Phase Portraits

Some notes and examples

1 What we're doing

We want to see what the solutions of the 2×2 system below look like.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Each solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ (remember how you parametrize curves in Calc 3) traces a curve in the xy-plane with time t as the parameter. We use arrows to indicate the direction of time.

In the case when the matrix has two eigenvalues, the general solution of the system looks like

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\, \mathbf{v}_1 + C_2 e^{\lambda_2 t}\, \mathbf{v}_2.$$

where $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors associated with the eigenvalues λ_1 and λ_2 of $\mathbf{A}$.

Looking at one of these guys for now:

$$e^{\lambda_1 t}\, \mathbf{v}_1$$

As what we've seen with the harmonic oscillator, the real part of the eigenvalue tells us the growth rate of the solutions, while the imaginary part dictates the oscillation.

Now we have a linear combination of two of these, by taking time to be big or very negative, depending on which eigenvalue has a larger real part, one of the terms would dominate, and the solution behavior would resemble the dominating term at those times.

Here's a summary table to sanity check your intuition.

(Assume the eigenvalue is in the form $\lambda = \alpha \pm i\beta$, with $\beta > 0$.)

Part to look at	Value	What it tells us (forward in time)
Real part α	$\alpha > 0$	Exponential growth: trajectories move away from the origin (source or spiral source).
	$\alpha < 0$	Exponential decay: trajectories move toward the origin (sink or spiral sink).
	$\alpha = 0$	No growth or decay. A complex pair gives a center; a zero real eigenvalue gives an equilibrium direction.
Imaginary part β	$\beta = 0$	No rotation; the eigenvalue is real.
	$\beta \neq 0$	Rotation present. Check a_{21} for the direction of rotation.[1]

[1] The velocity vector $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ tells us the direction in which the solution is moving. Evaluate it at a convenient point, say $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, to see which vertical direction the solution initially moves. There, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, so a_{21} tells us whether the trajectory moves up or down. This means counterclockwise motion when $a_{21} > 0$ and clockwise motion when $a_{21} < 0$.

You can also use the point $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and check a_{12}; in that case, the sign-direction correspondence is reversed.

2 Categories of solutions

There's a whole zoo of examples, but don't feel pressured — you'll get a feel of the pattern very soon...

In the graphs, blue curves are solution trajectories, and the orange lines mark invariant eigenlines or lines of equilibria (eigenvalue 0).

(a) Two distinct, real, nonzero eigenvalues

i. Both eigenvalues positive: nodal source

Eigenvalue test: $\lambda_1, \lambda_2 > 0$

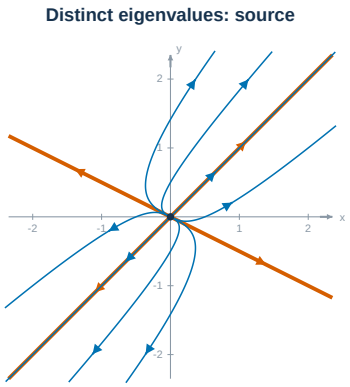
All trajectories move away from the origin. The two eigenlines give four straight-line half-trajectories. Other paths are tangent near the origin to the eigenvector for the eigenvalue closer to zero, and far away to the eigenvector for the eigenvalue with larger magnitude.

Example 1. $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$

Eigenpairs: $(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix})$ and $(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

The corresponding solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



Here the orange eigenlines are $x = -2y$ and $y = x$. Every other blue trajectory bends from the $\mathbf{v}_1$-direction in the far past toward the $\mathbf{v}_2$-direction in the far future.

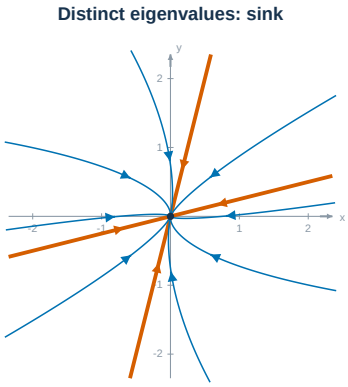
ii. Both eigenvalues negative: nodal sink

Eigenvalue test: $\lambda_1, \lambda_2 < 0$

All trajectories approach the origin along one of the two eigenlines or along a curved path. Near the origin, generic paths are tangent to the eigenvector whose eigenvalue is closest to zero; farther away, they align with the faster-decaying eigenvector.

Example 2. $\mathbf{A} = \begin{bmatrix} -46 & 4 \\ -4 & -29 \end{bmatrix}$

Eigenpairs: $(-45, \begin{bmatrix} 4 \\ 1 \end{bmatrix})$ and $(-30, \begin{bmatrix} 1 \\ 4 \end{bmatrix})$.



