Randomness in Computing: Applying Random Numbers

Yesterday:

- Randomness is hard to define
- Randomness is harder to achieve
- Define tests for acceptable randomness
- Often Pseudo Random is random enough:

Yesterday

- Linear Congruential Generators (LCGs)
- We can generate a series of numbers, all different, that looks random even though it isn't
- If we choose appropriate constants for our LCG, then we can generate a very long sequence before numbers begin to repeat. The length of the sequence is its period
- To generate random numbers in Python we can use randint(x,y), which generates a random integer between x and y.

$$x_{i+1} = (a\, x_i + c) \;\%\; m$$

Picking the constants a, c, m

(1) *c* and *m* relatively prime

(2) *a*-1 divisible by all prime factors of *m*

(3) if *m* a multiple of 4, so is *a*-1

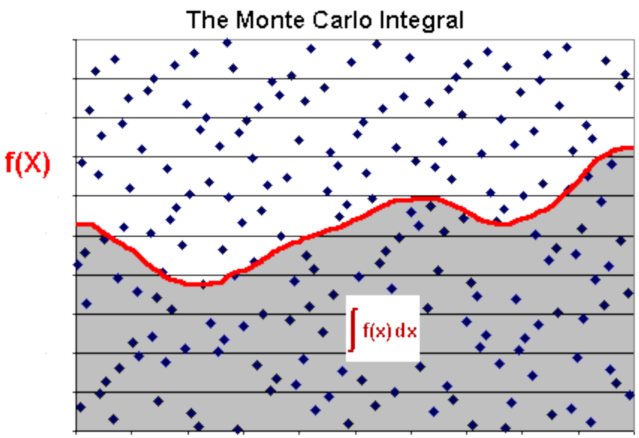
- Example: prng_fn (*a* = 1, *c* = 7, *m* = 12)
  - Factors of 7: 1, 7   Factors of 12: 1, 2, 3, 4, 6, 12
  - 0 is divisible by all prime factors of 12 → true
  - if 12 is a multiple of 4 , then 0 is also a multiple of 4 → true
- prng_fn will have a period of 12

Today: Monte Carlo Methods

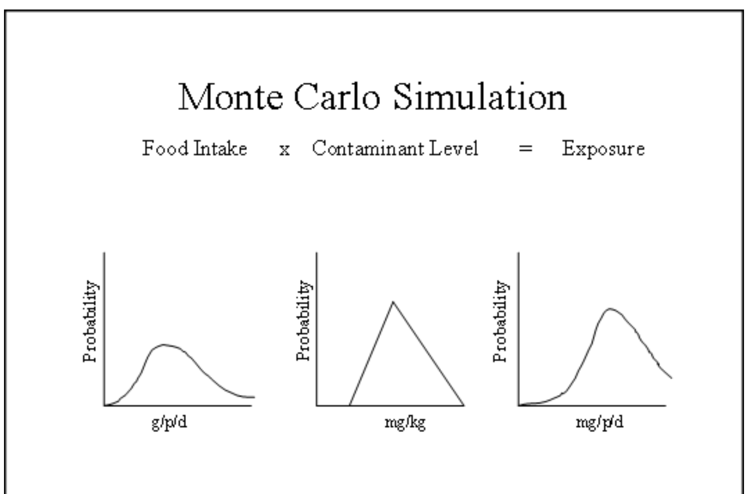
Idea: run many experiments with random inputs to approximate an answer to a question.

We might be unable to answer the question any other way, or an *analytical* (logical, mathematical, exact) solution might be too expensive.

