Optimal Rating Design

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Introduction

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  - Peer-to-peer platforms: eBay and Airbnb
  - Regulating insurance markets: Community ratings in health insurance exchanges under ACA
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  - Certification of doctors and restaurants

- Common feature:
  - Adverse selection and moral hazard
  - Intermediary observes information
  - Decides what to transmit to the other side

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- Key questions:
  - How should the intermediary transmit the information?
  - When is it optimal to hide some information?
  - When is it optimal to partition the state?

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  - How should the intermediary transmit the information?
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- Try to answer them in a model with adverse selection and moral hazard
  - Moral hazard: sellers choose quality at a cost
  - Adverse selection: sellers are het. w.r.t. cost of quality

Overview of Results

- Provide a full characterization of the set of achievable equilibrium payoffs under arbitrary rating systems

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    - Key difficulty: have to characterize sellers' expectation of buyers' conditional expectation
    - i.e., second order expectation
- Characterize Pareto optimal rating systems:
    - Buyer optimal or more weight on low quality sellers: hide information
    - More weight on high quality sellers: reveal everything

Related Literature

- Bayesian Persuasion: Kamenica and Gentzkow (2011), Rayo and Segal (2010), Gentzkow and Kamenica (2016), Dworczak and Martini (2019), Mathevet, Perego and Taneva (2019), ...
  - Characterize second order expectations + endogenous state
- Certification and disclosure: Lizzeri (1999), Ostrovsky and Schwartz (2010), Harbough and Rasmusen (2018), Hopenhayn and Saeedi (2019), Vellodi (2019), ...
  - Joint mechanism and information design
- (Dynamic) Moral Hazard and limited information/memory: Ekmekci (2011), Liu and Skrzpacz (2014), Horner and Lambert (2018), Bhaskar and Thomas (2018), ...

Roadmap

- The Model
- Characterization for arbitrary rating system
- Pareto optimal ratings

The Model

- Competitive model of adverse selection and moral hazard

The Model

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- Unit continuum of buyers
  - Payoffs:
    $$q - t$$
    q: quality of the good purchased
    t: transfer
  - Outside option: u

The Model

- Unit continuum of sellers
  - Produce one vertically differentiated product
  - Choose quality q
  - Differ in cost of quality provision

$$\text{Cost}: C(q,\theta);\theta \sim F(\theta)$$

  - Payoffs

$$t - C(q,\theta)$$

  - outside option: 0

The Model

Assumption. Cost function satisfies: $C_q > 0, C_\theta < 0, C_{qq} > 0, C_{\theta q} \leq 0$.

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- First Best Efficient: maximize total surplus $q - C(q, \theta)$

$$C_q\left(q^{FB}(\theta), \theta\right) = 1$$

- Submodularity: $q^{FB}(\theta)$ is increasing in θ.
    - Higher θ's have lower marginal cost

Information Design

- Sellers know their θ and q
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  - Alternative: commit to a machine that uses q as input and produces random signal

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  - S: set of signals
  - $\pi(\cdot|q) \in \Delta(S)$
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- Key statistic from the buyers perspective

$$\mathbb{E}[q|s]$$

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- Sellers entry: $\theta \in \Theta^*$

$$\int p(s)\,\pi\left(ds|q(\theta)\right) - C\left(q(\theta), \theta\right) \geq 0 \tag{3}$$

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$$\int p(s)\,\pi(ds|q(\theta)) - C(q(\theta), \theta) \geq 0 \tag{3}$$

Equilibrium: $\left(\{q(\theta)\}_{\theta\in\Theta^*}, v, p(s)\right)$ that satisfy (1), (2) and (3).

