On the Strengths of Connectivity and Robustness in General Random Intersection Graphs

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Abstract — Random intersection graphs have received much interest for nearly two decades, and have a wide range of applications including key predistribution in wireless sensor networks, and social network modeling. In this paper, we investigate the strengths of connectivity and robustness in a general random intersection graph model. Specifically, under such general model, we establish sharp zero–one laws for k-connectivity and k-robustness, as well as the asymptotically exact probability of k-connectivity, where k is an arbitrary positive integer. k-connectivity quantifies how resilient is the graph connectivity against node or edge failures; and k-robustness, recently introduced by Zhang and Sundaram [CDC 2012], measures the resiliency of message diffusion using only local information against node misbehavior. Our analytical findings constitute the first analysis of connectivity and robustness in a general random intersection graph. Moreover, for two specific instances of the general model, a binomial random intersection graph and a uniform random intersection graph, we provide the first results on the zero–one law for k-robustness, and on the asymptotically exact probability of k-connectivity.

I. INTRODUCTION

A. Graph Models

Random intersection graphs have been introduced in [12], [18], and several generalizations of the model are known [2]–[4], [10]. These models have been extensively analyzed in terms of various graph properties, including clustering [2], component evolution [3], [4] and degree distribution [10].

The model considered in this paper, hereafter referred to as a general random intersection graph, represents a prevalently studied generalization [2], [3], [10] of the random intersection graph. It is defined on an n-size node set $V = \{v_1, v_2, \ldots, v_n\}$ as follows. Each node $v_i$ ($i = 1, 2, \ldots, n$) is assigned an object set $S_i$ from an object pool $\mathcal{P}$ consisting of $P_n$ distinct objects, where $P_n$ is a function of $n$. Each object $S_i$ is constructed using the following two-step procedure: First, the size of $S_i$, $|S_i|$, is determined according to some probability distribution $\mathcal{D} : \{1, 2, \ldots, P_n\} \to [0, 1]$. Of course, we have $\sum_{x=1}^{P_n} \mathbb{P}[|S_i| = x] = 1$, with $\mathbb{P}[A]$ meaning the probability that event $A$ occurs. Next, $S_i$ is formed by selecting $|S_i|$ distinct objects uniformly at random from the object pool $\mathcal{P}$. In other words, conditioning on $|S_i| = s_i$, set $S_i$ is chosen uniformly among all $s_i$-size subsets of $\mathcal{P}$. This process is repeated independently for all object sets $S_1, \ldots, S_n$. Finally, an undirected edge is assigned between two nodes if and only if their corresponding object sets have at least one object in common; namely, distinct nodes $v_i$ and $v_j$ have an edge in between if and only if $S_i \cap S_j \neq \emptyset$. The graph defined through this adjacency notion is denoted by $G(n, P_n, \mathcal{D})$.

A specific instance of the general model $G(n, P_n, \mathcal{D})$, known as the binomial random intersection graph model, has been widely explored to date [2], [12], [16], [17]. Under this model, each object set $S_i$ is constructed by a Bernoulli-like mechanism, i.e., by adding each object to $S_i$ independently with probability $p_n$. Here $p_n$ is a function of $n$. The term “binomial” accounts for the fact that $|S_i|$ now follows a binomial distribution with $P_n$ as the number of trials and $p_n$ as the success probability in each trial. We denote the binomial random intersection graph by $G_b(n, P_n, p_n)$.

Another well-known instance of the general model $G(n, P_n, \mathcal{D})$ is the uniform random intersection graph [1], [8], [15], [19]. Under the uniform model, the probability distribution $\mathcal{D}$ concentrates on a single integer $K_n$, where $1 \leq K_n \leq P_n$; i.e., for each node $v_i$, the object set size $|S_i|$ equals $K_n$ with probability 1. We denote by $G_u(n, P_n, K_n)$ the uniform random intersection graph. Note that $P_n$ and $K_n$ are both functions of $n$.

B. (k-)Connectivity and (k-)Robustness

Connectivity and robustness are two fundamental graph properties. A graph is connected if there exists at least a path of edges between any two nodes [7]. To further characterize the strength of connectivity, a graph is said to be k-connected or have k-connectivity if each pair of nodes has no less than $k$ mutually node-disjoint path(s) in between [16]. By Menger’s theorem, an equivalent definition of a graph being k-connected is that after the deletion of any $(k - 1)$ nodes or edges, the remaining graph is still connected. Therefore, k-connectivity measures the resiliency of graph connectivity against node or edge failures.

For graph robustness, we use the definition introduced by Zhang and Sundaram [20], which has received much attention recently [13], [14], [21], [22]. Formally, a graph with a node set $V$ is k-robust or has k-robustness if at least one of (a) and (b) below hold for any non-empty and strict subset $T$ of $V$: (a) there exists at least a node $v_o \in T$ such that $v_o$ has no less than $k$ neighbors inside $\mathcal{V}\, T$; and (b) there exists at least a node $v_o \in \mathcal{V}\, T$ such that $v_o$ has no less than $k$ neighbors inside $T$. The property of k-robustness plays a key role in various dynamical processes over networks; e.g., in quantifying how resilient message dissemination is against
node misbehavior when only local information is used [20]. It is also tightly related to the global spreading probability of a contagion process [20].

C. Contributions and Organization

Our contributions in this paper are summarized as follows:

i) We derive sharp zero–one laws and asymptotically exact probabilities for $k$-connectivity in general random intersection graphs.

ii) We establish sharp zero–one laws for $k$-robustness in general random intersection graphs.

iii) For the two specific instances of the general graph model, a binomial random intersection graph and a uniform random intersection graph, our results on exact probabilities for $k$-connectivity and zero–one laws for $k$-robustness are the first in the literature.

