Finite State Machines 3

95-771 Data Structures and Algorithms for Information Processing

Carnegie Mellon

Notes taken with modifications from “Introduction to Automata Theory, Languages, and Computation” by John Hopcroft and Jeffrey Ullman, 1979

Deterministic Finite-State Automata (review)

A DFSA can be formally defined as $M = (Q, \Sigma, \delta, q_0, F)$:

Q, a finite set of states

Σ, the alphabet of input symbols

δ, $Q \times \Sigma \rightarrow Q$, a transition function

q_0, the initial state

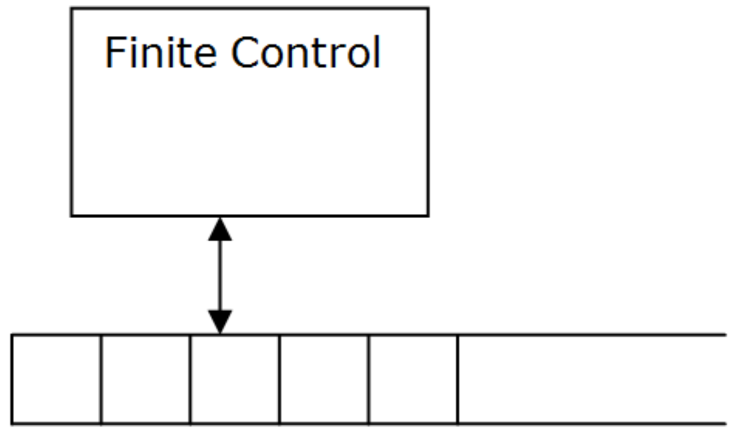
F, the set of final states

Pushdown Automata(review)

A pushdown automaton can be formally defined as M = $(Q, \Sigma, \Gamma, \delta, q_0, F)$:
Q, a finite set of states
Σ, the alphabet of tape symbols
Γ, the alphabet of stack symbols
δ, $Q \times \Sigma \times \Gamma \rightarrow Q \times \Gamma$
$q_{0,}$ the initial state
F, the set of final states

Turing Machines

- The basic model of a Turing machine has a finite control, an input tape that is divided into cells, and a tape head that scans one cell of the tape at a time.
- The tape has a leftmost cell but is infinite to the right.
- Each cell of the tape may hold exactly one of a finite number of tape symbols.
- Initially, the n leftmost cells, for some finite n >= 0, hold the input, which is a string of symbols chosen from a subset of the tape symbols called the input symbols.
- The remaining infinity of cells each hold the blank, which is a special symbol that is not an input symbol.



A Turing machine can be formally defined as M= (Q, Σ, Γ, δ, q_0 , B, F):
where
Q, a finite set of states
Γ, is the finite set of allowable tape symbols
B, a symbol from Γ is the blank
Σ, a subset of Γ not including B, is the set of input symbols
δ, Q x Γ $\rightarrow$ Q x Γ x { L,R} (may be undefined for some arguments)
q_0 in Q is the initial state
F $\subseteq$ Q is the set of final states

Turing Machine Example

The design of a Turing Machine M to decide the language $L = \{0^n1^n, n >= 1\}$. This language is decidable.

- Initially, the tape of M contains 0^n1^n followed by an infinity of blanks.
- Repeatedly, M replaces the leftmost 0 by X, moves right to the leftmost 1, replacing it by Y, moves left to find the rightmost X, then moves one cell right to the leftmost 0 and repeats the cycle.
- If, however, when searching for a 1, M finds a blank instead, then M halts without accepting. If, after changing a 1 to a Y, M finds no more 0's, then M checks that no more 1's remain, accepting if there are none.