Rating Design Problem

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  - Pareto optimality of outcomes
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- First step
  - What allocations are achievable for an arbitrary rating system
- Second Step:
  - Characterize what Pareto optimal outcomes look like

Second Order Expectations

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  - Contrast with Bayesian Persuasion literature: characterize properties of $\mathbb{E}\left[q|s\right]$
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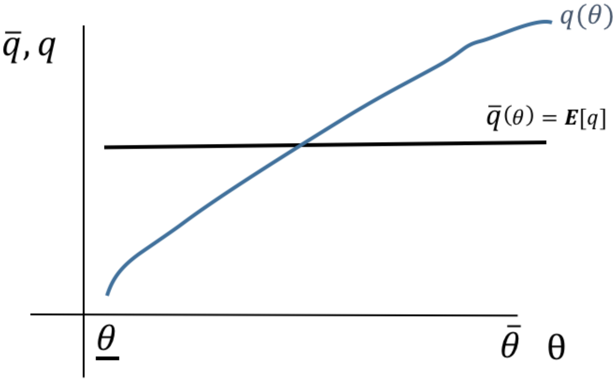
- Our main result: provide a characterization for it

Definition. **(Signaled qualities)** For a quality profile $\{q(\theta)\}_{\theta\in\Theta^*}$ and arbitrary rating system:

$$\overline{q}(\theta) = \int_S \mathbb{E}\left[q|s\right]\pi\left(ds|q(\theta)\right) = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|\, q(\theta)\right]$$

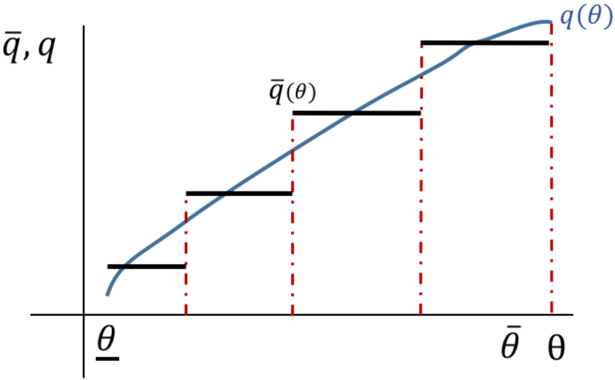
Examples

No information



Examples

Partition Information



Characterizing Rating Systems

- Start with discrete types $\Theta^* = \{\theta_1 < \cdots < \theta_N\}$ and distribution $F : \mathbf{f} = (f_1, \cdots, f_N)$
  - Boldface letters: vectors in $\mathbb{R}^N$
- Standard revelation-principle-type arugment leads to the following lemma

Lemma 1. If a vector of qualities, $\mathbf{q}$, and signaled qualities, $\overline{\mathbf{q}}$ arise from an equilibrium, then they must satisfy:

$$\overline{q}_N \geq \cdots \geq \overline{q}_1, q_N \geq \cdots \geq q_1$$
$$\overline{q}_i - C(q_i, \theta_i) \geq \overline{q}_j - C(q_j, \theta_i), \forall i, j$$

- Can ignore other deviations (off-path qualities): with appropriate out-of-equilibrium beliefs

Properties of Signaled Qualities

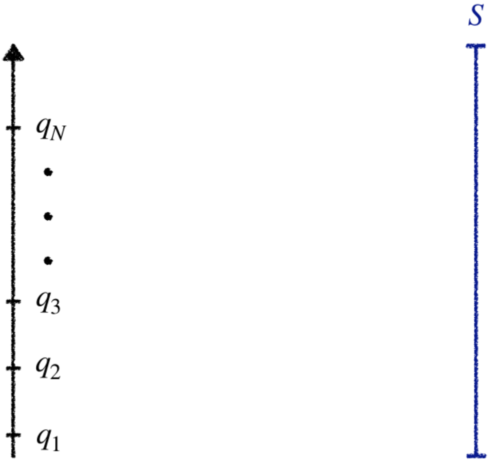
- First Key Property:
  - Equal in expectation:

$$\sum_i f_i \overline{q}_i = \sum_i f_i q_i$$

    - Implied by Bayes Plausibility

Feasible Signaled Qualities

- What signaled qualities are feasible?



