Optimal Rating Design Under Moral Hazard

Maryam Saeedi and Ali Shourideh

Carnegie Mellon University

SITE: Market Design
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  - security rating, eBay, college grades, Google Ranking

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        - Google's ranking incentivizes higher quality content
    - Ratings often lead to window dressing: ESG Ratings, USNews, Google, ...
- Ratings are information structure
- How should we think about information design when it provides incentives for the rated?

Related Literature

- Bayesian Persuasion: Kamenica and Gentzkow (2011), Dworczak and Martini (2019), Duval and Smolin (2023), ...
- Falsification and muddled information: Perez-Richet and Skreta (2020), Frankel and Kartik (2020), Ball (2020)
- Optimal communications in the presence of incentives: Boleslavsky and Kim (2023), Mahzoon, Shourideh, Zetlin-Jones (2023), Best, Quigley, Saeedi, Shourideh (2023)

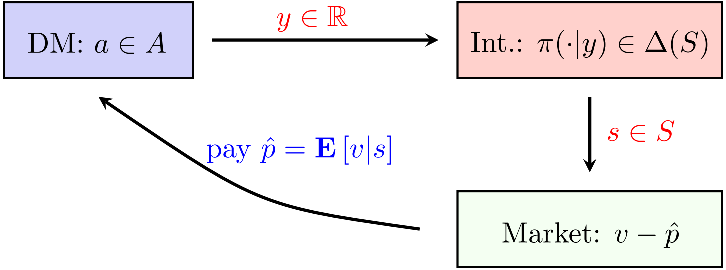
Roadmap

- The General Model
- General characterization of optimal rating system
- An Application:
  - Optimal Ratings in a Multi-tasking model a la Holmstrom and Milgrom

The General Model

The Model

- DM chooses an action $a \in A \subset \mathbb{R}^N$
- Induces $(y, v) \in \mathbb{R}^2$
  - y : indicator observed by intermediary
  - v : value for the market
  - $(y, v) \sim \sigma(y, v|a)$
- Intermediary observes y and sends a signal to the market:
  - Commits to $(S, \pi(\cdot|y))$ with $\pi(\cdot|y) \in \Delta(S)$



The Model

- Payoff of DM

$$\int_Y \int_S \mathbb{E}\left[v|s\right] d\pi\left(s|y\right) dG\left(y|a\right) - c\left(a\right)$$

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  - y observed by Int.
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- **Information:**
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  - y observed by Int.
  - s observed by market
- **Equilibrium**: $\phi \in \Delta\left(A\right)$ is a PBNE
  - Given π and market beliefs, a maximizes DM's payoff, a.e.–ϕ
  - Market beliefs are consistent with π, ϕ, and prior according to Bayes' rule

Feasible Outcomes

- What efforts, a, can be supported in some equilibrium?
- Incentive compatibility

$$a \in \arg\max_{a' \in A} \int_Y \underbrace{\int_S \mathbb{E}\left[v|s\right] d\pi\left(s|y\right)}_{p(y)} dG\left(y|a'\right) - c\left(a'\right)$$

Feasible Efforts

- Incentive compatibility

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- $p\left(y\right) = \mathbb{E}\left[\mathbb{E}\left[v|s\right]|y\right]$: interim prices

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- $p\left(y\right) = \mathbb{E}\left[\mathbb{E}\left[v|s\right]|y\right]$: interim prices

Proposition. If $p\left(\cdot\right)$ is an interim price function, then $p \preccurlyeq_{\text{maj}} \mathbb{E}\left[v|y\right]$. Moreover, if $p\left(\cdot\right)$ is co-monotone with $\mathbb{E}\left[v|y\right]$, i.e., $p\left(y\right) > p\left(y'\right) \Rightarrow \mathbb{E}\left[v|y\right] > \mathbb{E}\left[v|y'\right]$, and $p \preccurlyeq_{\text{maj}} \mathbb{E}\left[v|y\right]$, then $p\left(\cdot\right)$ is an interim price function.

▸ Example ▸ Proof ▸ Majorization

Recap

- Assuming that $\mathbb{E}[v|y]$ are comonotone allows us to significantly simplify the problem
  - Textbook moral hazard with an extra majorization constraint
  - interim prices play the role of transfers
- Given co-monotonicity, WLOG

Assumption. Full-info market values, $\mathbb{E}[v|y]$, are increasing in y.