Let $Q = \{ q_0, q_1, q_2, q_3, q_4 \}$, $\sum = \{0,1\}$, $\Gamma = \{0,1,X,Y,B\}$ and $F = \{q_4\}$
δ is defined with the following table:

INPUT SYMBOL

STATE	0	1	X	Y	B
q0	(q1,X,R)	-	-	(q3,Y,R)	-
q1	(q1,0,R)	(q2,Y,L)	-	(q1,Y,R)	-
q2	(q2,0,L)	-	(q0,X,R)	(q2,Y,L)	-
q3	-	-	-	(q3,Y,R)	(q4,B,R)
q4	-	-	-	-	-

As an exercise, draw a state diagram of this machine and trace its execution through 0011, 001101 and 001.

The Turing Machine as a computer of integer functions

- In addition to being a language acceptor, the Turing machine may be viewed as a computer of functions from integers to integers.
- The traditional approach is to represent integers in unary; the integer i >= 0 is represented by the string 0^i.
- If a function has more than one argument then the arguments may be placed on the tape separated by 1's.

For example, proper subtraction m – n is defined to be

m – n for m >= n, and

zero for m < n.

The TM M = ({q0,q1,...,q6}, {0,1}, {0,1,B}, δ, q0, B, {})

defined below, if started with 0^m10^n on its tape, halts with 0^{m-n} on its tape. M repeatedly replaces its leading 0 by blank, then searches right for a 1 followed by a 0 and changes the 0 to a 1. Next, M moves left until it encounters a blank and then repeats the cycle. The repetition ends if

- Searching right for a 0, M encounters a blank. Then, the n 0's in 0^m10^n have all been changed to 1's, and n+1 of the m 0's have been changed to B. M replaces the n+1 1's by a 0 and n B's, leaving m-n 0's on its tape.
- Beginning the cycle, M cannot find a 0 to change to a blank, because the first m 0's already have been changed. Then n >= m, so m – n = 0. M replaces all remaning 1's and 0's by B.

The function δ is described below.

δ(q0,0) = (q1,B,R)   Begin. Replace the leading 0 by B.

δ(q1,0) = (q1,0,R)   Search right looking for the first 1.
δ(q1,1) = (q2,1,R)

δ(q2,1) = (q2,1,R)   Search right past 1's until encountering a 0. Change that 0 to 1.
δ(q2,0) = (q3,1,L)

δ(q3,0) = (q3,0,L)   Move left to a blank. Enter state q0 to repeat the cycle.
δ(q3,1) = (q3,1,L)
δ(q3,B) = (q0,B,R)

If in state q2 a B is encountered before a 0, we have situation i described above. Enter state q4 and move left, changing all 1's to B's until encountering a B. This B is changed back to a 0, state q6 is entered and M halts.

δ(q2,B) = (q4,B,L)
δ(q4,1) = (q4,B,L)
δ(q4,0) = (q4,0,L)
δ(q4,B) = (q6,0,R)

If in state q0 a 1 is encountered instead of a 0, the first block of 0's has been exhausted, as in situation (ii) above. M enters state q5 to erase the rest of the tape, then enters q6 and halts.

δ(q0,1) = (q5,B,R)
δ(q5,0) = (q5,B,R)
δ(q5,1) = (q5,B,R)
δ(q5,B) = (q6,B,R)

As an exercise, trace the execution of this machine using an input tape with the symbols 0010.

Modifications To The Basic Machine

- It can be shown that the following modifications do not improve on the computing power of the basic Turing machine shown above:
  - Two-way infinite tape
  - Multi-tape Turing machine with k tape heads and k tapes
  - Multidimensional, Multi-headed, RAM, etc., etc.,…
  - Nondeterministic Turing machine
  - Let's look at a Nondeterministic Turing Machine…

Nondeterministic Turing Machine (NTM)

- The transition function has the form:
- δ: Q x Γ → *P*(Q x Γ x {L, R})
- So, the domain is an ordered pair, e.g., (q_0,1).
- Q x Γ x {L, R} looks like { (q_0,1,R),(q_0,0,R),(q_0,1,L),…}.
- *P*(Q x Γ x {L, R}) is the power set.
- *P*(Q x Γ x {L, R}) looks like { {}, {(q_0,1,R)}, {(q_0,1,R),(q_0,0,R)},…}
- So, if we see a 1 while in q_0 we might have to perform several activities…