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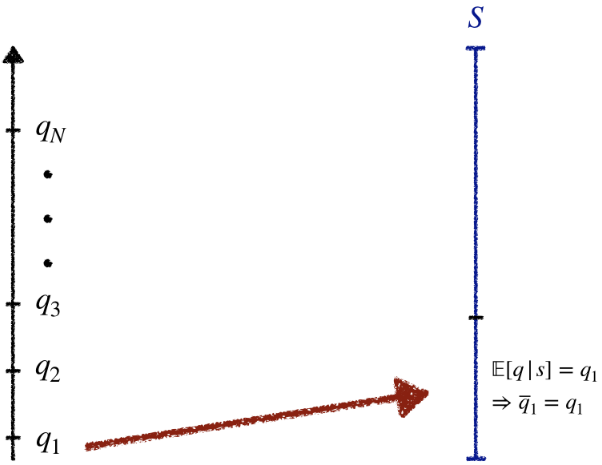

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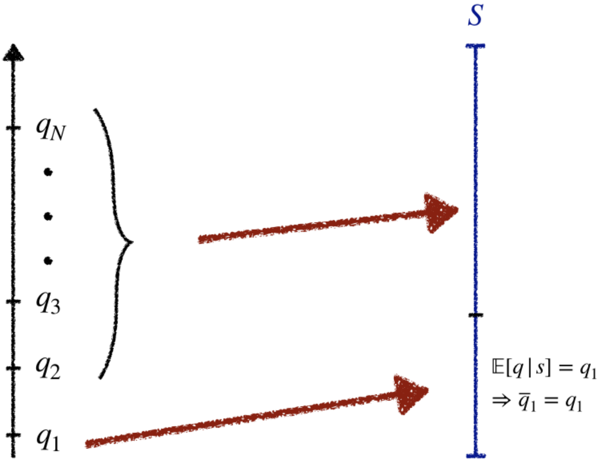

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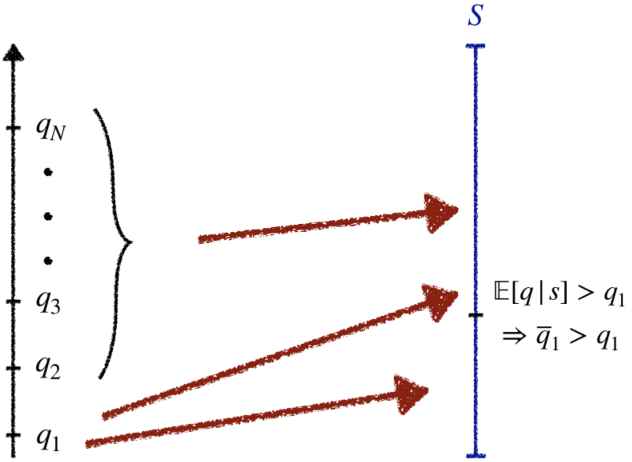

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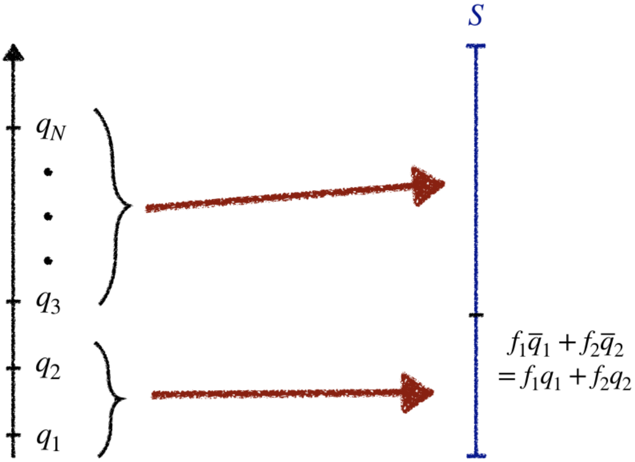

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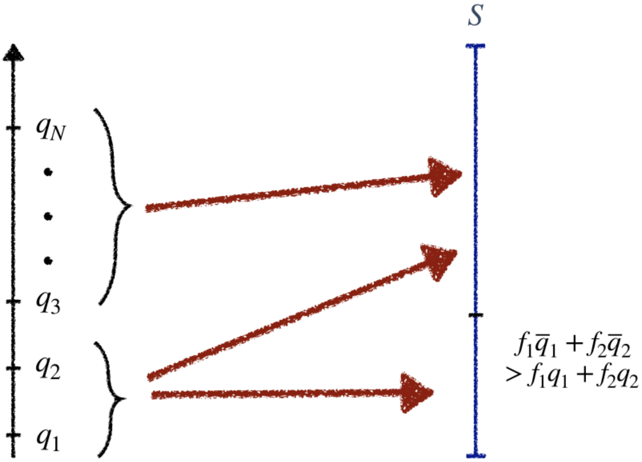