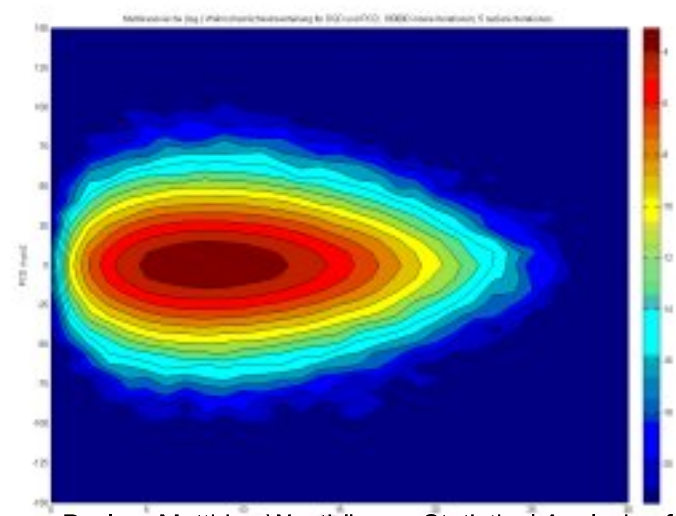
Some Applications



http://marcoagd.usuarios.rdc.puc-rio.br/quasi_mc.html



US Food and Drug Administration



Dr.-Ing. Matthias Westhäuser. Statistical Analysis of Fiber Optical Systems using Multicanonical Monte Carlo Methods (http://www.hft.e-technik.tu-dortmund.de/forschung/projekt.php?id=18&lang=en)

Monte Carlo methods

- The hungry dice player
- The clueless student*
- The umbrella quandary*
- A survey of applications

* Source: *Digital Dice* by Paul J. Nahin

What is a Monte Carlo method?

- An algorithm that uses a source of (pseudo) **random numbers**
- **Repeats an "experiment"** many times and calculates a statistic, often an average
- **Estimates** a value (often a probability)
- … usually a value that is **hard or *impossible*** to calculate analytically

A simple Monte Carlo method

(no computer needed!)

Simple example: dice statistics

- We can **analyze** throwing a pair of dice and get the following probabilities for the sum of the two dice:

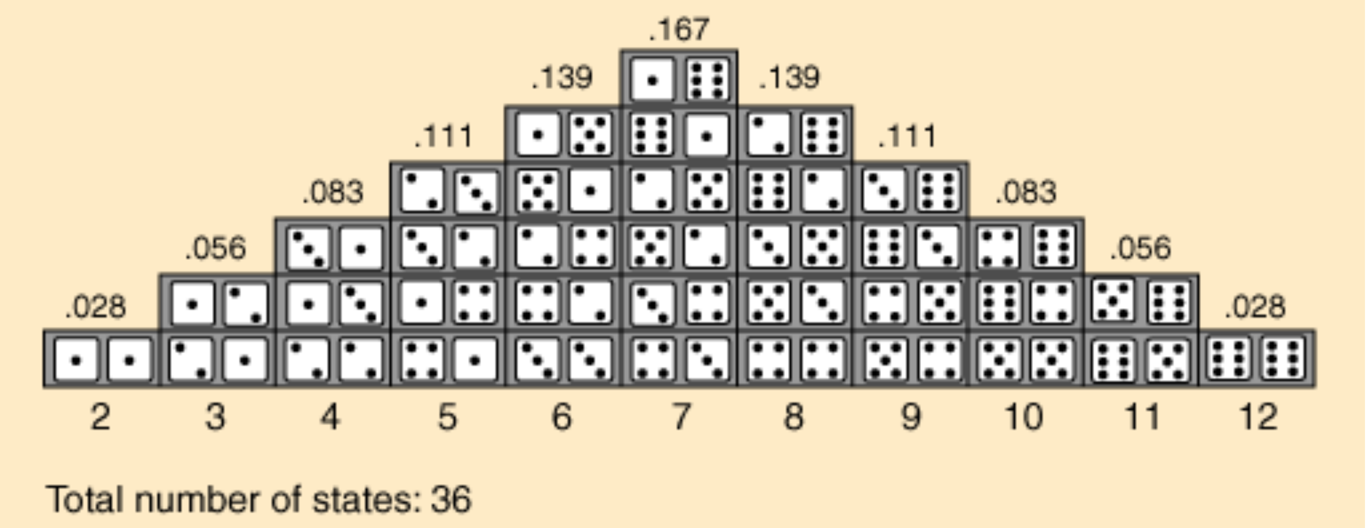
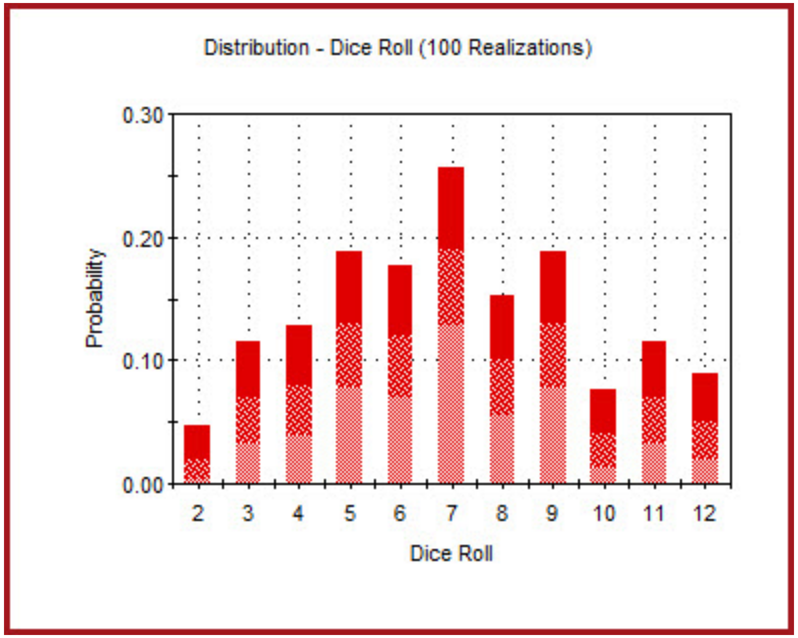


