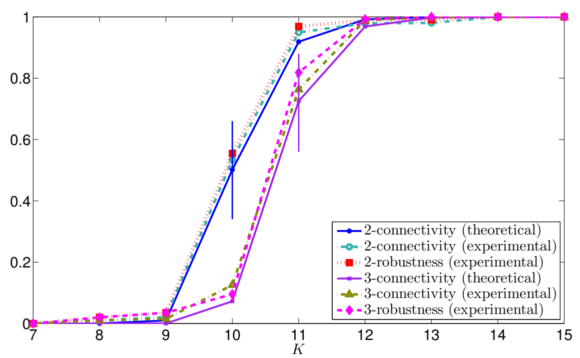

Fig. 3: A plot of the empirical probabilities that uniform random intersection graph $G_u(n, P, K)$ has k-connectivity or k-robustness as a function of K, with $n = 2000$, $P = 20000$ and $k = 3, 4$.

VIII. Numerical Experiments

We present numerical experiments in the non-asymptotic regime to confirm our theoretical results.

Figure 2 depicts the probability that binomial random intersection graph $G_b(n, P, p)$ has k-connectivity or k-robustness, for $k = 1, 2$. Similarly, Figure 3 illustrates the probability of k-connectivity or k-robustness for $k = 2, 3$ in uniform random intersection graph $G_u(n, P, K)$. In all set of experiments, we fix the number of nodes at $n = 2000$ and the object pool size $P = 20000$. For each pair (n, P, p) (resp., (n, P, K)), we generate 1000 independent samples of $G_b(n, P, p)$ (resp., $G_u(n, P, K)$) and count the number of times that the obtained graphs are k-connected or k-robust. Then the counts divided by 1000 become the corresponding empirical probabilities. As illustrated in Figures 2 and 3, there is an evident transition in the probabilities of k-connectivity and k-robustness. Also, for each k, the curves of k-connectivity and k-robustness are close to each other. Furthermore, the horizontal lines in Figure 2 specify p such that $p^2 P$ equals $\frac{\ln n+(k-1)\ln\ln n}{n}$, while the horizontal lines in Figure 3 specify K such that $\frac{K^2}{P}$ is closest to $\frac{\ln n+(k-1)\ln\ln n}{n}$ (since K and P are both integers, there might not exist K satisfying $\frac{K^2}{P} = \frac{\ln n+(k-1)\ln\ln n}{n}$).

The horizontal lines in Figure 2 are at 4.4×10^{-4} and 4.9×10^{-4} because under $n = 2000$ and $P = 20000$, $\sqrt{\frac{\ln n+(k-1)\ln\ln n}{nP}}$ is $\sqrt{\frac{\ln 2000}{2000\times 20000}} \approx 4.4 \times 10^{-4}$ for $k = 1$ and is $\sqrt{\frac{\ln 2000+\ln\ln 2000}{2000\times 20000}} \approx 4.9 \times 10^{-4}$ for $k = 2$. The horizontal lines in Figure 3 are at 10 and 11 because under $n = 2000$ and $P = 20000$, $\operatorname{argmin}_K \left|\frac{K^2}{P} - \frac{\ln n+(k-1)\ln\ln n}{n}\right|$ equals 10 for $k = 2$ from $\frac{\ln 2000+\ln\ln 2000}{2000} \approx 0.00481$, $\frac{9^2}{20000} \approx 0.00405$ and $\frac{10^2}{20000} \approx 0.005$, and equals 11 for $k = 3$ from $\frac{\ln 2000+2\ln\ln 2000}{2000} \approx 0.00583$, $\frac{10^2}{20000} \approx 0.005$ and $\frac{11^2}{20000} \approx 0.00605$.

IX. Parameter Conditions

Note that we impose conditions on the parameters in the theorems; e.g., $P_n = \omega\big(n(\ln n)^5\big)$ in Theorem 1, and $K_n = \Omega\big(\sqrt{\ln n}\,\big)$ in Theorem 2. These conditions are enforced to have the proofs get through and are not that conservative as explained below. We take a binomial random intersection graph as an example and note that Theorem 1 for k-connectivity in a binomial random intersection graph does not hold if the condition $P_n = \omega\big(n(\ln n)^5\big)$ in Theorem 1 is replaced by $P_n = n^\tau$ for a positive constant $\tau < 1$. Specifically, we use [16, Theorem 4] and [16, Conjecture 1] confirmed by later work [17] to have the following lemma:

Lemma 15 *Under $P_n = n^\tau$ for a positive constant $\tau < 1$, with a sequence γ_n for all n defined through*

$$p_n P_n = \ln n + \gamma_n, \tag{156}$$

then

$$\lim_{n\to\infty} \mathbb{P}[G_b(n, P_n, p_n) \textit{ is } k\textit{-connected.}] = \begin{cases} 0, & \textit{if } \lim_{n\to\infty}\gamma_n = -\infty, \\ 1, & \textit{if } \lim_{n\to\infty}\gamma_n = \infty. \end{cases}$$

Note that different from (1), the scaling condition (156) above does not depend on k.