General Characterization of Optimal Ratings

Optimal Ratings

- Notion of optimality: objective

$$\int W(a)\, d\phi + \int p(y)\, \alpha(y)\, dG(y|a)\, d\phi$$

with $\alpha(y) \geq 0$.

  - Recall ϕ: distribution of action $a \in A$

Optimal Ratings

$$\int W(a)\, d\phi + \int p(y)\, \alpha(y)\, dG(y|a)\, d\phi$$

- Examples:
  - **Correcting an externality :** $\alpha(y) = 0$ and $W(a) \neq \underbrace{V(a) = \mathbb{E}[v|a] - c(a)}_{\text{total surplus}}$

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    - **Learning Externality a la Holmstrom (1999):** $\alpha(y) = 0, W(y) = V(y)$
        - Under full information: market's belief about v, $\mathbb{E}[v|y]$, does not vary with DM's choice of a
        - Externality when $\frac{\partial}{\partial a}\mathbb{E}[v|y] \neq 0$.

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    - Externality when $\frac{\partial}{\partial a}\mathbb{E}[v|y] \neq 0$.
  - **Distributional concerns:** $\alpha(y)$ varies with y

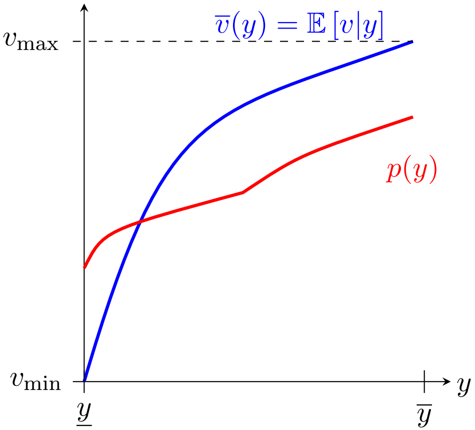
Optimality under Majorization

- Suppose mathematical problem of finding optimal interim prices was of the form (For now trust me that it is!!!):

$$\max_{p(y):\mathbb{E}[v|y]\succcurlyeq_{\text{maj}}p(y)} \int h(y)\,p(y)\,dG(y)$$

subject to monotonicity and given a ϕ

Majorization: A Reformulation



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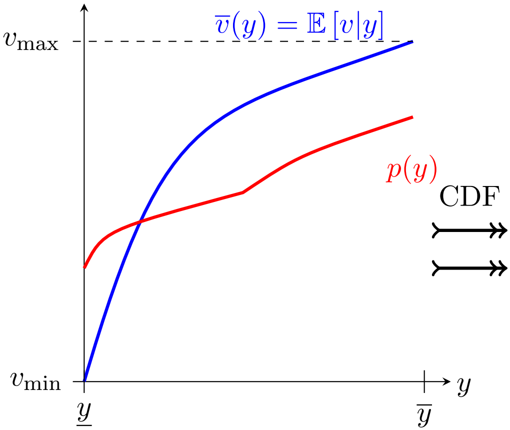

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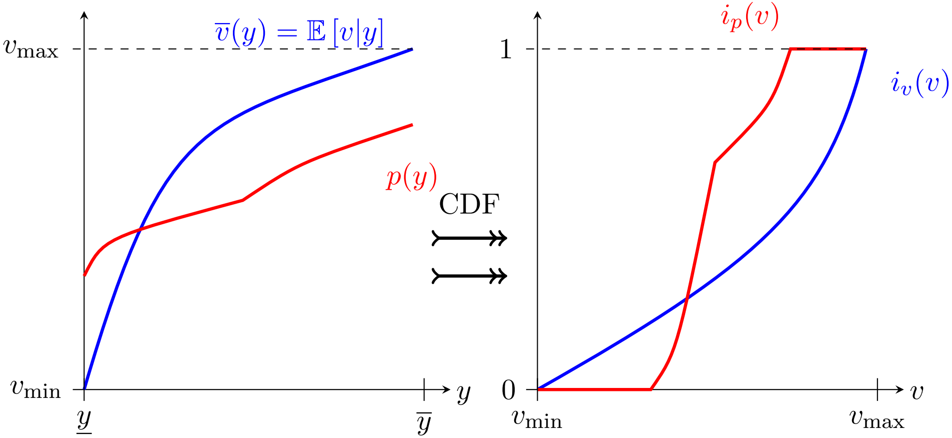