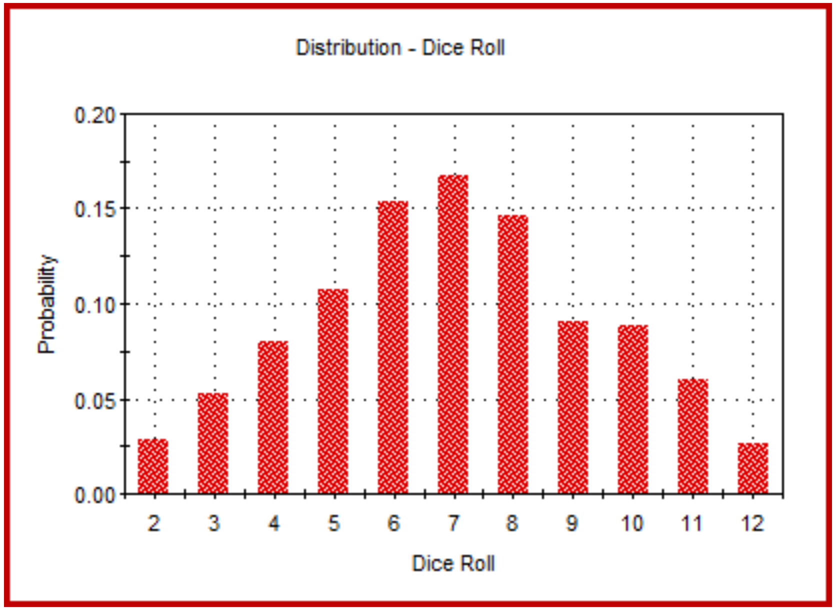
image source: http://hyperphysics.phy-astr.gsu.edu/hbase/math/dice.html via http://www.goldsim.com/Web/Introduction/Probabilistic/MonteCarlo/

Simple example: dice statistics

- … **or** we can throw a pair of dice 100 times and record what happens,



- or 10000 times for a more accurate estimate.