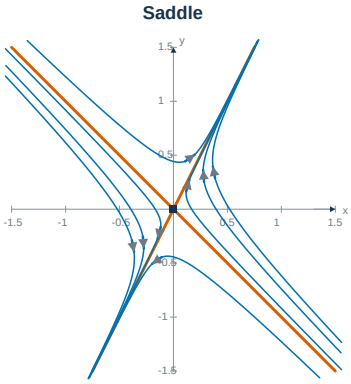
iii. Opposite signs: saddle

Eigenvalue test: $\lambda_1 \lambda_2 < 0$

There are two invariant eigenlines. Along the negative-eigenvalue eigenline, trajectories move toward the origin; along the positive-eigenvalue eigenline, they move away. Every other trajectory is asymptotic to the negative-eigenvalue eigenline as $t \to -\infty$ and to the positive-eigenvalue eigenline as $t \to +\infty$.

Example 3. $\mathbf{A} = \begin{bmatrix} -5 & 4 \\ 8 & -1 \end{bmatrix}$

Eigenpairs: $(-9, \begin{bmatrix} -1 \\ 1 \end{bmatrix})$ and $(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix})$.



(b) Complex-conjugate eigenvalues

i. Purely imaginary pair: center

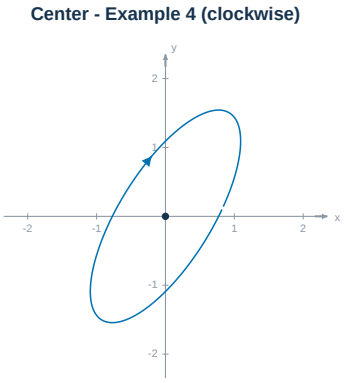
Eigenvalue test: $\lambda = \pm i\beta, \quad \beta > 0$

Trajectories are closed curves around the origin (circles for the canonical rotation system; in general they may be ellipses). To check rotation, look at the off-diagonal entries: for a center, a_{12} and a_{21} have opposite signs. At $(1,0)$, $y' = a_{21}$; at $(0,1)$, $x' = a_{12}$. Thus $a_{21} > 0$ (equivalently $a_{12} < 0$) means counterclockwise, while $a_{21} < 0$ (equivalently $a_{12} > 0$) means clockwise.

Example 4. $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$

Eigenvalues: $\pm 2i$.

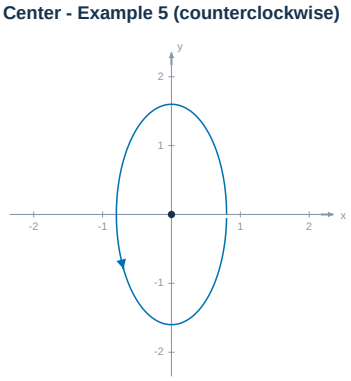
Here $a_{12} = 2 > 0$, so the rotation is clockwise.



Example 5. $\mathbf{A} = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$

Eigenvalues: $\pm 6i$.

Here $a_{12} = -3 < 0$, so the rotation is counterclockwise.



ii. Nonzero real part: spiral sink or source

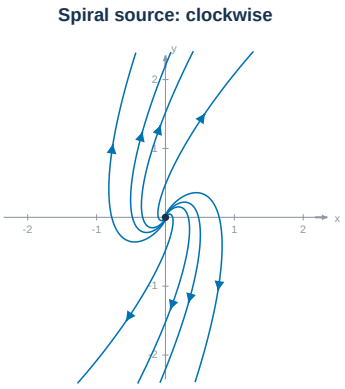
Eigenvalue test: $\lambda = \alpha \pm i\beta, \quad \beta > 0,\ \alpha \neq 0$

Trajectories spiral inward if $\alpha < 0$ and outward if $\alpha > 0$. To check rotation, use the same off-diagonal test as for a center: a_{12} and a_{21} have opposite signs. At $(0, 1)$, $x' = a_{12}$, so $a_{12} > 0$ (equivalently $a_{21} < 0$) gives clockwise motion; $a_{12} < 0$ (equivalently $a_{21} > 0$) gives counterclockwise motion. Examples 6 and 7 rotate in opposite directions, as do Examples 8 and 9.

Example 6. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$

Eigenvalues: $3 \pm 2i$ (spiral source).