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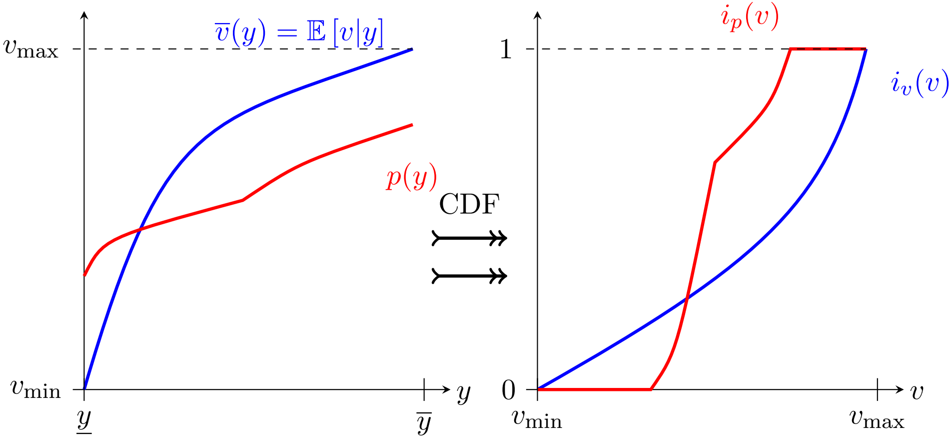


$$\mathbb{E}[v|y] \succcurlyeq_{\text{maj}} p(y) \Leftrightarrow i_p(v) \succcurlyeq_{\text{maj}} i_v(v)$$

Majorization: A Reformulation

$$\max_{p(y)} \int h(y)\, p(y)\, dG(y)$$

subject to

$$\mathbb{E}[v|y] \succcurlyeq_{\text{maj}} p(y)$$

$=$

$$\int \operatorname{cav} H(i)\, dv_Q(i)$$

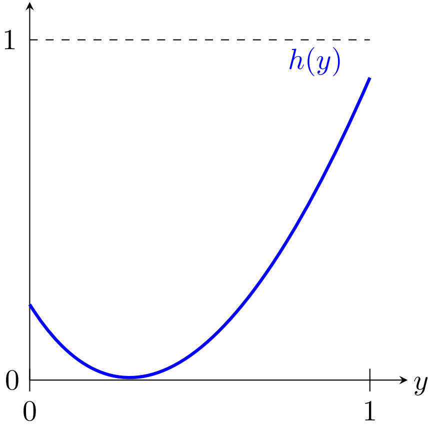
with

$$v_Q(i) = i_v^{-1}(i) \text{ Quantiles } \overline{v}(y)$$

$$H(i) = \int \mathbf{1}\left[\{y : \overline{v}(y) > v_Q(i)\}\right] h(y)\, dG \text{ Cumulative weight above } i$$

Majorization: A Reformulation

- **Example:** $v = y \in [0,1]$, $i_v(v) = v$, $v_Q(i) = i$; v, y: uniformly distributed



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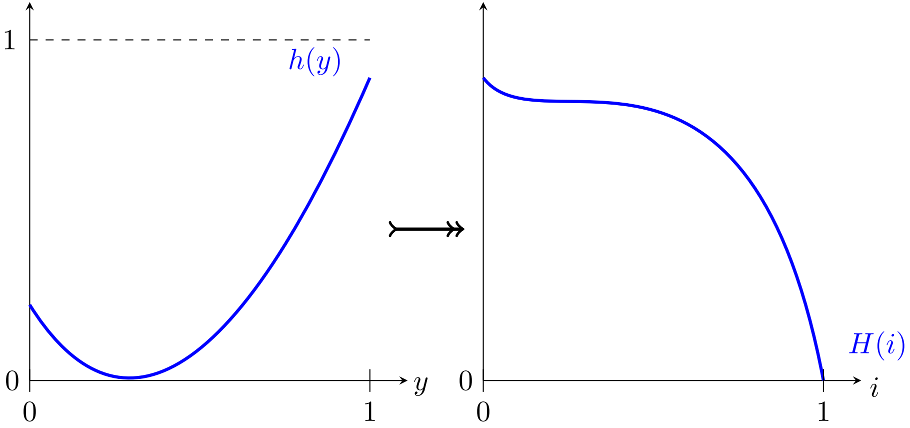