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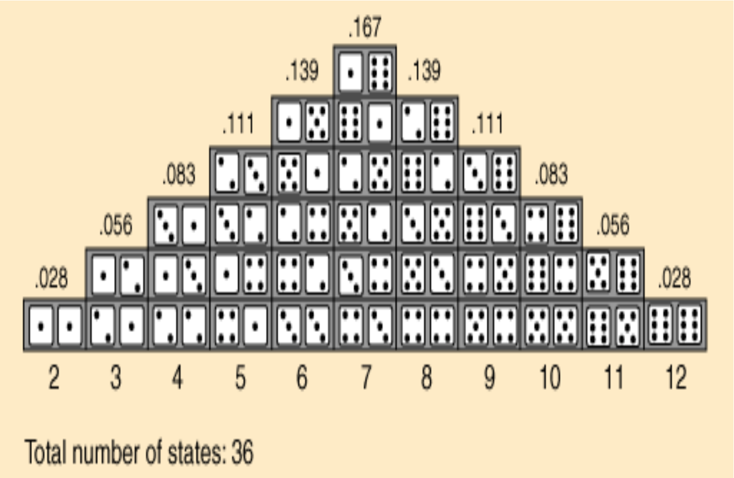


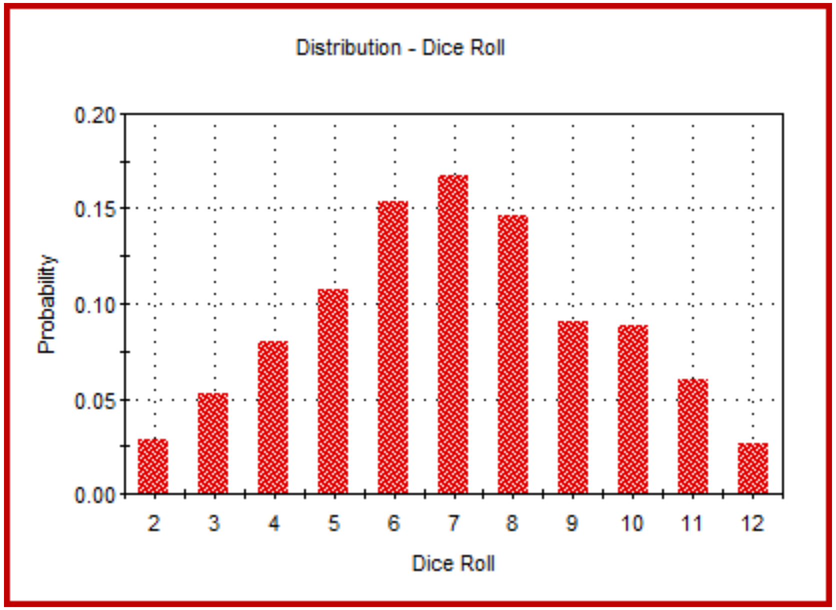


image source:
http://www.goldsim.com/Web/Introduction/Probabilistic/MonteCarlo/

The Hungry Dice Player

estimating the expected value of a simple game

A game of dice

```
def dice_game():
    strikes = 0
    winnings = 0
    while strikes < 3: # 3 strikes and you're out
        # get 2 random numbers (1..6)
        die1 = roll()
        die2 = roll()
        # strike or win?
        if die1 == die2:
            strikes = strikes + 1
        else:
            winnings = winnings + die1 + die2
    return winnings   # in cents
```

The Hungry Dice Player

- In our simple game of dice:
  Can I expect to make enough money playing it to buy lunch?
- That is, what is the expected (average) value won in the game?
- We could figure it out by applying laws of probability

…or use a Monte Carlo method

Monte Carlo method for the hungry dice player

```
def average_winnings(samples) :
    # samples is the number of experiments to run
    total = 0
    for n in range(samples) :
        total = total + dice_game()
    return total / samples
```

```
>>> [round(average_winnings(10),2) for i in range(5)]

[85.8, 94.8, 120.7, 123.3, 90.0]

>>> [round(average_winnings(100),2) for i in range(5)]

[105.97, 102.95, 107.74, 134.4, 114.54]

>>> [round(average_winnings(1000),2) for i in range(5)]

[106.84, 107.11, 105.59, 104.28, 106.41]

>>> [round(average_winnings(10000),2) for i in range(5)]

[104.94, 105.71, 105.81, 105.74, 104.62]
```

The Clueless Student

a famous matching problem

The Clueless Student

A clueless student faced a pop quiz:

a list of the 24 Presidents of the 19th century and

another list of their terms in office, but scrambled.