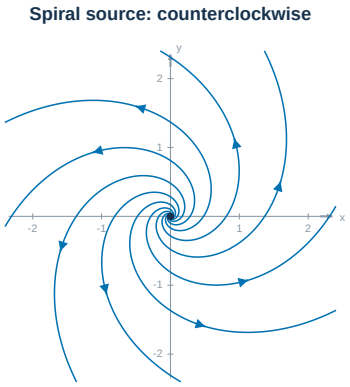
Here $a_{12} = 2 > 0$, so the rotation is clockwise.



Example 7. $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$

Eigenvalues: $2 \pm 3i$ (spiral source).

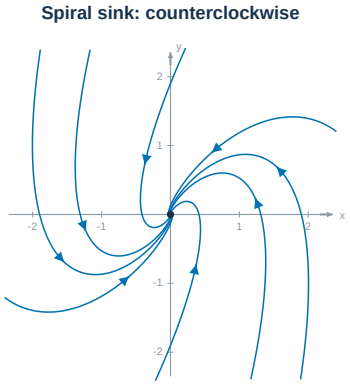
Here $a_{12} = -3 < 0$, so the rotation is counterclockwise.



Example 8. $\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}$

Eigenvalues: $-3 \pm 2i$ (spiral sink).

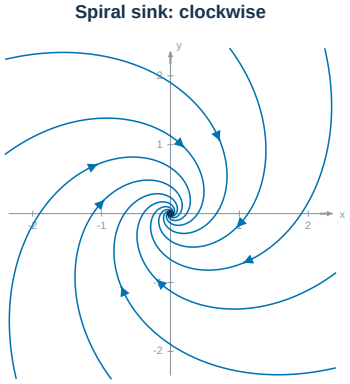
Here $a_{12} = -2 < 0$, so the rotation is counterclockwise.



Example 9. $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$

Eigenvalues: $-2 \pm 3i$ (spiral sink).

Here $a_{12} = 3 > 0$, so the rotation is clockwise.



(c) One zero eigenvalue and one nonzero eigenvalue

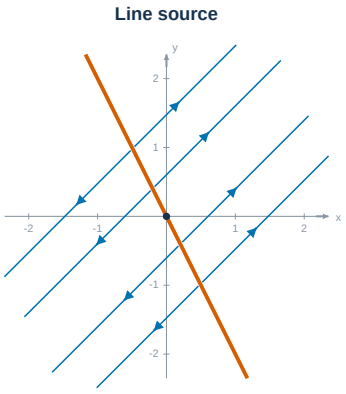
i. The other eigenvalue is positive: linear source

Eigenvalue test: $0, \lambda, \quad \lambda > 0$

The eigenline for eigenvalue zero is a line of equilibria. Other solutions move away from that line, parallel to the eigenvector for λ.

Example 10. $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

Eigenpairs: $(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix})$ and $(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.



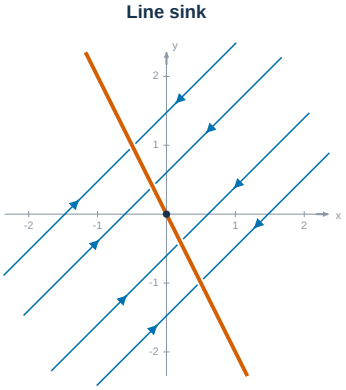
ii. The other eigenvalue is negative: linear sink

Eigenvalue test: $0, \lambda, \quad \lambda < 0$

The zero-eigenvalue eigenline is again a line of equilibria. Nearby solutions contract toward that line, parallel to the eigenvector for λ.

Example 11. $\mathbf{A} = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$

Eigenpairs: $(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix})$ and $(-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.



(d) Repeated eigenvalue and two independent eigenvectors

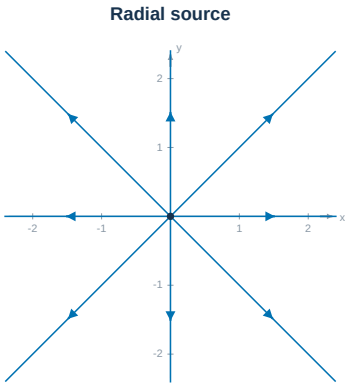
Here $\mathbf{A} = \lambda \mathbf{I}$, so every line through the origin is invariant. When $\lambda \neq 0$, nonzero trajectories are straight radial lines.

i. Positive eigenvalue: radial source

Eigenvalue test: $\lambda > 0$

Every nonzero solution moves directly away from the origin.