Computing using a NTM

- A tree corresponds to the different possibilities. If some branch leads to an accept state, the machine accepts. If all branches lead to a reject state, the machine rejects.
- Solve subset sum in linear time with NTM:
- Set A = {a,b,c} and sum = x. Is there a subset of A summing to x? Suppose A = {1,2}, x = 3.   /   \
- for each element e of A   1 no 1

    take paths with and without e   / \   /\

accept if any path sums to x   2 no 2 2 no 2

accept reject reject reject

<u>Church-Turing Hypothesis</u>

Notes taken from "The Turing Omnibus", A.K. Dewdney

- Try as one might, there seems to be no way to define a mechanism of any type that computes more than a Turing machine is capable of computing.
- Note: On the previous slide we answered an NP-Complete problem in linear time with a non-deterministic algorithm.
- Quiz? Why does this not violate the Church-Turing Hypothesis?
- With respect to computability, non-determinism does not add power.

The Halting Problem

Notes taken from “Algorithmics The Sprit of Computing” by D. Harel

Consider the following algorithm A:

```
while(x != 1) x = x – 2;
stop
```

Assuming that its legal input consists of the positive integers <1,2,3,...>,It is obvious that A halts precisely for odd inputs. This problem can be expressed as a language recognition problem. How?

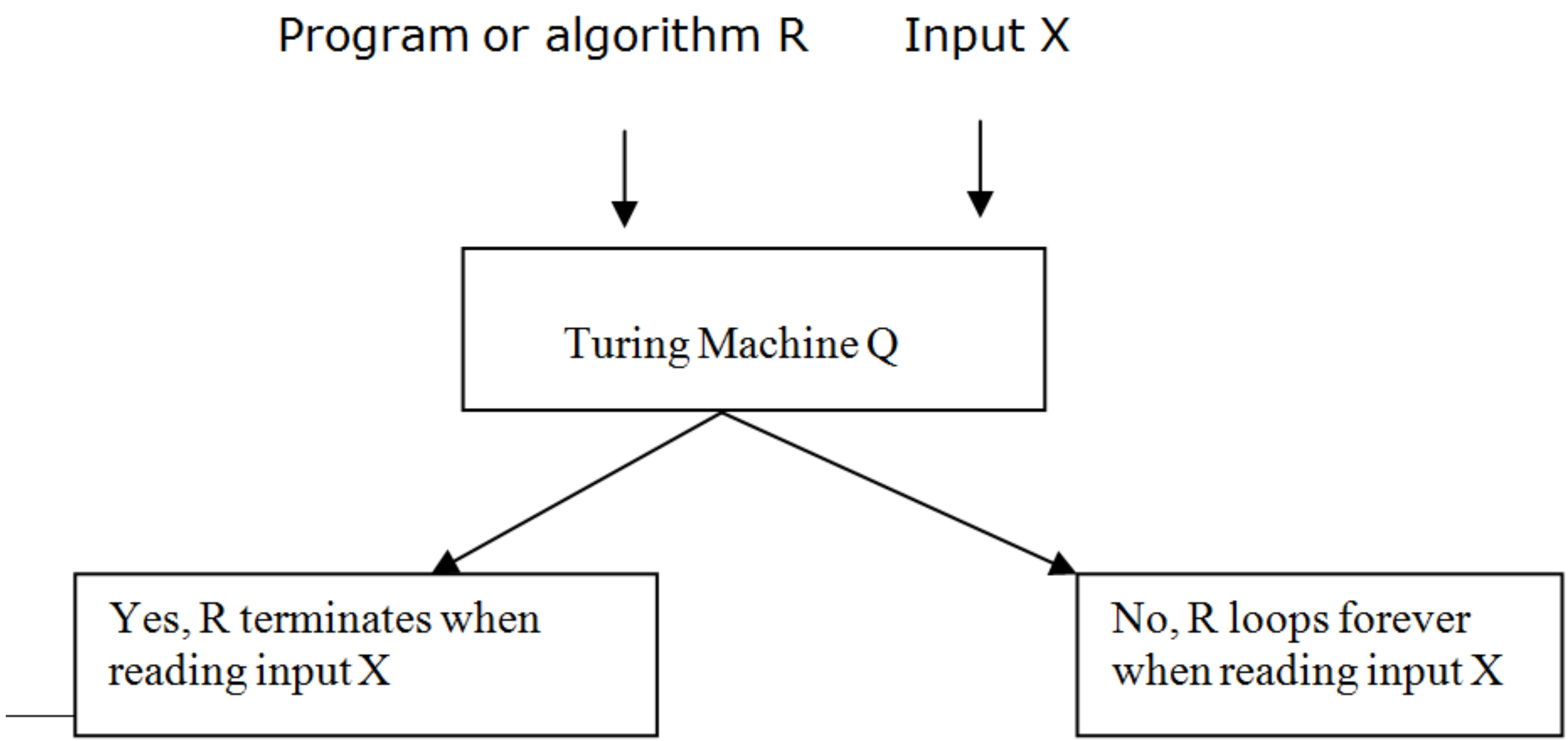
Now, consider Algorithm B:

```
while (x != 1) {
        if (x % 2 == 0) x = x / 2;
          else x = 3 * x + 1;
   }
```