The object was to match the President with the term.

If the student guesses a random one-to-one matching,

how many matches will be right out of the 24, on average?

The quiz

1. Monroe	a. 1801-1809
2. Jackson	b. 1869-1877
3. Arthur	c. 1885-1889
4. Madison	d. 1850-1853
5. Cleveland	e. 1889-1893
6. Jefferson	f. 1845-1849
7. Lincoln	g. 1837-1841
8. Van Buren	h. 1853-1857
9. Adams	i. 1809-1817
etc.	etc.

Solving the problem

- The problem (1710, Pierre de Montmort) was important in development of probability theory

- The mathematical analysis is, um, interesting

  (see http://www.randomservices.org/random/urn/Matching.html)

- But we're not that smart. Let's just simulate the situation, randomly selecting guesses and checking to see how many correct match-ups they contain.

Representing a guess

values

0	1	2	3	4	5	6	7	8	…
1801-09	1809-17	1817-25	1825-29	1829-37	1837-41	1841-41	1841-45	1845-49	…

6	**0**	**2**	**3**	**1**	**4**	**5**	**7**	**8**	**…**

0	1	2	3	4	5	6	7	8	…
Jefferson	Madison	Monroe	Adams	Jackson	Van Buren	Harrison	Tyler	Polk	…

indexes

Representing a guess

- **What is a guess?**

E.g., [0, 1, 2, 3, 4, 5, …, 23] represents a completely correct guess
[1, 0, 2, 3, 4, 5, …, 23] represents a guess that is correct

except that it gets the first two presidents wrong.

  - A guess is just a **permutation** (shuffling) of the numbers 0 … 23.

- Let's define a ***match*** in a guess to be any number *k* that occurs in position *k*. (*E.g.,* 0 in position 0, 10 in position 10)

- With this representation, our question becomes:
  if I pick a random shuffling of the numbers 0…23,
  how many (on average) matches occur?

Randomly permuting a list

To get a random shuffling of the numbers 0 to 23 we use the `shuffle` function from module `random`:

```
>>> nums = list(range(10))
>>> nums
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
>>> shuffle(nums)
>>> nums
[4, 5, 3, 2, 0, 9, 6, 1, 8, 7]
>>> shuffle(nums)
>>> nums
[3, 6, 1, 4, 5, 8, 2, 9, 0, 7]
```

Algorithm

- **Input:**
  - *pairs* (number of things to be matched),
  - *samples* (number of experiments to run)

Consider as a more general problem

- **Output:** average number of correct matches per sample

- **Method:**

1. Set *num_correct* = 0
2. Do the following *samples* times:
   a. Set *matching* to a random permutation of the numbers 0…*pairs*-1
   b. For *k* in 0…*pairs*, if *matching*[*k*] = *k* add one to *num_correct*
3. The result is *num_correct* / *samples*

Code for the clueless student

```
from random import shuffle
# pairs is the number of pairs to be guessed
# samples is the number of experiments to run
def student(pairs, samples):
    num_correct = 0
    matching = list(range(pairs))
    for i in range(samples):     # experiment samples
times
        shuffle(matching)     # generate a guess
        # count matches
        for k in range(pairs):
            if matching[k] == k:
                num_correct = num_correct + 1
    return num_correct / samples  # average correct
```

Running the code

- The mathematical analysis says the expected value is exactly 1 (**no matter how many matches are to be guessed**).

```
>>> student(24, 10000)
0.9924
>>> student(24, 10000)
1.0071
>>> student(10, 10000)
1.0224
>>> student(10, 10000)
0.9999
>>> student(5, 10000)
1.0039
>>> student(5, 10000)
0.9826
```

More samples – smaller error

```
>>> 1 - student(5, 1000)
0.036000000000000003

>>> 1 - student(5, 10000)
0.0059000000000000016

>>> 1 - student(5, 100000)
0.0014100000000000223

>>> 1 - student(5, 1000000)
-0.00066799999999998909
```

The Umbrella Quandary

simulating a system

The Umbrella Quandary

- Mr. X walks between home and work every day
- He likes to keep an umbrella at each location
- But he always forgets to carry one if it’s not raining

- If the probability of rain is *p,* how many trips can he expect to make before he gets caught in the rain because all his umbrellas are at the other location?