Lemma 15 has $P_n = n^\tau$ for a positive constant $\tau < 1$, while Theorem 1 has $P_n = \omega\big(n(\ln n)^5\big)$. We let δ denote an arbitrary constant with $\tau < \delta < \frac{\tau+1}{2}$ below. Lemma 15 shows that the probability of $G_b(n, n^\tau, n^{-\delta})$ (i.e., $G_b(n, P_n, p_n)$ with $P_n = n^\tau$ and $p_n = n^{-\delta}$) being k-connected asymptotically converges to 0 since γ_n specified by (156) satisfies

$$\gamma_n = p_n P_n - \ln n = n^{\tau-\delta} - \ln n \to -\infty, \text{ as } n \to \infty.$$

In contrast, Theorem 1 with $P_n = \omega\big(n(\ln n)^5\big)$ replaced by $P_n = n^\tau$ for a positive constant $\tau < 1$ presents that the probability of $G_b(n, n^\tau, n^{-\delta})$ (i.e., $G_b(n, P_n, p_n)$ with $P_n = n^\tau$ and $p_n = n^{-\delta}$) being k-connected asymptotically approaches to 1 because α_n defined by (1) satisfies

$$\begin{aligned}\alpha_n &= n{p_n}^2 P_n - [\ln n + (k-1)\ln\ln n] \\ &= n^{1+\tau-2\delta} - [\ln n + (k-1)\ln\ln n] \to \infty, \text{ as } n \to \infty.\end{aligned}$$

Hence, Lemma 15 shows the erroneousness of Theorem 1 with $P_n = \omega\big(n(\ln n)^5\big)$ replaced by $P_n = n^\tau$ for a positive constant $\tau < 1$. A future work is to investigate if we can use the intermediate range $\omega\big(n^\tau\big) = P_n = o\big(n(\ln n)^5\big)$ (e.g., $P_n = \Theta(n), \Theta\big(n(\ln n)^{-1}\big), \Theta(n \ln n)$) to replace $P_n = \omega\big(n(\ln n)^5\big)$ in Theorem 1. Finally, although the above discussion is for k-connectivity of a binomial random intersection graph, similar arguments also apply to the property of minimum node degree being at least k as well as k-robustness.

We have discussed the parameter conditions of Theorems 1 and 4 for binomial random intersection graphs. It is unclear whether $K_n = \Omega\big(\sqrt{\ln n}\,\big)$ in Theorem 2 and $K_n = \Omega\big((\ln n)^3\big)$ in Theorem 5 can be weakened since these conditions are also often enforced in related work [4], [22], [29]. Moreover, these conditions are applicable to secure sensor networks since it has been shown that K_n is at least on the order of $\ln n$ to have reasonable connectivity and resiliency simultaneously [19]. For a general random intersection graph, Yağan [21] recently obtains a zero–one law for connectivity and shows in [21, Section 3.3] that Theorem 3 with $\mathrm{Var}[X_n] = o\Big(\frac{\{\mathbb{E}[X_n]\}^2}{n(\ln n)^2}\Big)$ replaced by a broader condition does not hold.

To conclude, the parameter conditions in our theorems are not that conservative.

X. Related Work

For connectivity (i.e., k-connectivity with $k = 1$) in binomial random intersection graph $G_b(n, P_n, p_n)$, Rybarczyk establishes the exact probability [17] and a zero–one law [16], [17]. She further shows a zero–one law for k-connectivity [16], [17]. Our Theorem 1 provides not only a zero–one law, but also the exact probability to understand k-connectivity precisely.

For connectivity in uniform random intersection graph $G_u(n, P_n, K_n)$, Rybarczyk [15] derives the exact probability and a zero–one law, while Blackburn and Gerke [2], and Yağan and Makowski [22] also obtain zero–one laws. Rybarczyk [16] implicitly shows a zero–one law for k-connectivity in $G_u(n, P_n, K_n)$. Our Theorem 2 also gives a zero–one law. In addition, it gives the exact probability to provide an accurate understanding of k-connectivity.

For general random intersection graph $G(n, P_n, \mathcal{D}_n)$, Godehardt and Jaworski [12] investigate its degree distribution and Bloznelis *et al.* [4] explore its component evolution. Recently, Yağan [21] obtains a zero–one law for connectivity.

To date, there have not been results on (k-)robustness of random intersection graphs reported by others. As noted in Lemma 2, Zhang and Sundaram [24] present a zero–one law for k-robustness in an Erdős–Rényi graph.

For random intersection graphs in this paper, two nodes have an edge in between if their object sets share at least one object. A natural variant is to define graphs with edges only between nodes which have at least s objects in common (instead of just 1) for some positive integer s. Recent researches [5], [30] investigate k-connectivity in graphs under this definition.

XI. Conclusion and Future Work

Under a general random intersection graph model, we derive sharp zero–one laws for k-connectivity and k-robustness, as well as the asymptotically exact probability of k-connectivity, where k is an arbitrary positive integer. A future direction is to obtain the asymptotically exact probability of k-robustness for a precise characterization on the robustness strength.

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