The rest of the paper is organized as follows. We describe the applications of general random intersection graphs in the next subsection. Section II presents the results as theorems. Afterwards, we introduce auxiliary facts and lemmas in Section III, before establishing the theorems in Sections IV and V. Section VI details the proofs of the lemmas. We provide numerical experiments in Section VII. Section VIII reviews related work; and Section IX concludes the paper.

D. Applications

We explain below two different applications of general random intersection graphs, one in modeling key predistribution schemes in secure wireless sensor networks, the other in analyzing common-interest relations in social networks.

1) Key Predistribution in Secure Wireless Sensor Networks:

The uniform random intersection graph model $G_u(n, P_n, K_n)$ represents the Eschenauer–Gligor (EG) random key predistribution scheme [8], which is a typical solution to secure communications in wireless sensor networks. With $n$ nodes in graph $G_u(n, P_n, K_n)$ standing for $n$ sensors in a wireless network, each object in the object set of a sensor is a cryptographic key so that each sensor has $K_n$ keys selected uniformly at random from a pool comprising $P_n$ keys. Two sensors have to share at least one key to establish a communication link in between.

To capture the heterogeneousness among the numbers of key possession on sensors, we need to use the general random intersection graph model instead of the uniform model above. Such heterogeneousness are observed in the following scenarios: (a) the assigned numbers of keys on sensors vary prior to deployment in consideration of the heterogeneity of sensors’ memory [11]; (b) the numbers of keys on some sensors decrease since certain keys are revoked after being compromised [5]; and (c) the numbers of keys on sensors increase because of path key establishment, in which new path keys are generated on sensors [6], [8].

Our theoretical results on $k$-connectivity and $k$-robustness in general random intersection graphs enable us to design large-scale wireless sensor networks in terms of choosing the parameters to achieve certain levels of connectivity and robustness, and determining whether reliable message dissemination can be achieved when some sensors fail due to battery depletion or are captured by attackers [5], [6].

2) Common-Interest Relations in Social Networks:

We introduce in a recent work [23] the application of uniform random intersection graphs in modeling common-interest relation in social networks. A common interest between two individuals is represented by their selection of at least one common object from a large pool of available objects. Examples of objects include books, hobbies, movies, and professional activities. The uniform random intersection graph model assumes that people have the same number of interests, which may not hold in practice. By using the general random intersection graph model, we remove such assumption so that different individuals can have varying number of interests from a vast set of possibilities.

Our analytical findings on $k$-connectivity and $k$-robustness in general random intersection graphs allow us to answer what values should the parameters take so that the graph induced by the common-interest relation exhibits a desired strength of connectivity and robustness.

II. THE RESULTS

We present the results in theorems below. We defer the proofs of all theorems to Sections IV and V. Throughout the paper, $k$ is a positive integer and does not scale with $n$; and $e$ is the base of the natural logarithm function, $\ln$. All limits are understood with $\lim_{n \to \infty}$.

We use the standard Landau asymptotic notation $o(\cdot), O(\cdot), \omega(\cdot), \Omega(\cdot), \Theta(\cdot)$ and $\sim$; in particular, for two positive functions $f(n)$ and $g(n)$, $f(n) \sim g(n)$ signifies $\lim_{n \to \infty} f(n)/g(n) = 1$. For a random variable $X$, the terms $\mathbb{E}[X]$ and $\text{Var}[X]$ stand for its expected value and variance, respectively.

A. Zero–One Laws and Exact Probabilities for Asymptotic $k$-Connectivity

We describe zero–one laws and exact probabilities for asymptotic $k$-connectivity in different graphs below.

1) $k$-Connectivity in General Random Intersection Graphs:

The following Theorem 1 gives a zero–one law and the exact probability for asymptotic $k$-connectivity in a general random intersection graph.

**Theorem 1:** Consider a general random intersection graph $G(n, P_n, D)$. Let $X$ be a random variable following probability distribution $D$. With a sequence $\alpha_n$ for all $n$ defined through

$$\frac{\mathbb{E}[X]}{\text{Var}(X)} = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},$$

if $\mathbb{E}[X] = \Omega(\sqrt{\ln n})$, $\text{Var}[X] = o\left(\frac{(\mathbb{E}[X])^2}{\ln n \ln n}\right)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \to \infty} \mathbb{P}\left[\text{Graph } G(n, P_n, D) \text{ is } k\text{-connected.}\right] = \begin{cases} 0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty, \\ e^{-\frac{\alpha_n}{(k-1)n}}, & \text{if } \lim_{n \to \infty} \alpha_n = \alpha^* \in (-\infty, \infty). \end{cases}$$

\[\square\]
2) $k$-Connectivity in Binomial Random Intersection Graphs:
Theorem 2 below describes a zero–one law and the exact probability for asymptotic $k$-connectivity in a binomial random intersection graph.

**Theorem 2:** Consider a binomial random intersection graph $G_b(n, P_n, p_n)$. With a sequence $\alpha_n$ for all $n$ defined through

$$p_n^2 P_n = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},$$  \hspace{1cm} (2)

if $P_n = \omega(n(\ln n)^3)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \to \infty} \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-connected}.] = \left\{ \begin{array}{ll}
0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\
1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty, \\
e^{-\frac{\alpha_n}{n}}, & \text{if } \lim_{n \to \infty} \alpha_n = \alpha^* \in (-\infty, \infty).
\end{array} \right.$$  \hspace{1cm} \square

**Remark 1:** As we will explain in Section IV-B within the proof of Theorem 2, for the zero–one law, the condition $P_n = \omega(n(\ln n)^3)$ can be weakened as $P_n = \Omega(n(\ln n)^3)$, while we enforce $P_n = \omega(n(\ln n)^3)$ for the asymptotically exact probability result.