(*Assuming that if it’s not raining when he starts a trip, it doesn’t rain during the trip.*)

The trivial cases

- What if it always rains?
- What if it never rains (ok, that was too easy)
- So we only need to think about a probability of rain greater than zero and less than one

Solving the umbrella quandary

- Analysis of the problem can be done with Markov chains

- But we’re just humble programmers;

  we’ll simulate and measure

Simulating an event with a given probability

- In contrast to the clueless student problem we're given a probability of an event

- We want to simulate that the event rain happens, with the given **probability *p*** (where *p* is a number between 0 and 1)

Technique: Get a random float between 0 and 1;

If it's less than *p* simulate that the event happened

```
if random() < p:
raining = True
```

Representing home, work, and umbrellas

- Use **0** for **home**,

  1 for **work**

- A list for the **number of umbrellas** at each location (2 locations)

- How should we initialize?

```
location = 0 # start at home
umbrellas = [1, 1]
```

Recall: he likes to keep an umbrella at each location

Figuring out when to stop

- We want to count the number of trips before Mr. X gets wet, so we want to keep simulating trips until he does.

- To keep track:

```
wet = False
trips = 0
while (not wet) :
  …
```

Changing locations

Mr. X walks between home (0) and work (1)

- To keep track of where he is:
```
location = 0 # start at home
```
- To move to the other location:
```
location = 1 – location
```
- To find how many umbrellas at current location:
```
umbrellas[location]
```

Putting it together

```
from random import random

def umbrella(p):             # p is the probability of rain
    wet = False
    trips = 0
    location = 0
    umbrellas = [1, 1]       # index 0 stands for home, 1 stands for work
    while (not wet):
        if random() < p:     # it's raining
            if umbrellas[location] == 0:  # no umbrella
                wet = True
            else:
                trips = trips + 1
                umbrellas[location] -= 1          # take an umbrella
                location = 1 - location           # switch locations
                umbrellas[location] += 1          # put umbrella
        else:     # it's not raining, leave umbrellas where they are
            trips = trips + 1
            location = 1 - location
    return trips
```

Running simulations

```
>>> umbrella(.5)
22
>>> umbrella(.5)
4
>>> umbrella(.5)
13
>>> umbrella(.5)
2
>>> umbrella(.5)
2
```

Great, but we want averages

- One experiment doesn't tell us much–we want to know, **on average**, if the probability of rain is *p*, how many trips can Mr. X make without getting wet?