No one has been able to offer a proof that B always terminates. This is an open question in number theory. This too may be expressed as a language recognition problem.

The halting problem is “undecidable”, meaning that there is no algorithm that will tell, in a finite amount of time, whether a given arbitrary program R, will terminate on a data input X or not.

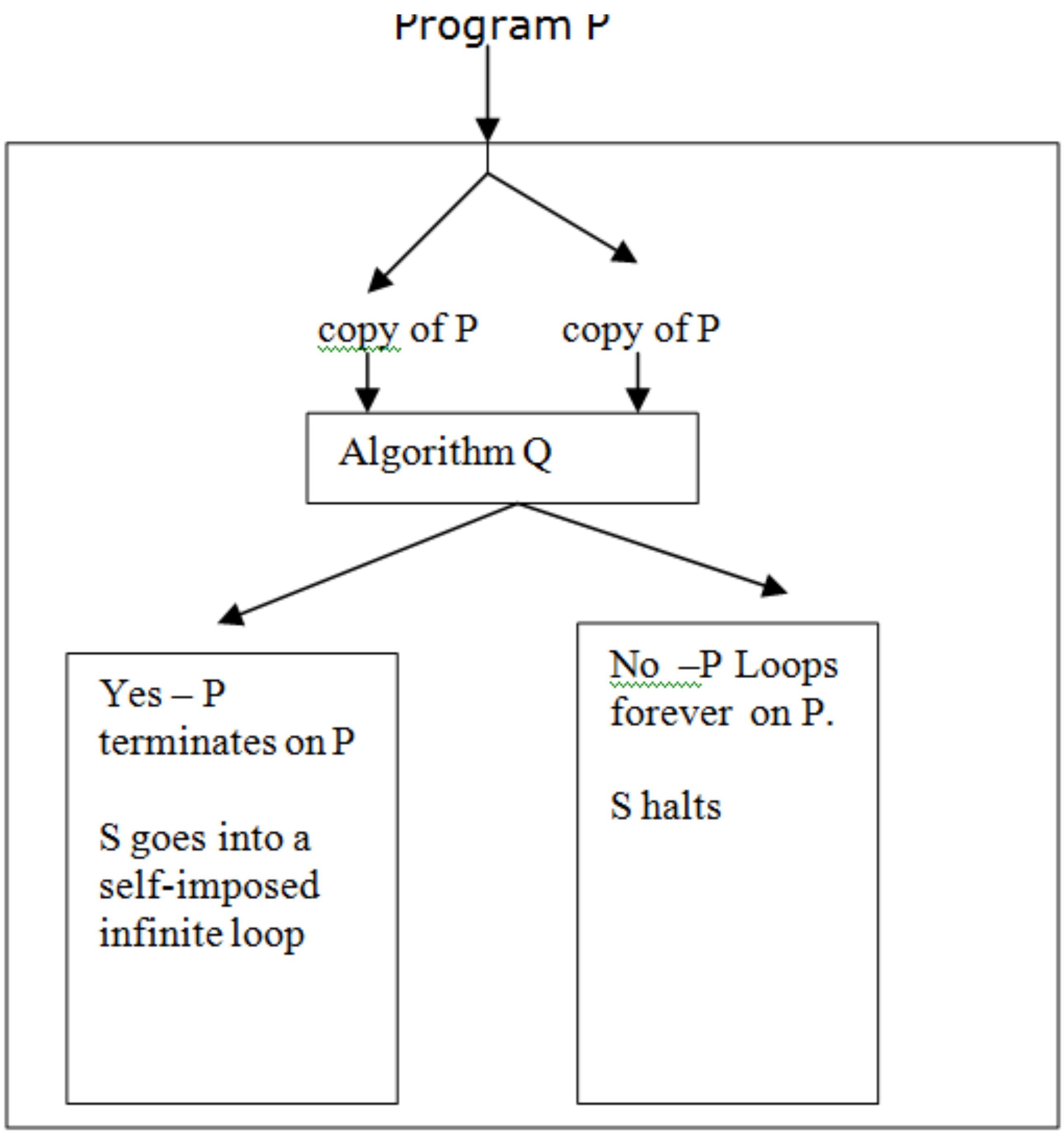
But let's build such a device anyway…



And let's use it as a subroutine…

- Build a new program S that uses Q in the following way.
- S first makes a copy of its input. It then passes both copies (one as a program and another as its input) to Q.
- Q makes its decision as before and gives its result back to S.
- S halts if Q reports that Q's input would loop forever.
- S itself loops forever if Q reports that Q's input terminates.