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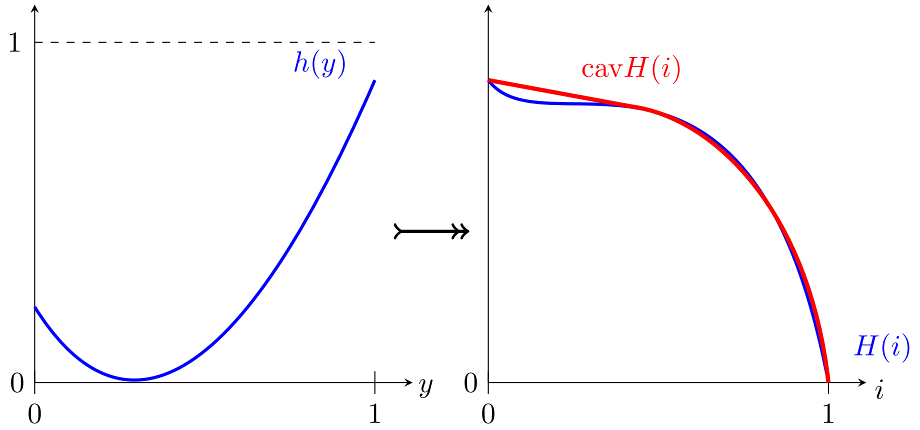

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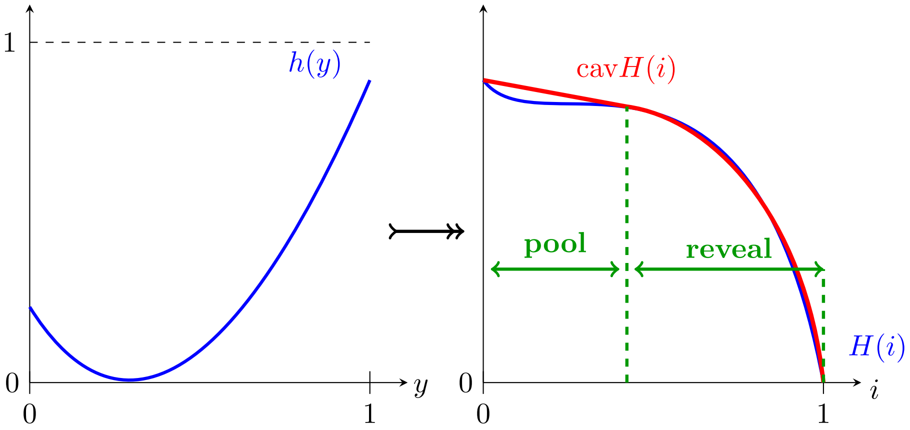

Theorem 1. The problem of optimal rating design is solved by solving the following

$$\min_{\Lambda} \max_{\phi, v_Q} \int W(a)\, d\phi + \int \mathrm{cav} H(i; \Lambda, \phi)\, dv_Q(i)$$

where

$$\begin{aligned} H(i; \Lambda, \phi) = & \int \mathbf{1}\left[\{y : \overline{v}(y) > v_Q(i)\}\right] \alpha(y)\, dG + \int \int_{\hat{a} \in A} \left[F(i|\hat{a}) - i\right] d\Lambda d\phi \\ & + \int \int \left[c(\hat{a}) - c(a)\right] d\Lambda d\phi \end{aligned}$$

and

$$F(i|\hat{a}) = \int \mathbf{1}\left[y : \overline{v}(y) \leq v_Q(i)\right] dG(y|\hat{a})$$

Optimal Ratings

- Theorem 1 is a mouthful!
- Some unpacking:
  - Identifies the function to concavify:
    - changes in quantile distribution from binding deviation weighted by their shadow value

$$\int \int_{\hat{a} \in A} \left[F \left(i | \hat{a} \right) - i \right] d\Lambda d\phi$$

    - Cumulative welfare weights

$$\int \mathbf{1} \left[\left\{ y : \overline{v} \left(y \right) > v_Q \left(i \right) \right\} \right] \alpha \left(y \right) dG$$

  - No need for first order approach
- Proof: Uses Rockefellar-Fenchel duality
  - used also in Dworczak-Koloilin (2023), Corrao-Kolotilin-Wolitzky (2024), Farboodi-Haghpanah-Shourideh (2024)

Simple Ratings are Optimal

Assumption 1. Distribution $G(y|a)$ satisfies:

1. **Interval Support (IS):** $\forall a \in A, \text{Supp} G(\cdot|a) = I \subseteq \mathbb{R}$,
2. **Independence (I).** For any subinterval $I' \subset I$ and $a \neq a' \in A$, there exists $y_1, y_2 \in I'$ such that $G(y_1|a)/G(y_1|a') \neq G(y_2|a)/G(y_2|a')$.