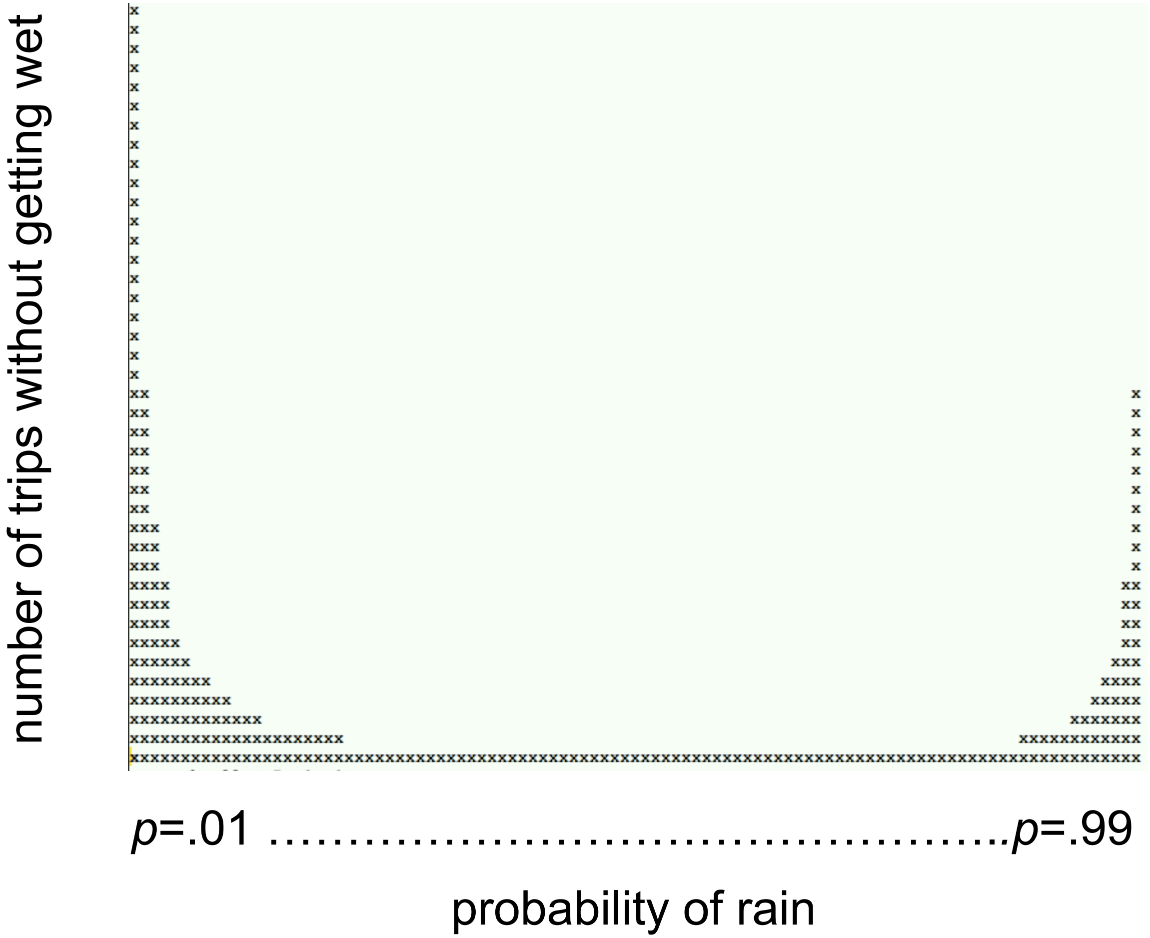
- We add code to run `umbrella(p)` 10,000 times for different probabilities of rain, from
  $p = .01$ to .99 in increments of .01

- We accumulate the results in a list that will show us how the average number of trips is related to the probability of rain.

Running the experiments

```
# 10,000 experiments for each probability .01 to .99
# Accumulate averages in a list
def test() :
    results = [None]*99 # Initialize: 99 probabilities
    p = 0.01            # probability starts at .01
    for i in range(99) :
        trips = 0
        # find average of 10000 experiments
        for k in range(10000) :
            trips = trips + umbrellas(p)
        results[i] = trips/10000
        p = p + .01      # next probability
    return results
```

Crude plot of results



Applications

many, many, many

Finance

- Investment portfolio analysis
- Stock option analysis
- Personal financial planning

Engineering

- Reliability engineering
- Wireless network design
- Wind farm yield prediction
- Fluid dynamics
- Robotics

Mathematics and physics

- Multi-dimensional partial differentiation and integration
- Optimization
- Simulating quantum systems (pioneered by Fermi in 1930)

Many others

- Computational biology
- Physical chemistry
- Applied statistics where data distributions are difficult to analyze
- Game playing