Program S

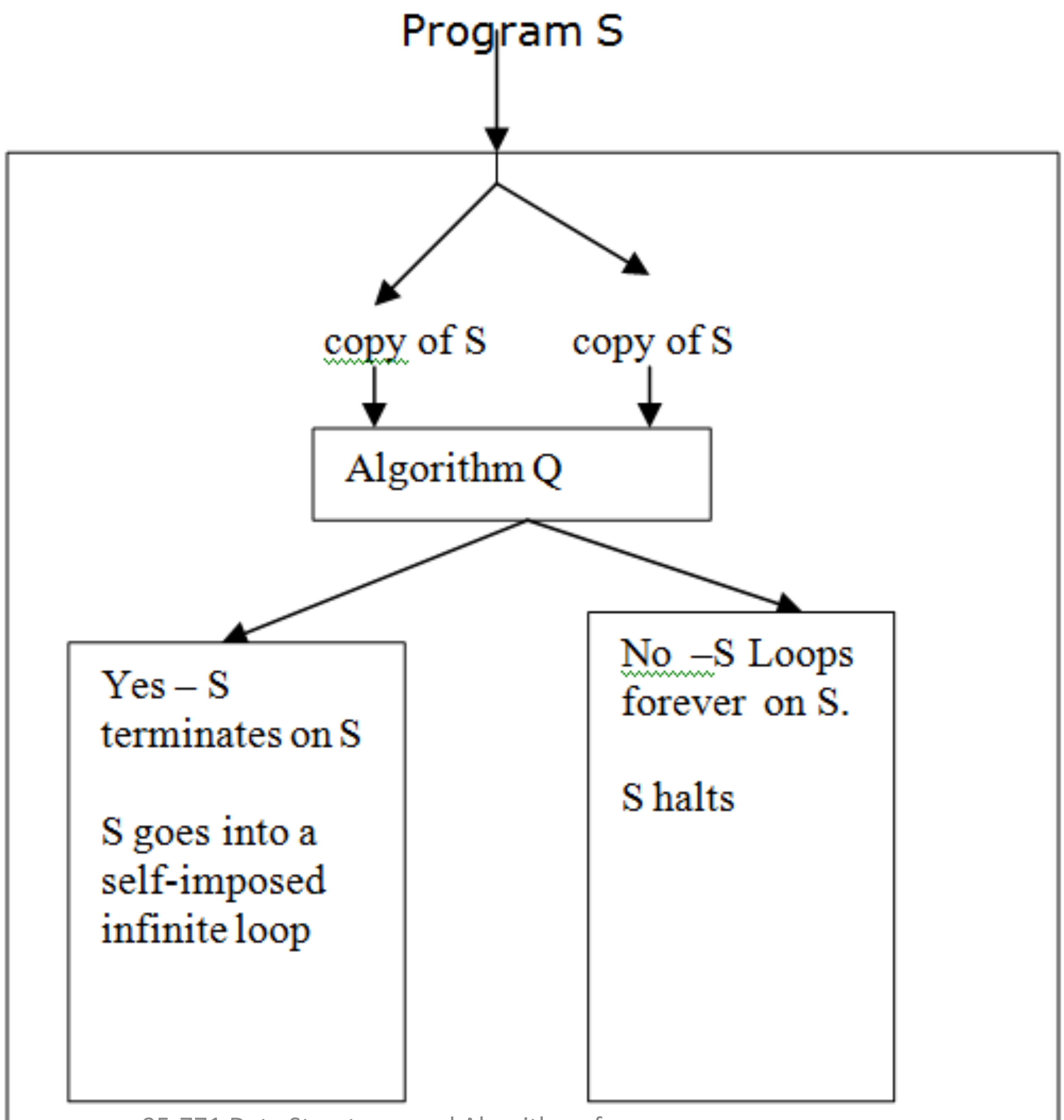


How much effort would It require for you to write S?

Assuming, of course, that Q is part of the Java API?

Program S

OK, so far so good. Now, pass S in to S as input.



- The existence of S leads to a logical contradiction. If S terminates when reading itself as input then Q reports this fact and S starts looping and never terminates. If S loops forever when reading itself as input then Q reports this to be the case and S terminates.
- The construction of S seems to be reasonable in many respects. It makes a copy of its input. It calls a function called Q. It gets a result back and uses that result to decide whether or not to loop (a bit strange but easy to program). So, the problem must be with Q. Its existence implies a contradiction. So, Q does not exist. The halting problem is undecidable.