Feasible Signaled Qualities

- Extend this insight: theory of majorization a la Hardy, Littlewood and Polya (1934)

Definition. $\overline{\mathbf{q}}$ F- majorizes $\mathbf{q}$ or $\overline{\mathbf{q}} \succcurlyeq_F \mathbf{q}$ if

$$\sum_{i=1}^{k} f_i \overline{q}_i \geq \sum_{i=1}^{k} f_i q_i, \forall k = 1, \cdots N-1$$

$$\sum_{i=1}^{N} f_i \overline{q}_i = \sum_{i=1}^{N} f_i q_i$$

Majorization: Main Result

Theorem. Consider vectors of signaled and true qualities, $\overline{\mathbf{q}}, \mathbf{q}$ and suppose that they satisfy

$$\overline{q}_1 \leq \cdots \leq \overline{q}_N, q_1 \leq \cdots \leq q_N$$

where equality in one implies the other. Then $\overline{\mathbf{q}} \succcurlyeq_F \mathbf{q}$ if and only if there exists a rating system (π, S) so that

$$\overline{q}_i = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q_i\right]$$

Majorization: Basic Properties

- $\overline{\mathbf{q}} \succcurlyeq_F \mathbf{q}$: dispersion in $\overline{\mathbf{q}}$ < dispersion in $\mathbf{q}$
- Relationship with Second Order Stochastic Dominance:
  - It is equivalent
- Why majorization?
  - Easier to work with: do not have to work with distributions
  - Easier to interpret binding majorization constraint: seperation of signals

Majorization: Proof of The Main Result

- First direction: If $\overline{q}_i = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q_i\right]$, then an argument similar to the above can be used to show that $\overline{\mathbf{q}} \succcurlyeq_F \mathbf{q}$.
    - If all states below k have separate signals from those above, then $\sum_{i=1}^{k} f_i \overline{q}_i = \sum_{i=1}^{k} f_i q_i$.
    - With overlap, $\sum_{i=1}^{k} f_i \overline{q}_i$ can only go up.

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  - First step: show that the set of signaled qualities $\mathcal{S}$ is convex ▸ Proof
  - Second step: Illustration for $N = 2$.

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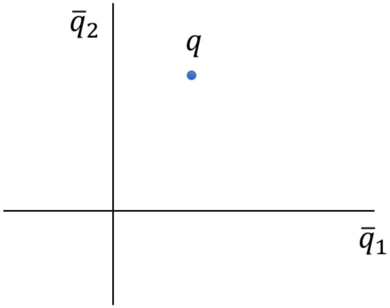

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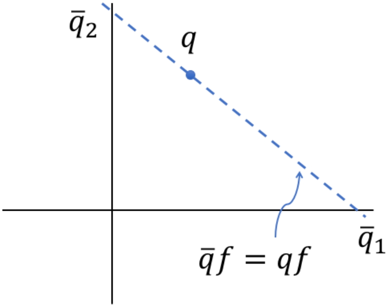

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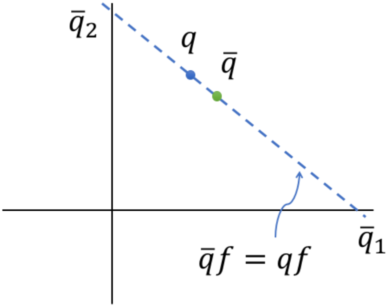

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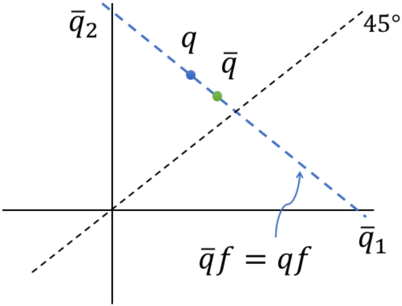