Graphics: path tracing

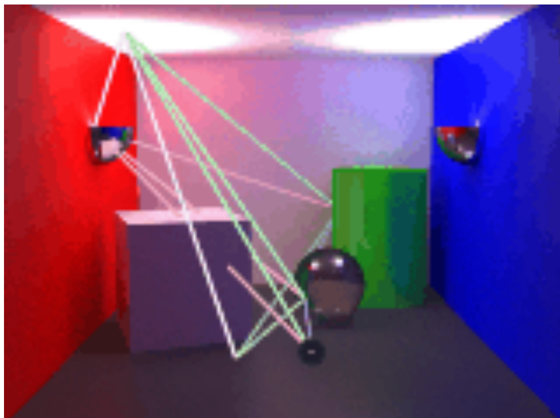


image: http://www.graphics.cornell.edu/~eric/thesis/images.html

image: http://2.bp.blogspot.com/-cUQu1ym3krA/UPYw6qhsZPI/AAAAAAAADeU/YnqtyJjBJJc/s1600/cubecity9b.png

Summary

- Monte Carlo methods use random number generator to “run experiments” in software

- Operations we used:
  - get random integer in a given range
  - get a random permutation of a list
  - use random float between 0 and 1 to decide if an event with probability `p` happens

```
if random() < p : # event happened
```

Next time: Simulation

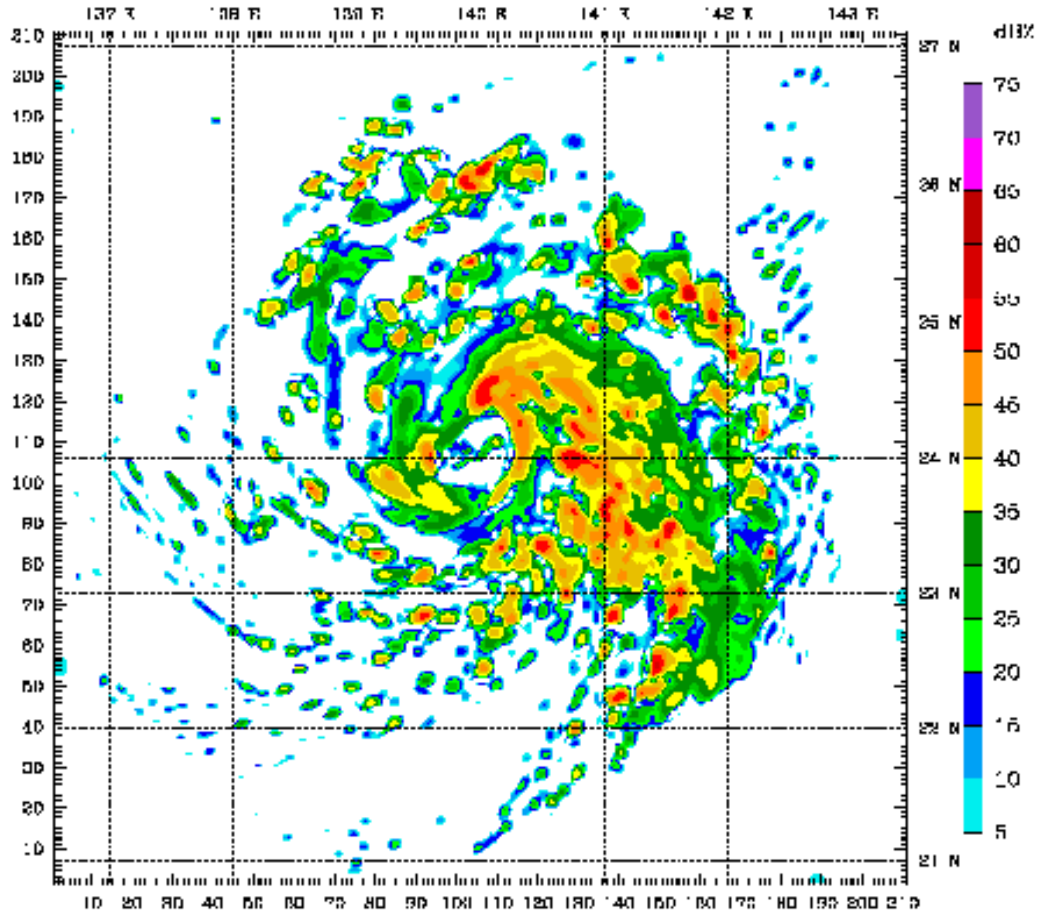


Image: Wikipedia