3) $k$-Connectivity in Uniform Random Intersection Graphs:
Theorem 3 below presents a zero–one law and the exact probability for asymptotic $k$-connectivity in a uniform random intersection graph.

**Theorem 3:** Consider a uniform random intersection graph $G_u(n, P_n, K_n)$. With a sequence $\alpha_n$ for all $n$ defined through

$$K_n^2 P_n = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},$$  \hspace{1cm} (3)

if $K_n = \Omega(\sqrt{\ln n})$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \to \infty} \mathbb{P}[\text{Graph } G_u(n, P_n, K_n) \text{ is } k\text{-connected}.] = \left\{ \begin{array}{ll}
0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\
1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty, \\
e^{-\frac{\alpha_n}{(k-1)n}}, & \text{if } \lim_{n \to \infty} \alpha_n = \alpha^* \in (-\infty, \infty).
\end{array} \right.$$  \hspace{1cm} \square

**B. Zero–One Laws for Asymptotic k-Robustness**

We detail zero–one laws for asymptotic $k$-robustness in different graphs below.

1) $k$-Robustness in General Random Intersection Graphs:
Theorem 4 as follows describes a zero–one law for asymptotic $k$-robustness in a general random intersection graph.

**Theorem 4:** Consider a general random intersection graph $G(n, P_n, D)$. Let $X$ be a random variable following probability distribution $D$. With a sequence $\alpha_n$ for all $n$ defined through

$$\frac{\mathbb{E}[X]^2}{P_n} = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},$$  \hspace{1cm} (4)

if $\mathbb{E}[X] = \Omega((\ln n)^3)$, $\mathbb{V}[X] = o\left(\frac{\mathbb{E}[X]^2}{n(\ln n)^2}\right)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \to \infty} \mathbb{P}[\text{Graph } G(n, P_n, D) \text{ is } k\text{-robust}.] = \left\{ \begin{array}{ll}
0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\
1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty.
\end{array} \right.$$  \hspace{1cm} \square

2) $k$-Robustness in Binomial Random Intersection Graphs:
Theorem 5 below shows a zero–one law for asymptotic $k$-robustness in a binomial random intersection graph.

**Theorem 5:** Consider a binomial random intersection graph $G_b(n, P_n, p_n)$. With a sequence $\alpha_n$ for all $n$ defined through

$$p_n^2 P_n = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},$$  \hspace{1cm} (5)

if $P_n = \Omega(n(\ln n)^3)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \to \infty} \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-robust}.] = \left\{ \begin{array}{ll}
0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\
1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty.
\end{array} \right.$$  \hspace{1cm} \square

3) $k$-Robustness in Uniform Random Intersection Graphs:
The following Theorem 5 elaborates a zero–one law for asymptotic $k$-robustness in a uniform random intersection graph.

**Theorem 6:** Consider a uniform random intersection graph $G_u(n, P_n, K_n)$. With a sequence $\alpha_n$ for all $n$ defined through

$$K_n^2 P_n = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},$$  \hspace{1cm} (6)

if $K_n = \Omega((\ln n)^3)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \to \infty} \mathbb{P}[\text{Graph } G_u(n, P_n, K_n) \text{ is } k\text{-robust}.] = \left\{ \begin{array}{ll}
0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\
1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty.
\end{array} \right.$$  \hspace{1cm} \square

In view of Theorems 1–6, for each general/binomial/uniform random intersection graph, its $k$-connectivity and $k$-robustness asymptotically obey the same zero–one laws. Moreover, these zero–one laws are all sharp since $|\alpha_n|$ can be much smaller compared to $\ln n$; e.g., even $\alpha_n = \pm e \cdot \ln \ln \cdots \ln n$ with an arbitrary positive constant $c$ and an arbitrary number of $\ln$ satisfies $\lim_{n \to \infty} \alpha_n = \pm \infty$.

By [20, Lemma 1], $k$-robustness implies $k$-connectivity; i.e., any $k$-robust graph is also $k$-connected. It is also worth noting that a $k$-connected graph is not necessarily $k$-robust; and an example of such graph is given in [20, Figure 1].

**III. AUXILIARY FACTS AND LEMMAS**

We present a few facts and lemmas which are used to demonstrate the theorems. To begin with, recalling that $k$ does not scale with $n$, we obtain the following Facts 1 and 2.
Their proofs are omitted here since they are straightforward to derive.

**Fact 1:** For \( |\alpha_n| = o(\ln n) \), it holds that  
\[
\frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n} \sim \frac{\ln n}{n}.
\]

**Fact 2:** For \( |\alpha_n| = o(\ln n) \), we have  
\[
\frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n} \cdot \left(1 \pm O \left(\frac{1}{\ln n}\right)\right) = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n} \pm O(1),
\]
and  
\[
\frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n} \cdot \left(1 \pm o \left(\frac{1}{\ln n}\right)\right) = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n} \pm o(1).
\]

We detail the lemmas below and defer their proofs to Section VI. Lemma 1 below based on [20, Theorem 3] and [16, Facts 3 and 7] provides the result on \( k \)-robustness of an Erdős-Rényi graph, a model introduced by Erdős and Rényi [7] and Gilbert [9]. An Erdős-Rényi graph \( G(n, \hat{p}_n) \) is defined on a set of \( n \) nodes such that any two nodes establish an edge in between independently with probability \( \hat{p}_n \).