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Proposition. Suppose that IS and I hold, then the optimal rating is monotone partition.

Moreover, whenever $\text{cav}H(i;\Lambda,\phi) = H(i;\Lambda,\phi)$, optimal rating reveals the value $\overline{v} = v_Q(i)$ to the market. When $\text{cav}H(i;\Lambda,\phi) < H(i;\Lambda,\phi)$, then there exists an interval $i \in [i_1, i_2]$ such that optimal rating reveals that $\overline{v} \in [v_Q(i_1), v_Q(i_2)]$.

Distribution Independent Optimal Ratings

Implementable Efforts

- When $\alpha(y) = 0$, only relevant question is what subset A^* of A is implementable by some rating.
- Common case: $A \subset \mathbb{R}$, $g(y|a)$ satisfies MLRP, i.e., $g(y|a)$: log–supermodular

Proposition. Suppose that $A \subset \mathbb{R}_+$ and $G(y|a)$ satisfies IS, I and MLRP. Then, $\max A^* = a^*_{FI}$ where a^*_{FI} is the highest level of equilibrium effort when y is fully revealed.

- The change in quantile distribution is concave
- See also: Dewatripont, Jewitt and Tirole (1999)

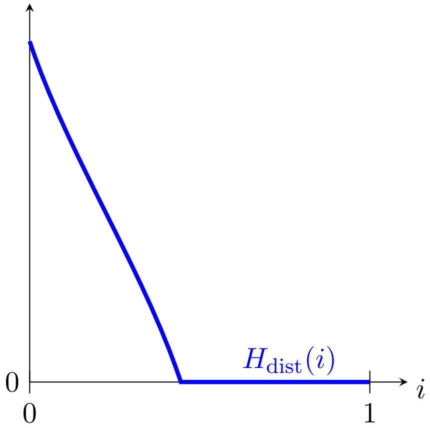
Implementable Efforts

- Other specifications:
  - $y \sim N(a, ka)$, $a \geq 0$, $\max A^* = \max a^*_{LS}$: the highest value of effort among all lower censorship policies.
  - $G(y|a) = e^{-y^{-1/a}}, a \leq 1/2$. $\max A^* = \max a^*_{HS}$: the highest value of effort among all upper censorship policies.
- Both among a class of distribution function where $\frac{\partial^2}{\partial a \partial y} \log g(y|a)$ switches sign only once.

Redistributive Optimal Ratings

Redistributive Motives

- Suppose that $\alpha(y)$'s are positive and decreasing



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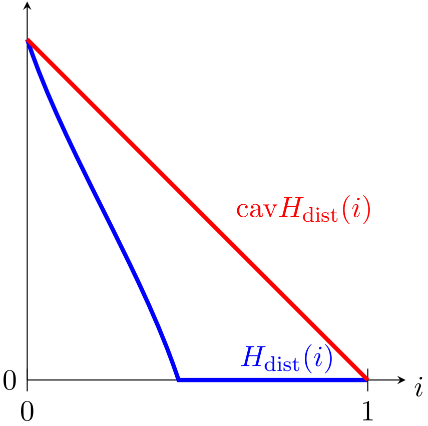

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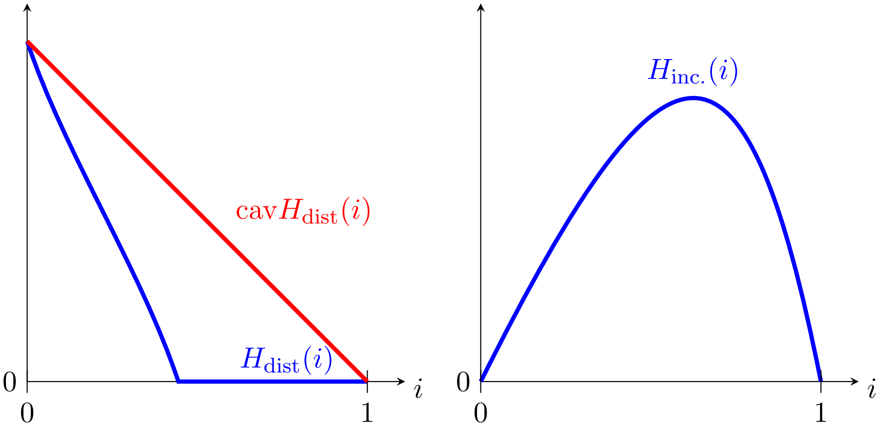