Example: Malware Detection

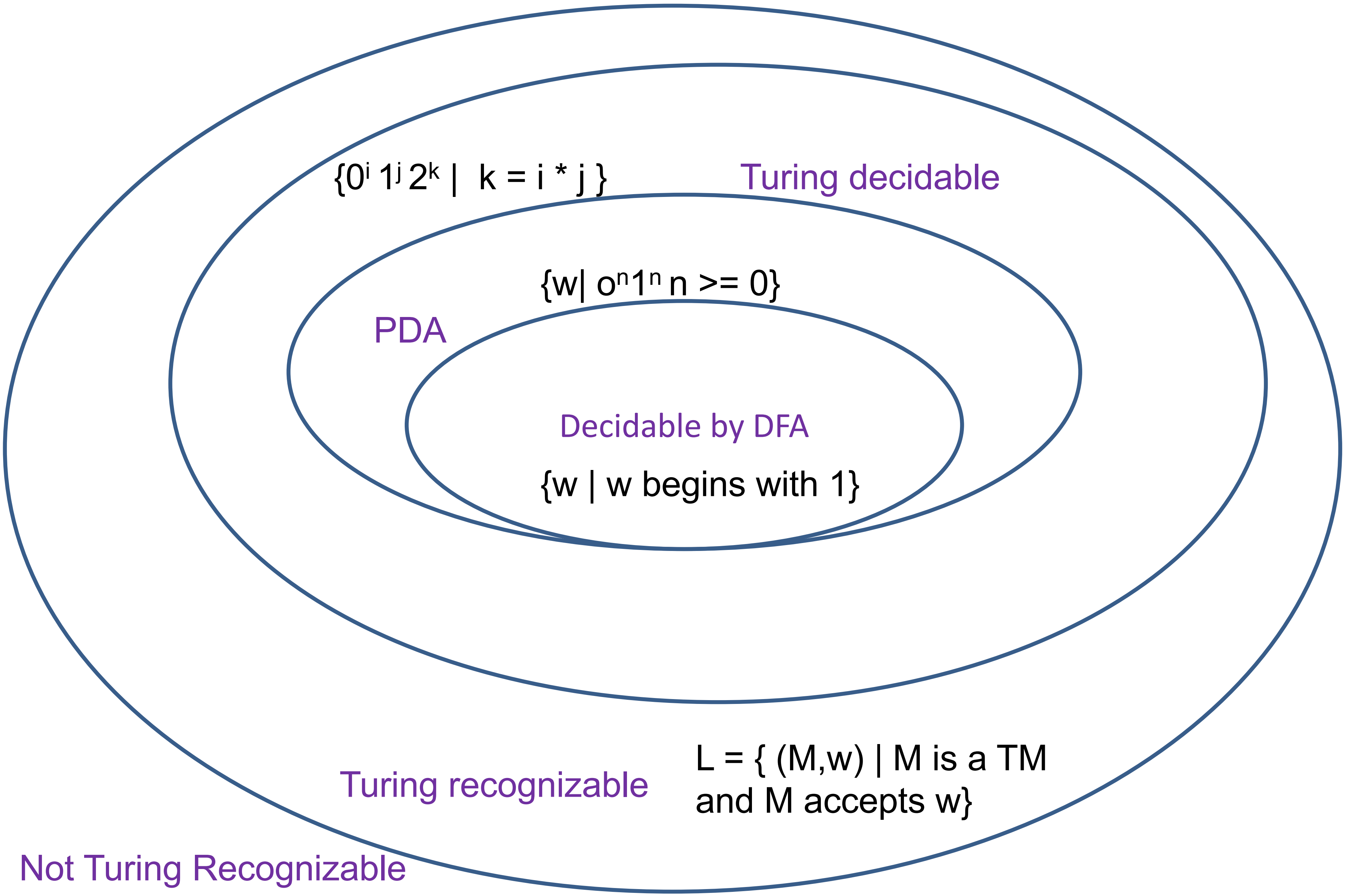
- Shown to be undecidable
- Do we give up?
- No – monitoring output of processes can still be fruitful

Terminology: Recursive and Recursively Enumerable notes from Wikipedia

- A formal language is ***recursive*** if there exists a Turing machine which halts for every given input and always either accepts or rejects candidate strings. This is also called a ***decidable*** language.
- A ***recursively enumerable*** language requires that some Turing machine halts and accepts when presented with a string <u>in the language</u>. It may either halt and reject or loop forever when presented with a string not in the language. A machine can ***recognize*** the language.
- The set of halting program integer pairs is in R.E. but is not recursive. We can't decide it but we can recognize it.
- All recursive (decidable) languages are recursively enumerable.

Recursive and Recursively Enumerable

- The set of halting program integer pairs is in R.E. but is not recursive.
- Are there any languages that are not recursively enumerable?
- Yes. Let L be { w = (program p, integer i) | p loops forever on i}.
- L is not recursively enumerable.
- We can't even recognize L.
- The set of languages is bigger than the set of Turing machines.



Co-L = {(M,w) | M is a TM and M rejects w or loops}

Some Results First

Computing Model	Finite Automata	Pushdown Automata	Linear Bounded Automata	Turing Machines
Language Class	Regular Languages	Context-Free Languages	Context-Sensitive Languages	Recursively Enumerable Languages
Non-determinism	Makes no difference	Makes a difference	No one knows	Makes no difference

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