**Lemma 1:** For an Erdős-Rényi graph \( G(n, \hat{p}_n) \), if  
\[
\hat{p}_n = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n}{n},
\]
then  
\[
\lim_{n \to \infty} \mathbb{P}[G(n, \hat{p}_n) \text{ is } k\text{-robust.}] = \begin{cases} 0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty. \end{cases}
\]

**Remark 2:** We will only use the one law of Lemma 1. □

Throughout Lemmas 2–5, \( \mathcal{I} \) is an arbitrary monotone increasing graph property, where a graph property is called monotone increasing if it holds under the addition of edges.

**Lemma 2:** Let \( X \) be a random variable with probability distribution \( \mathcal{D} \). If \( \text{Var}[X] = o \left( \frac{\mathbb{E}[X]^2}{n \ln(n)} \right) \), then there exists \( \epsilon_n = o \left( \frac{1}{\ln n} \right) \) such that  
\[
\mathbb{P}[\text{ Graph } G(n, P_n, \mathcal{D}) \text{ has } \mathcal{I}.] 
\geq \mathbb{P}[\text{ Graph } G_u(n, P_n, (1 - \epsilon_n)\mathbb{E}[X]) \text{ has } \mathcal{I}.] - o(1),
\]
and  
\[
\mathbb{P}[\text{ Graph } G(n, P_n, \mathcal{D}) \text{ has } \mathcal{I}.] 
\leq \mathbb{P}[\text{ Graph } G_u(n, P_n, (1 + \epsilon_n)\mathbb{E}[X]) \text{ has } \mathcal{I}.] + o(1).
\]

**Lemma 3:** If \( p_n = O \left( \frac{1}{\ln n} \right) \) and \( p_n^2 P_n = O \left( \frac{1}{\ln n} \right) \), then there exists \( \hat{p}_n = p_n^2 P_n \cdot \left[1 - O \left( \frac{1}{\ln n} \right)\right] \) such that  
\[
\mathbb{P}[\text{ Graph } G_b(n, P_n, \hat{p}_n) \text{ has } \mathcal{I}.] 
\geq \mathbb{P}[\text{ Graph } G(n, \hat{p}_n) \text{ has } \mathcal{I}.] - o(1).
\]

**Lemma 4** ([3, Lemma 4]): If \( p_n P_n = \omega(\ln n) \), and for all \( n \) sufficiently large\(^1\),  
\[
K_{n,a} \leq p_n P_n - \sqrt{3(p_n P_n + \ln n) \ln n},
\]
\[
K_{n,b} \geq p_n P_n + \sqrt{3(p_n P_n + \ln n) \ln n},
\]
then  
\[
\mathbb{P}[\text{ Graph } G_u(n, P_n, K_{n,a}) \text{ has } \mathcal{I}.] - o(1) 
\leq \mathbb{P}[\text{ Graph } G_b(n, P_n, p_n) \text{ has } \mathcal{I}.] 
\leq \mathbb{P}[\text{ Graph } G_u(n, P_n, K_{n,b}) \text{ has } \mathcal{I}.] + o(1).
\]

**Lemma 5:** If \( K_n = \omega(\ln n) \) and \( p_n = K_n \left( 1 - \sqrt{\frac{3 \ln n}{K_n}} \right) \), then  
\[
\mathbb{P}[\text{ Graph } G_u(n, P_n, K_n) \text{ has } \mathcal{I}.] 
\geq \mathbb{P}[\text{ Graph } G_b(n, P_n, p_n) \text{ has } \mathcal{I}.] - o(1).
\]

**IV. Establishing Theorems 1–3**

Theorems 1–3 describe results on \( k \)-connectivity for various graphs. We will apply Theorem 3 to prove Theorems 1 and 2, respectively.

**A. The Proof of Theorem 1**

We demonstrate Theorem 1 with the help of Theorem 3, the proof of which is detailed in Section IV-C.

For any \( \epsilon_n = o \left( \frac{1}{\ln n} \right) \), it is clear that  
\[
(1 \pm \epsilon_n)^2 = 1 \pm 2\epsilon_n + \epsilon_n^2 = 1 \pm o \left( \frac{1}{\ln n} \right).
\]

We recall conditions (1) and \( |\alpha_n| = o(\ln n) \), which together with (10) and Fact 2 yield  
\[
\frac{(1 \pm \epsilon_n)\mathbb{E}[X]^2}{P_n} = \frac{\ln n + (k - 1) \ln \ln n + \alpha_n \pm o(1)}{P_n}\]

With \( \mathbb{E}[X] = \Omega \left( \sqrt{\ln n} \right) \) and \( \epsilon_n = o \left( \frac{1}{\ln n} \right) \), it follows that  
\[
(1 \pm \epsilon_n)\mathbb{E}[X] = \left[1 \pm o \left( \frac{1}{\ln n} \right)\right] \cdot \Omega \left( \sqrt{\ln n} \right) = \Omega \left( \sqrt{\ln n} \right).
\]