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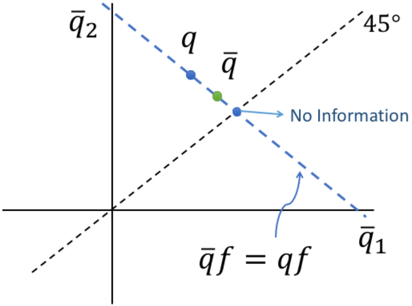

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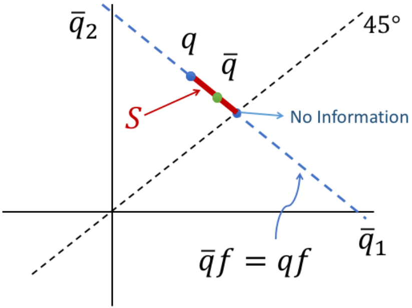

Majorization: Proof of The Main Result

- Second steps for higher dimensions:

Majorization: Proof of The Main Result

- Second steps for higher dimensions:
  - For every direction $\boldsymbol{\lambda} \neq \mathbf{0}$, find two points in S, $\tilde{\mathbf{q}}_1$ and $\tilde{\mathbf{q}}_2$ such that
  $$\boldsymbol{\lambda} \cdot \tilde{\mathbf{q}}_1 \leq \boldsymbol{\lambda} \cdot \overline{\mathbf{q}} \leq \boldsymbol{\lambda} \cdot \tilde{\mathbf{q}}_2$$
    - Proven by induction

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    $$\boldsymbol{\lambda} \cdot \tilde{\mathbf{q}}_1 \leq \boldsymbol{\lambda} \cdot \overline{\mathbf{q}} \leq \boldsymbol{\lambda} \cdot \tilde{\mathbf{q}}_2$$
    - Proven by induction
  - Since $\mathcal{S}$ is convex, separating hyperplane theorem implies that $\overline{\mathbf{q}}$ must belong to $\mathcal{S}$.

Majorization: Continuous Case

- We can extend the results to the case with continuous distribution
    - Use the fact that discrete distributions are dense in the space of distributions (and that the space of distributions over distributions is compact!!)

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- We can extend the results to the case with continuous distribution
  - Use the fact that discrete distributions are dense in the space of distributions (and that the space of distributions over distributions is compact!!)
- We say $\overline{q}(\cdot) \succcurlyeq_F q(\cdot)$ if

$$\int_{\theta^*}^{\theta} \overline{q}\left(\theta'\right) dF\left(\theta'\right) \geq \int_{\theta^*}^{\theta} q\left(\theta'\right) dF\left(\theta'\right), \forall \theta \in \Theta^* = \left[\theta^*, \overline{\theta}\right]$$

$$\int_{\theta^*}^{\overline{\theta}} \overline{q}\left(\theta\right) dF\left(\theta\right) = \int_{\theta^*}^{\overline{\theta}} q\left(\theta\right) dF\left(\theta\right)$$

Majorization: Continuous Case

Corollary. Let $\overline{q}(\theta)$ and $q(\theta)$ be two functions representing signaled and true quality. Then, these functions arise from an equilibrium for some rating system if and only if they satisfy the following:

1. The surplus function $S(\theta) = \overline{q}(\theta) - C(q(\theta), \theta)$ is differentiable and satisfies

$$S'(\theta) = -C_{\theta}(q(\theta), \theta)$$

2. The functions $\overline{q}(\theta)$ and $q(\theta)$ are increasing in θ and satisfy $\overline{q} \succcurlyeq_F q$.

Constructing Signals

- Given $\overline{q}(\theta)$ and $q(\theta)$ that satisfy majorization: What is (π, S)?
- For the discrete case, we have an algorithm that constructs (π, S)
    - Starts from q
    - A sequence of signals: convex combination of
        - pooling adjacent states
        - full information
    - Repeat until it reaches $\overline{q}$
- For the continuous case: use discrete approximation