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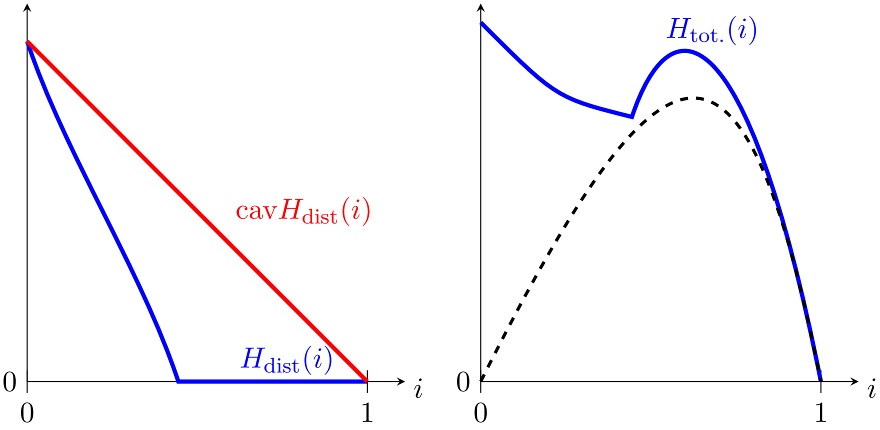

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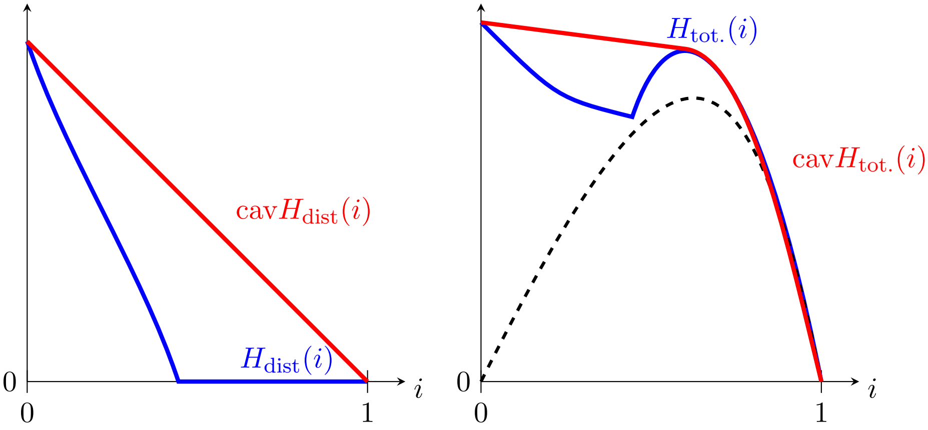

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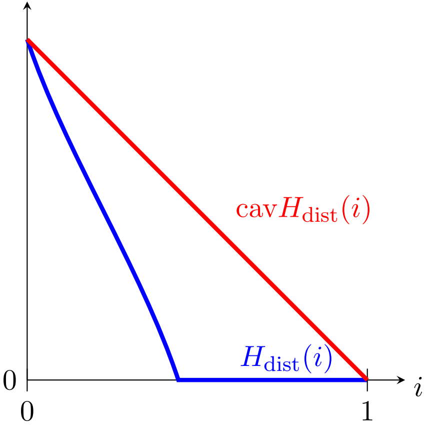

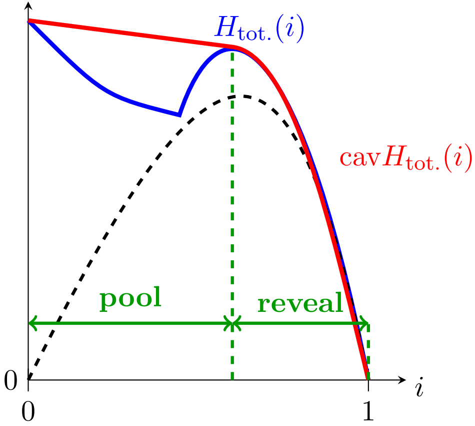

Redistributive Motives

- Suppose that $\alpha(y)$'s are positive and decreasing
- Typical case: optimality of lower censorship
- Has implications for the design of tests for admission into college