Given (11) (12) and \( |\alpha_n| = o(\ln n) \), we use Theorem 3 to obtain  
\[
\lim_{n \to \infty} \mathbb{P}[G_u(n, P_n, (1 \pm \epsilon_n)\mathbb{E}[X]) \text{ is } k\text{-connected.}] 
\geq \begin{cases} 0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty, \\ e^{-\frac{\alpha^*}{(k-1)n}}, & \text{if } \lim_{n \to \infty} \alpha_n = \alpha^* \in (-\infty, \infty). \end{cases}
\]

Since \( k \)-connectivity is a monotone increasing graph property [16], Theorem 1 is proved by (13) and Lemma 2. □

\(^1\)The term “for all \( n \) sufficiently large” means “for any \( n \geq N \), where \( N \) is a positive integer selected appropriately”. 
B. The Proof of Theorem 2

From Lemma 4 and Theorem 3, the proof of Theorem 2 is completed once we show that with $K_{n,±}$ defined by

$$K_{n,±} = p_n P_n \pm \sqrt{3(p_n P_n + \ln n) \ln n}, \quad (14)$$

under conditions of Theorem 2, we have $K_{n,±} = \Omega(\sqrt{\ln n})$ and with $\alpha_{n,±}$ defined by

$$\frac{K_{n,±}^2}{P_n} = \ln n + (k - 1) \ln \ln n + \alpha_{n,±}, \quad (15)$$

then

$$\alpha_{n,±} = \alpha_n \pm o(1). \quad (16)$$

From conditions (2) and $|\alpha_n| = o(\ln n)$, and Fact 1, it is clear that

$$p_n^2 P_n \sim \frac{\ln n}{n}. \quad (17)$$

Substituting (17) and condition $P_n = \omega(n(\ln n)^3)$ into (14), it holds that

$$K_{n,±} = \omega((\ln n)^3) = \Omega(\sqrt{\ln n}), \quad (18)$$

and

$$\frac{K_{n,±}^2}{P_n} = p_n^2 P_n \cdot \left[1 + o\left(\frac{1}{\ln n}\right)\right]. \quad (19)$$

Then from (2) (15) (19) and Fact 2, we obtain (16). As explained before, with (15) (16) and (18), Theorem 2 is proved from Lemma 4 and Theorem 3.

As noted in Remark 1 after Theorem 2, to prove only the zero–one law but not the asymptotically exact probability result in Theorem 2, the condition $P_n = \omega(n(\ln n)^3)$ can be weakened as $P_n = \Omega(n(\ln n)^5)$. This can be seen by the argument that under $P_n = \Omega(n(\ln n)^5)$, (16) can be weakened as $\alpha_{n,±} = \alpha_n \pm O(1)$, which still used to establish the zero–one law.

C. The Proof of Theorem 3

We derive in [24] the asymptotically exact probability and a zero–one law for $k$-connectivity in graph $G(n, \tilde{p}_n)$, which is the superposition of an Erdős–Rényi graph $G(n, \tilde{p}_n)$ on a uniform random intersection graph $G_u(n, P_n, K_n)$. Setting $\tilde{p}_n = 1$ in $G(n, \tilde{p}_n)$, it is straightforward to see $G(n, \tilde{p}_n) \cap G_u(n, P_n, K_n) = G_u(n, P_n, K_n)$. Then with $\tilde{p}_n = 1$, we obtain from [24, Theorem 1] that if $P_n = \Omega(n)$ and

$$1 - \left(\frac{P_n - K_n}{K_n}\right) \left(\frac{P_n}{K_n}\right) = \frac{\ln n + (k - 1) \ln \ln n + \beta_n}{n}, \quad (20)$$

then

$$\lim_{n \to \infty} P\left[G_u(n, P_n, K_n) \text{ is } k\text{-connected.}\right] = \begin{cases} 0, & \text{if } \lim_{n \to \infty} \beta_n = -\infty, \\ 1, & \text{if } \lim_{n \to \infty} \beta_n = \infty, \\ e^{-e^{-s}}, & \text{if } \lim_{n \to \infty} \beta_n = \beta^* \in (-\infty, \infty). \end{cases} \quad (21)$$

Note that if $\beta_n = \alpha_n \pm o(1)$, then (i) $\lim_{n \to \infty} \beta_n$ exists if and only if $\lim_{n \to \infty} \alpha_n$ exists; and (ii) when they both exist, $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \alpha_n$. Therefore, Theorem 3 is proved once we show $P_n = \Omega(n)$ and (20) with $\beta_n = \alpha_n \pm o(1)$ given conditions $K_n = \Omega(\sqrt{\ln n})$, $|\alpha_n| = o(\ln n)$ and (3).

From $|\alpha_n| = o(\ln n)$, (3) and Fact 1, it holds that

$$\frac{K_n^2}{P_n} \sim \frac{\ln n}{n},$$

which along with $K_n = \Omega(\sqrt{\ln n})$ yields

$$P_n \sim \frac{nK_n^2}{\ln n} = \Omega(n).$$

We derive in [23, Lemma 8] that

$$1 - \left(\frac{P_n - K_n}{K_n}\right) \left(\frac{P_n}{K_n}\right) = \frac{K_n^2}{P_n} \cdot \left[1 + o\left(\frac{K_n^2}{P_n}\right)\right]. \quad (23)$$

Applying (22) to (23),

$$1 - \left(\frac{P_n - K_n}{K_n}\right) \left(\frac{P_n}{K_n}\right) = \frac{K_n^2}{P_n} \cdot \left[1 + o\left(\frac{1}{\ln n}\right)\right],$$

which together with (3) and Fact 2 leads to (20) with condition $\beta_n = \alpha_n \pm o(1)$. Since we have proved $P_n = \Omega(n)$ and (20) with $\beta_n = \alpha_n \pm o(1)$, Theorem 3 follows from (21).