Optimal Rating Systems

- Pareto optimal allocations

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- Approach:

$$\max v\left(1-F\left(\theta^{*}\right)\right)+uF\left(\theta^{*}\right)+\int\lambda\left(\theta\right)\Pi\left(\theta\right)dF\left(\theta\right)$$

subject to

(PC),(IC),(Maj)

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- Our focus is on
  - $\lambda\left(\theta\right)=1$: Total surplus
  - $\lambda\left(\theta\right)=0$: Buyer optimal
  - $\lambda\left(\theta\right)$: decreasing
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  - $\lambda\left(\theta\right)$: decreasing
  - $\lambda\left(\theta\right)$: increasing; higher weight on higher quality sellers
- Possible Interpretation:
  - Competition between intermediaries lead to Pareto optimality

Total Surplus

- First Best allocation: maximizes total surplus ignoring all the constraints

$$C_q\left(q^{FB}(\theta),\theta\right)=1$$

$$\left[q^{FB}(\theta^*)-u-C\left(q^{FB}(\theta^*),\theta^*\right)\right](\theta^*-\underline{\theta})=0$$

- Incentive constraint:

$$\overline{q}'(\theta)=C_q(q(\theta),\theta)\,q'(\theta)=q'(\theta)$$

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- Set $\overline{q}(\theta)=q(\theta)$
  - Satisfies IC
  - Satisfies majorization

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  - Satisfies IC
  - Satisfies majorization
- Maximizing total surplus: full information about quality

Buyer Optimal Allocations

- Suppose that $\lambda(\theta) = 0$
  - Textbook mechanism design problem: all types have the same outside option; only binding for the entrant, θ^*

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- Suppose that $\lambda(\theta) = 0$
  - Textbook mechanism design problem: all types have the same outside option; only binding for the entrant, θ^*
- Tradeoff: information rents vs. reallocation of profits
  - Want to allocate resources to the buyers
  - All higher quality types want to lie downward
- Reduce qualities relative to First Best

Buyer Optimal Allocations

Relaxed problem - w/o majorization constraint

$$\max uF\left(\theta^*\right) + v\left(1 - F\left(\theta^*\right)\right)$$

subject to

$$\Pi'\left(\theta\right) = -C_\theta\left(q\left(\theta\right), \theta\right)$$

$$q\left(\theta\right) : \text{increasing}$$

$$\int_{\theta^*}^{\overline{\theta}} \left(v + \Pi\left(\theta\right)\right) dF\left(\theta\right) = \int_{\theta^*}^{\overline{\theta}} \left[q\left(\theta\right) - C\left(q\left(\theta\right), \theta\right)\right] dF\left(\theta\right)$$

$$\Pi\left(\theta\right) \geq 0$$

Proposition. A quality allocation $q\left(\theta\right)$ is buyer optimal if and only if it is a solution to the relaxed problem. Moreover, if the cost function $C\left(\cdot, \cdot\right)$ is strictly submodular, then a buyer optimal rating system never features a separation.

Buyer Optimal Allocations: Intuition

- The solution of the relaxed problem (with or without ironing)

$$C_q\left(q\left(\theta\right),\theta\right) \leq 1$$

- Incentive constraint

$$\overline{q}'\left(\theta\right) = C_q\left(q\left(\theta\right),\theta\right) q'\left(\theta\right)$$

- $\overline{q}\left(\theta\right)$ flatter than $q\left(\theta\right)$: majorization constraint holds and is slack
    - If $C_q < 1$ for a positive measure of types, no separation of qualities

Constructing Signals: Buyer Optimal

- When $\overline{q}(\theta)$ is flatter than $q(\theta)$ and majorization constraint never binds:
  - Finding signals is very straightforward

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- Signal:

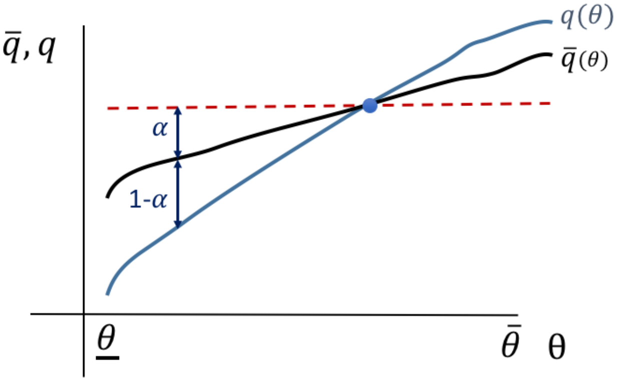
$$S = \left\{ q(\theta) : \theta \in \Theta^* \right\} \cup \{\emptyset\}$$

$$\pi\left(\{s\} | q\right) = \begin{cases} \alpha(q) & s = q \\ 1 - \alpha(q) & s = \emptyset \end{cases}$$

  - Reveal quality or say nothing!

Non-separating signal

When $\bar{q}(\theta)$ is flatter than $q(\theta)$



Decreasing Welfare Weights

Corollary. If $\lambda(\theta)$ is decreasing in θ, then the majorization inequality is slack at the optimum. Furthermore, if $C(\cdot,\cdot)$ is strictly sub-modular, then the optimal rating system never features any separation.

High Quality Seller Optimal

- Now suppose $\lambda(\theta)$ is increasing in θ
- Solution of the relaxed mechanism design problem satisfies

$$C_q(q(\theta), \theta) \geq 1$$

- IC:

$$\overline{q}'(\theta) = C_q(q(\theta), \theta) q'(\theta) > q'(\theta)$$

High Quality Seller Optimal

- Now suppose $\lambda(\theta)$ is increasing in θ
- Solution of the relaxed mechanism design problem satisfies

$$C_q(q(\theta), \theta) \geq 1$$

- IC:

$$\overline{q}'(\theta) = C_q(q(\theta), \theta) q'(\theta) > q'(\theta)$$

- Majorization inequality will be violated
  - Intuition: overprovision of quality to prevent low θ's from lying upwards; signaled quality must be steep

High Quality Seller Optimal

Proposition. Suppose that $\lambda(\theta)$ is increasing and $\lambda(\overline{\theta}) > \lambda(\underline{\theta})$. Then optimal rating system is full information.

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  - First step: majorization cannot bind in isolated points
  - Second step: majorization cannot be slack over an interval

High Quality Seller Optimal

Proposition. Suppose that $\lambda(\theta)$ is increasing and $\lambda(\overline{\theta}) > \lambda(\underline{\theta})$. Then optimal rating system is full information.

- Proof: extensive use of calculus of variation
  - First step: majorization cannot bind in isolated points
  - Second step: majorization cannot be slack over an interval
- Intuition:
  - Want to allocate profits to high θ's
  - Constrained by majorization: must always bind

Conclusion

- Rating Systems in a competitive model of adverse selection and moral hazard
- Provide full characterization of feasible allocations:
  - Majorization
- Pareto optimal rating systems

Convexity of $\mathcal{S}$

- Discrete signal space:

$$\overline{q}_i = \sum_s \pi\left(\{s\}|q_i\right) \frac{\sum_j \pi\left(\{s\}|q_j\right) f_j q_j}{\sum_j \pi\left(\{s\}|q_j\right) f_j}$$

- Alternative representation of the RS:

$$\tau \in \Delta\left(\Delta\left(\Theta\right)\right) : \mu_j^s = \frac{\pi\left(\{s\}|q_j\right) f_j}{\sum_j \pi\left(\{s\}|q_j\right) f_j}, \tau\left(\{\boldsymbol{\mu}^s\}\right) = \sum_j \pi\left(\{s\}|q_j\right) f_j$$

- Bayes plausibility

$$\mathbf{f} = \int_{\Delta(\Theta)} \boldsymbol{\mu} d\tau$$

- We can write signaled quality as

$$\overline{\mathbf{q}} = \mathrm{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \mathbf{q} = \mathbf{A}\mathbf{q}$$

Convexity of $\mathcal{S}$

- The set $\mathcal{S}$ is given by

$$\mathcal{S} = \left\{ \overline{\mathbf{q}} : \exists \tau \in \Delta\left(\Delta\left(\Theta\right)\right), \int \boldsymbol{\mu} d\tau = \mathbf{f}, \overline{\mathbf{q}} = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \right\}$$