Application: A Multi-Tasking Model a la Holmstrom and Milgrom (1991)

A Multi-Tasking Model

- Holmstrom and Milgrom (1991)
- Two tasks: $a = (e_1, e_2)$
  - e_1: value generating
  - e_2 : window dressing
  - cost: $k_1 e_1^2/2 + k_2 e_2^2/2$
- Market values and indicators:
  - values: $v = \beta \cdot e_1 + \varepsilon_v$
  - indicator: $y = \alpha_1 e_1 + \alpha_2 e_2 + \varepsilon_y$

$$\begin{pmatrix} \varepsilon_v \\ \varepsilon_y \end{pmatrix} \sim N\left(0, \Sigma\left(a\right)\right)$$

  - $\alpha_i, \beta > 0$

A Multi-Tasking Model

- Inefficient action: window dressing
- Conditional expectation of v:

$$\mathbb{E}\left[v|y\right] = \beta e_1 + \frac{\sigma_{yv}\left(a\right)}{\sigma_v\left(a\right)^2}\left(y - \alpha_1 e_1 - \alpha_2 e_2\right)$$

- **Holmstrom and Milgrom (1991):** Assuming linear wage contracts, a decline in k_2 leads to lower power incentives.

A Multi-Tasking Model

Proposition. Suppose that $\frac{\partial}{\partial a}\Sigma(a) = 0$, then total surplus maximizing rating is always full information.

- $\frac{\partial}{\partial a}\Sigma(a) = 0$ implies MLRP

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Proposition. 1. Suppose that FOA holds, then total surplus maximizing rating is either lower censorship or higher censorship.
2. If $\frac{\partial}{\partial e_1}\sigma_y = 0, \frac{\partial}{\partial e_2}\sigma_y > 0$, HM's result holds: as k_2 goes down, optimal rating becomes less informative.

Conclusion

- Studied optimal rating design in presence of incentives
- General Characterization of optimal ratings
- Our Techniques can be used to shed light on several design questions of interest:
  - HM's result on changes in window dressing costs
  - Possible to think about the redistributive design of exams and tests

Majorization

Definition. For a r.v. $y \sim H$, satisfy $f(y) \succcurlyeq_{\text{maj}} g(y)$ (equivalently, $g(y) \succcurlyeq_{\text{cv}} f(y)$ or $f(y) \succcurlyeq_{\text{cx}} g(y)$) if and only if

$$\int u(f(y))\, dH \geq \int u(g(y))\, dH, \forall u : \text{convex}, u : X \to \mathbb{R}$$

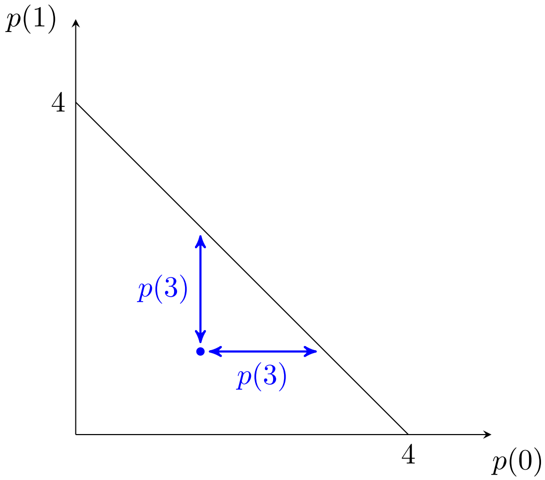
or equivalently

$$\int u(g(y))\, dH \geq \int u(f(y))\, dH, \forall u : \text{concave}, u : X \to \mathbb{R}.$$

Back

Example

- $Y = A = \{0, 1, 3\}$, $v = y$, prior: $\mu(\{y\}) = 1/3$
- Interim prices: $\frac{p(0)+p(1)+p(3)}{3} = \frac{4}{3}$