V. Establishing Theorems 4–6

Theorems 4–6 present results on $k$-robustness for various graphs. We will use Theorem 5 to demonstrate Theorem 6, and then apply Theorem 6 to establish Theorem 4.

A. The Proof of Theorem 4

Similar to the process of proving Theorem 1 with the help of Theorem 3, we demonstrate Theorem 4 using Theorem 6, the proof of which is given in Section V-C.

Note that condition (4) is the same as (1); and condition $|\alpha_n| = o(\ln n)$ holds. Then as shown in Theorem 1, for any $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, from (1) (10), $|\alpha_n| = o(\ln n)$ and Fact 2, we obtain (11) here. With $E[X] = \Omega((\ln n)^3)$ and $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, it follows that

$$(1 \pm \epsilon_n)E[X] = 1 \pm o\left(\frac{1}{\ln n}\right) \cdot \Omega((\ln n)^3) = \Omega((\ln n)^3).$$

(24)

Given (11) and (24), we use Theorem 6 to obtain that for $E[X] = \Omega((\ln n)^3)$ and any $\epsilon_n = o\left(\frac{1}{\ln n}\right)$,

$$\lim_{n \to \infty} P\left[G_u(n, P_n, (1 \pm \epsilon_n)E[X]) \text{ is } k\text{-robust.}\right] = \begin{cases} 0, & \text{if } \lim_{n \to \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \to \infty} \alpha_n = \infty. \end{cases} \quad (25)$$

Since $k$-robustness is a monotone increasing graph property according to [13, Lemma 3], Theorem 4 is proved by (25) and Lemma 2.


B. The Proof of Theorem 5

Since \( k \)-robustness implies \( k \)-connectivity by [20, Lemma 1], the zero law of Theorem 5 is clear from Theorem 2 and Remark 1 in view that under conditions of Theorem 5, if \( \lim_{n \to \infty} \alpha_n = -\infty \),

\[
P \left[ G_b(n, P_n, p_n) \text{ is } k \text{-robust} \right] 
\leq P \left[ G_b(n, P_n, p_n) \text{ is } k \text{-connected} \right] \to 0, \text{ as } n \to \infty.
\]

(26)

Below we prove the one law of Theorem 5. Note that (5) is the same as (2); and we have condition \( |\alpha_n| = o(\ln n) \). Then as proved in Theorem 2, given (2) and \( |\alpha_n| = o(\ln n) \), we obtain (17), which together with condition \( P_n = \Omega(n(\ln n)^5) \) leads to

\[
p_n \sim \sqrt{\frac{\ln n}{n P_n}} = O\left(\sqrt{\frac{\ln n}{n^2(\ln n)^5}}\right) = O\left(\frac{1}{n(\ln n)^2}\right).
\]

(27)

Then we apply Lemmas 1 and 3, and condition (5) to derive that there exists \( \tilde{p}_n \)

\[
p_n \sim \sqrt{\frac{\ln n}{n \tilde{p}_n}} = O\left(\sqrt{\frac{\ln n}{n^2 \tilde{p}_n}}\right) = O\left(\frac{1}{n(\ln n)^2}\right).
\]

(28)

The proof of Theorem 5 is completed via (26) and (28). ■

C. The Proof of Theorem 6

The zero law of Theorem 6 is proved below by an approach similar to that of Theorem 5. Since \( k \)-robustness implies \( k \)-connectivity by [20, Lemma 1], the zero law of Theorem 6 is clear from Theorem 3 in view that under conditions of Theorem 6, if \( \lim_{n \to \infty} \alpha_n = -\infty \),

\[
P \left[ G_u(n, P_n, K_n) \text{ is } k \text{-robust} \right] 
\leq P \left[ G_u(n, P_n, K_n) \text{ is } k \text{-connected} \right] \to 0, \text{ as } n \to \infty.
\]

(29)

Below we establish the one law of Theorem 6 with the help of Theorem 5. Given \( K_n = \Omega((\ln n)^3) = \omega(\ln n) \), we use Lemma 5 to obtain that with \( p_n \) set by

\[
p_n = \frac{K_n}{\tilde{p}_n} \left(1 - \frac{3 \ln n}{K_n}\right), \quad (30)
\]

it holds that

\[
P \left[ \text{Graph } G_u(n, P_n, K_n) \text{ is } k \text{-robust} \right] 
\geq P \left[ \text{Graph } G_b(n, P_n, p_n) \text{ is } k \text{-robust} \right] - o(1).
\]

(31)

Note that (6) is the same as (3); and \( |\alpha_n| = o(\ln n) \) holds as a condition. Then as shown in Theorem 3, from (3), \( |\alpha_n| = o(\ln n) \) and Fact 2, we obtain (22) here, which together with \( K_n = \Omega((\ln n)^3) \) results in

\[
p_n \sim \frac{n K_n^2}{\ln n} = \Omega(n(\ln n)^5), \quad (32)
\]

From \( K_n = \Omega((\ln n)^3) \) and (30), it follows that

\[
p_n^2 P_n = \left[ \frac{K_n}{\tilde{p}_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right)\right]^2 \cdot P_n
\]

\[
= \frac{K_n^2}{\tilde{p}_n} \left[1 - O\left(\frac{1}{\ln n}\right)\right]. \quad (33)
\]

By (6) (33) and Fact 2, it is clear that

\[
p_n^2 P_n = \frac{\ln n + (k - 1) \ln n + \alpha_n - O(1)}{n}
\]