- For any τ_1, τ_2 satisfying Bayes plausibility, i.e., $\int \boldsymbol{\mu} d\tau = \mathbf{f}$,their convex combination also satisfies BP since integration is a linear operator.
- Therefore

$$\begin{aligned} \lambda \overline{\mathbf{q}}_1 + \left(1-\lambda\right) \overline{\mathbf{q}}_2 = {} & \lambda \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau_1 + \\ & \left(1-\lambda\right) \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau_2 \\ & = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\left(\lambda \tau_1 + \left(1-\lambda\right)\tau_2\right) \end{aligned}$$

- Since $\lambda \tau_1 + \left(1-\lambda\right)\tau_2$ satisfies BP, $\lambda \overline{\mathbf{q}}_1 + \left(1-\lambda\right) \overline{\mathbf{q}}_2 \in \mathcal{S}$

Back

Majorization: Basic Properties

- $\succcurlyeq_F$ is transitive.
- The set of $\overline{\mathbf{q}}$ that F-majorize $\mathbf{q}$ is convex.
- Can show that there exists a positive matrix $\mathbf{A}$ such that $\overline{\mathbf{q}} = \mathbf{A}\mathbf{q}$ where

$$\mathbf{f}^T\mathbf{A} = \mathbf{f}^T, \mathbf{A}\mathbf{e} = \mathbf{e}$$

  with $\mathbf{e} = (1, \cdots, 1)$ and $\mathbf{f} = (f_1, \cdots, f_N)$.
- We refer to $\mathbf{A}$ as an F-stochastic matrix.
  - Set of F-stochastic matrices is closed under matrix multiplication.

Back

Constructing Signals

- One easy case: $\overline{q}(\theta)$ flatter than $q(\theta)$, i.e., $\overline{q}'(\theta) < q'(\theta)$
  - majorization constraint never binds.

- Signal:

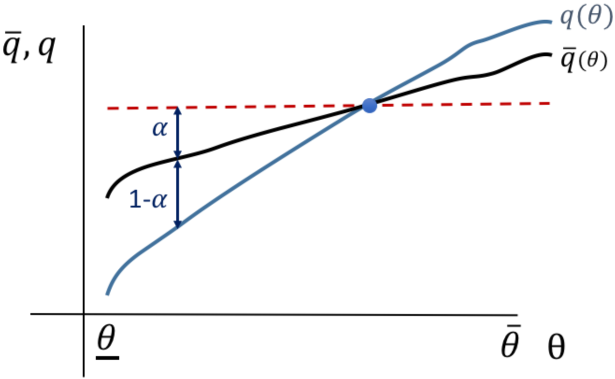
$$S = \{q(\theta) : \theta \in \Theta\} \cup \{\emptyset\}$$

$$\pi(\{s\} | q) = \begin{cases} \alpha(q) & s = q \\ 1 - \alpha(q) & s = \emptyset \end{cases}$$

  - Reveal quality or say nothing!

Non-separating signal

When $\bar{q}(\theta)$ is flatter than $q(\theta)$



Constructing Signals: Algorithm

- For the discrete case, we can give an algorithm to construct the signals (rough idea; much more details in the actual proof)
  1. Start from $\mathbf{q}$
  2. Consider a convex combination of two signals:
     2.1 Full revelation: $\pi^{FI}\left(\{q\}\,|q\right) = 1$
     2.2 Pooling signal: pool two qualities q_i and q_j

$$S = \{q_1, \cdots, q_N\} - \{q_i, q_j\} \cup \{q_{ij}\}$$

$$\pi^{i,j}\left(\{s\}\,|q\right) = \begin{cases} 1 & s = q, q \neq q_i, q_j \\ 1 & s = q_{ij}, q = q_i, q_j \end{cases}$$

     2.3 Send π^{FI} with probability α and $\pi^{i,j}$ with probability $1-\alpha$
  3. Choose α so that the resulting signaled quality has one element in common with $\overline{\mathbf{q}}$
  4. Repeat the same procedure on resulting signaled quality until reaching $\overline{\mathbf{q}}$ Back