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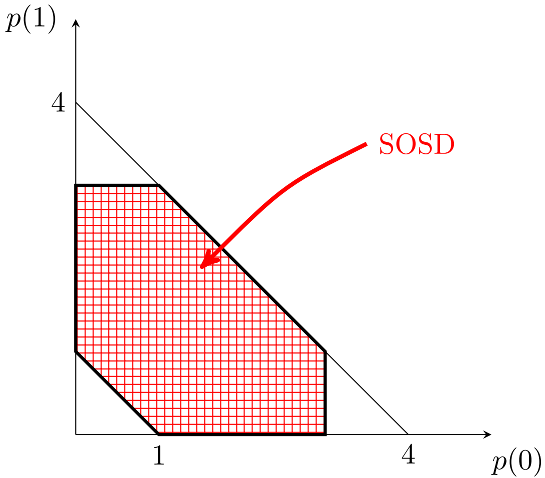

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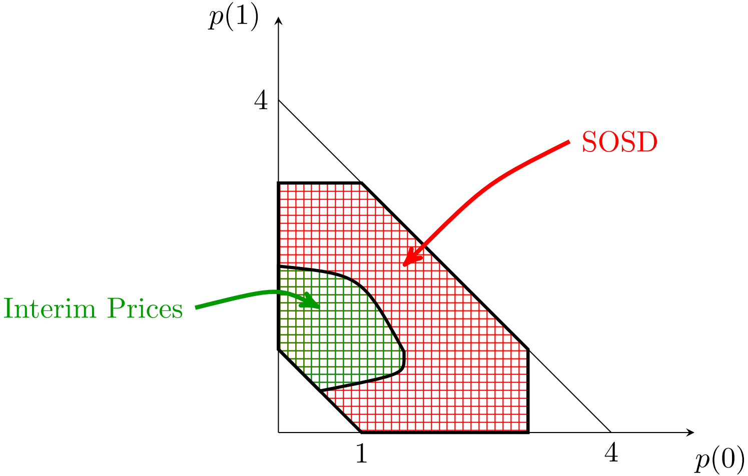

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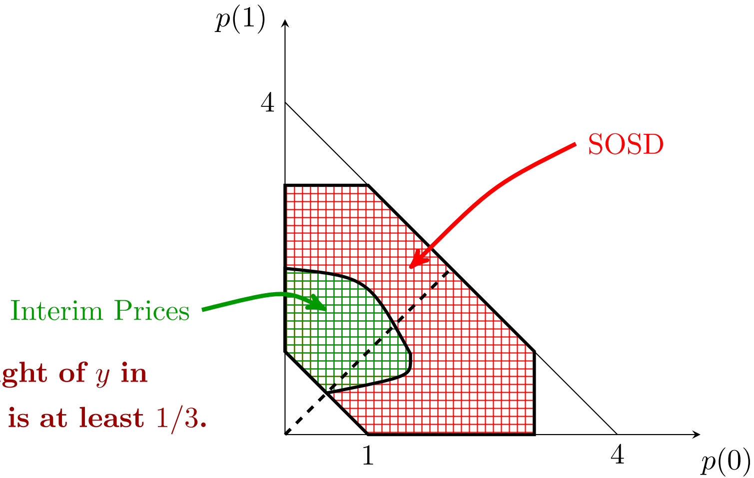

Weight of y in $p(y)$ is at least $1/3$.

Idea of Proof

- Steps:
  - Assume support y's, Y, is finite,
  - Use induction to construct π,
  - Approximate compact Y's
- Suppose Y is finite, Market values $\{\overline{v}_1 < \cdots < \overline{v}_n\}$.
- Co-monotonicity: $p_1 \leq \cdots \leq p_n$

Idea of Proof

- A class of signal structures: for a given $i : 1 \leq i \leq n-1$

$$\pi\left(\{s\}|y\right) = \begin{cases} \lambda & s = y \\ (1-\lambda)\,\hat{\pi}\left(\{s\}|y\right) & s \in \hat{S} \end{cases},\ \hat{\pi}\left(\{s\}|y_i\right) = \hat{\pi}\left(\{s\}|y_{i+1}\right), \forall s \in \hat{S}$$

- Reveals the state with probability $\lambda \in [0,1]$; otherwise pools i and $i+1$.
- Can always choose i and λ so that the implied interim price for $\hat{\pi}$ is co-monotone and satisfies SOSD
  - Use induction hypothesis

▸ Back