(34)

Given (32) (34) and \( |\alpha_n| = o(\ln n) \), we use Theorem 5 and (31) to get that if \( \lim_{n \to \infty} \alpha_n = -\infty \),

\[
P \left[ G_u(n, P_n, K_n) \text{ is } k \text{-robust} \right] \to 1, \text{ as } n \to \infty. \quad (35)
\]

The proof of Theorem 6 is completed via (29) and (35). ■

VI. Establishing Lemmas in Section III

A. The Proof of Lemma 1

To establish Lemma 1, we note the following two arguments. (a) Under condition (7), the desired result (8) with \( |\alpha_n| = o(\ln n) \) is demonstrated by [20, Theorem 3]. (b) By [16, Facts 3 and 7], for any monotone increasing graph property \( I \), the probability that graph \( G(n, \tilde{p}_n) \) has property \( I \) is non-decreasing as \( \tilde{p}_n \) increases. Then in view of (a) and (b) above, it is straightforward to derive (8) given (7), regardless of \( |\alpha_n| = o(\ln n) \).

B. The Proof of Lemma 2

According to [3, Lemma 3], for any monotone increasing graph property \( I \) and any \( |\epsilon_n| < 1 \),

\[
P \left[ G(n, P_n, D) \text{ has } I \right] - P \left[ G_u(n, P_n, (1 - \epsilon_n)E[X]) \text{ has } I \right]
\geq \left\{ 1 - P[X < (1 - \epsilon_n)E[X]] \right\}^n - 1, \quad (36)
\]

and

\[
P \left[ G(n, P_n, D) \text{ has } I \right] - P \left[ G_u(n, P_n, (1 + \epsilon_n)E[X]) \text{ has } I \right]
\leq 1 - \left\{ 1 - P[X > (1 + \epsilon_n)E[X]] \right\}^n. \quad (37)
\]

By (36) (37) and the fact that \( \lim_{n \to \infty} (1 - m_n)^n = 1 \) for \( m_n = o\left(\frac{1}{n}\right) \) (this can be proved by a simple Taylor series expansion as in [23, Fact 2]), the proof of Lemma 2 is completed once we demonstrate that with \( \text{Var}[X] = o\left(\frac{E[X]^2}{n(\ln n)^2}\right) \), there exists \( \epsilon_n = o\left(\frac{1}{\ln n}\right) \) such that

\[
P \left[ X < (1 - \epsilon_n)E[X] \right] = o\left(\frac{1}{n}\right), \quad (38)
\]

and

\[
P \left[ X > (1 + \epsilon_n)E[X] \right] = o\left(\frac{1}{n}\right). \quad (39)
\]

To prove (38) and (39), we apply Chebyshev’s inequality to have

\[
P \left[ |X - E[X]| > \epsilon_n E[X] \right] \leq \frac{\text{Var}[X]}{\epsilon_n^2 E[X]^2}. \quad (40)
\]
We set \( \epsilon_n \) by
\[
\epsilon_n = \sqrt{\frac{n \text{Var}[X]}{E[X]}} \cdot \frac{1}{\sqrt{\ln n}}.
\]
Then given condition
\[
\text{Var}[X] = o\left(\frac{E[X]}{n\ln n}\right)^2 
\]
we obtain
\[
\epsilon_n = o\left(\frac{1}{(\ln n)^2}\right) \cdot \frac{1}{\sqrt{\ln n}} = o\left(\frac{1}{\ln n}\right), \tag{41}
\]
and
\[
\text{Var}[X] \left(\epsilon_n E[X]\right)^2 = \frac{\text{Var}[X]}{n \{E[X]\}^2} \cdot \ln n = o\left(\frac{1}{n}\right). \tag{42}
\]
By (40) (41) and (42), it is straightforward to see that (38) and (39) hold with
\( \epsilon_n = o\left(\frac{1}{\ln n}\right) \). Therefore, we have completed the proof of Lemma 2.

C. The Proof of Lemma 3

In view of [16, Lemma 3], if \( p_n^2 P_n < 1 \) and \( p_n = O\left(\frac{1}{n}\right) \), with \( \hat{p}_n \) defined through
\[
\hat{p}_n := p_n^2 P_n \cdot \left(1 - n p_n + 2 p_n - \frac{p_n^2 P_n}{2}\right), \tag{43}
\]
then (9) follows. Given conditions \( p_n = O\left(\frac{1}{n \ln n}\right) \) and \( p_n^2 P_n = O\left(\frac{1}{n \ln n}\right) \) in Lemma 3, \( p_n^2 P_n < 1 \) and \( p_n = O\left(\frac{1}{n}\right) \) clearly hold. Then Lemma 3 is proved once we show \( \hat{p}_n \) defined by (43) satisfies \( \hat{p}_n = p_n^2 P_n \cdot \left[1 - O\left(\frac{1}{n \ln n}\right)\right] \), which is easy to see via
\[
-n p_n + 2 p_n - \frac{p_n^2 P_n}{2}
= (-n + 2) \cdot O\left(\frac{1}{n \ln n}\right) - \frac{1}{2} \cdot O\left(\frac{1}{n \ln n}\right) = -O\left(\frac{1}{n \ln n}\right).
\]
Hence, the proof of Lemma 3 is completed.

D. The Proof of Lemma 5

We use Lemma 4 to prove Lemma 5. From conditions
\( K_n = \omega\left(\ln n\right) \) and \( p_n = \frac{K_n}{T_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right) \), we obtain
\[
p_n P_n = K_n \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right) = \omega\left(\ln n\right);
\]
and for all \( n \) sufficiently large,
\[
K_n - \left[p_n P_n + \sqrt{3(p_n P_n + \ln n) \ln n}\right]
= K_n \sqrt{\frac{3 \ln n}{K_n}} - \left[3 \left[K_n \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right) + \ln n\right]\ln n
= \sqrt{3 K_n \ln n} - \sqrt{3 K_n \ln n \left(\sqrt{\ln n - \sqrt{3 K_n}}\right)}\ln n
\geq \sqrt{3 K_n \ln n} - \sqrt{3 K_n \ln n}
= 0.
\]
Then by Lemma 4, Lemma 5 is now established.

VII. NUMERICAL EXPERIMENTS

We present numerical experiments in the non-asymptotic regime to confirm our theoretical results.

Figure 1 depicts the probability that binomial random intersection graph \( G_b(n, P, p) \) has \( k \)-connectivity or \( k \)-robustness, for \( k = 2, 6 \). Figure 2 does similar with \( k = 3, 4 \) for uniform random intersection graph \( G_u(n, P, K) \). In all set of experiments, we fix the number of nodes at \( n = 2,000 \) and the object pool size \( P = 20,000 \). For each pair \((n, P, p)\) (resp., \((n, P, K)\)), we generate 1,000 independent samples of \( G_b(n, P, p) \) (resp., \( G_u(n, P, K) \)) and count the number of times that the obtained graphs are \( k \)-connected or \( k \)-robust.

Then the count divided by 1,000 becomes the corresponding empirical probability. As illustrated in Figures 1 and 2, we notice the evident threshold behavior in the probabilities of \( k \)-connectivity and \( k \)-robustness. Also, for each \( k \), the curves of \( k \)-connectivity and \( k \)-robustness are close to each other. These observations are in agreement with our analytical results.
VIII. RELATED WORK

For connectivity (i.e., \( k \)-connectivity with \( k = 1 \)) in binomial random intersection graph \( G(n, P_n, p_n) \), Rybarczyk establishes the exact probability [17] and a zero–one law [16], [17]. She further shows a zero–one law for \( k \)-connectivity [16], [17], proving that if \( P_n = n^\beta \) for some constant \( \beta > 1 \) and \( p_n^2 P_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{\ln n} \), then graph \( G(n, P_n, p_n) \) is almost surely \( k \)-connected (resp., not \( k \)-connected) if \( \lim \alpha_n = \infty \) (resp., \( \lim \alpha_n = -\infty \)).

Our Theorem 2 demonstrates not only a zero–one law as well, but also the exact probability to deliver a precise understanding of \( k \)-connectivity; namely, if \( P_n = \omega(n(\ln n)^5) \) and \( p_n^2 P_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{\ln n} \), then graph \( G(n, P_n, p_n) \) is \( k \)-connected asymptotically converges to \( e^{-\frac{1}{\alpha_n^2}} \) (resp, 1 and 0) if \( \lim \alpha_n = \alpha^* \in (-\infty, \infty) \) (resp., \( \lim \alpha_n = 0 \) and \( \lim \alpha_n = -\infty \)).

For connectivity in uniform random intersection graph \( G_u(n, P_n, K_n) \), Rybarczyk [15] derives the exact probability and a zero–one law, while Blackburn and Gerke [1], and Yağan and Makowski [19] also obtain zero–one laws. Rybarczyk [16] implicitly shows a zero–one law for \( k \)-connectivity in \( G_u(n, P_n, K_n) \). Specifically, her implicit result is that if \( K_n \geq n^\gamma \sqrt{\ln n} \) for some positive constant \( \gamma \) and \( K_n^2 = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{\ln n} \), then graph \( G_u(n, P_n, K_n) \) is almost surely \( k \)-connected (resp., not \( k \)-connected) if \( \lim \alpha_n = \infty \) (resp., \( \lim \alpha_n = -\infty \)). Our Theorem 3 describes not only a zero–one law as well, but also the exact probability to provide an accurate understanding of \( k \)-connectivity; namely, under conditions \( K_n = \Omega(\sqrt{\ln n}) \) and \( K_n^2 = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{\ln n} \), with \( \alpha_n = o(\ln n) \), the probability that \( G_u(n, P_n, K_n) \) is \( k \)-connected asymptotically tends to \( e^{-\frac{1}{\alpha_n^2}} \) (resp, 1 and 0) if \( \lim \alpha_n = \alpha^* \in (-\infty, \infty) \) (resp., \( \lim \alpha_n = 0 \) and \( \lim \alpha_n = -\infty \)).

For general random intersection graph \( G(n, P_n, D) \), Goedehardt; and Jaworski [10] investigate its degree distribution and Bloznelis et al. [3] explore its component evolution, but neither a zero–one law nor the asymptotically exact probability of its \( k \)-connectivity property has been reported prior to our work.

To the best of our knowledge, there has not been any work on \((k)\)-robustness of random intersection graphs by others. As noted in Lemma 1, Zhang and Sundaram [20] present a zero–one law for \( k \)-robustness in an Erdős-Rényi graph.

IX. CONCLUSION AND FUTURE WORK

Under a general random intersection graph model, we derive sharp zero–one laws for \( k \)-connectivity and \( k \)-robustness, as well as the asymptotically exact probability of \( k \)-connectivity, where \( k \) is an arbitrary positive integer. A future direction is to obtain the asymptotically exact probability of \( k \)-robustness for a precise characterization on the robustness strength.

References