Example 12. $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

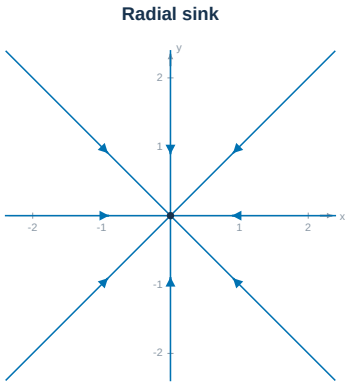


ii. Negative eigenvalue: radial sink

Eigenvalue test: $\lambda < 0$

Every nonzero solution moves directly toward the origin.

Example 13. $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

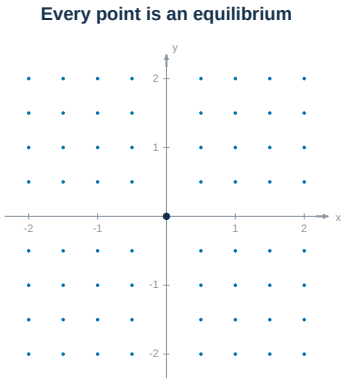


iii. Zero eigenvalue: every point is an equilibrium

Eigenvalue test: $\lambda = 0, \quad \mathbf{A} = 0$

Every nonzero vector is an eigenvector, and $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at every point. Every solution is constant.

Example 14. $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$



(e) Repeated real eigenvalue with only one eigenvector

This is the defective case: there is one eigenline. After a change of basis, every such matrix has the same Jordan form, $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Examples 15–16 show the shearing trajectories in Jordan coordinates; Examples 17–20 show the same behavior in the original coordinates. The zero-eigenvalue version gives parallel straight lines, as in Examples 21–22.

i. Jordan form for a nonzero repeated eigenvalue

Eigenvalue test: $\mathbf{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \lambda \neq 0$

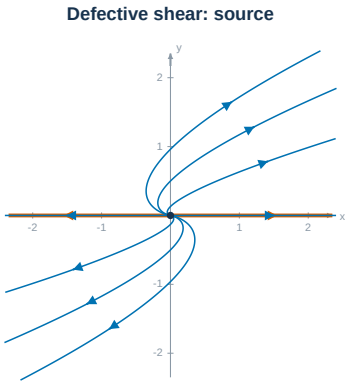
In these coordinates, let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be the eigenvector and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the generalized eigenvector, so $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$. The solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t}(C_1\mathbf{v} + C_2(\mathbf{w} + t\mathbf{v})) = e^{\lambda t}\begin{bmatrix} C_1 + tC_2 \\ C_2 \end{bmatrix}.$$

The $C_2\mathbf{w}$ term puts a trajectory off the eigenline, while $tC_2\mathbf{v}$ shifts it along that line as time changes. That is the shear: the sign of C_2 selects which side of the eigenline the trajectory lies on, and the generalized eigenvector tells us the transverse direction.

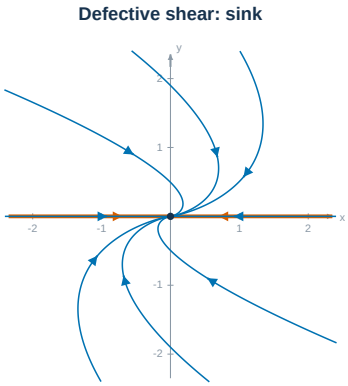
Example 15. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenpair: $(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$.



Example 16. $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

Eigenpair: $(-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$.



ii. Positive eigenvalue in the original coordinates

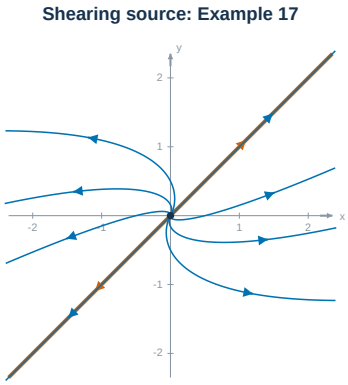
Eigenvalue test: $\lambda > 0$

These trajectories move away from the origin and bend toward the eigenline as time moves forward.

Example 17. $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

Eigenpair: $(3, \mathbf{v})$ with $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

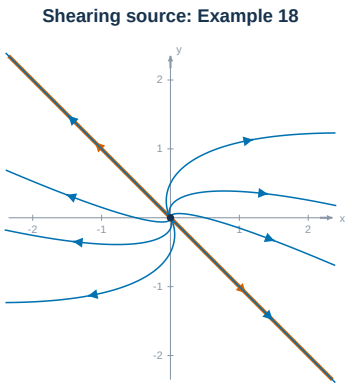
Generalized eigenvector: $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $(\mathbf{A} - 3\mathbf{I})\mathbf{w} = \mathbf{v}$.



Example 18. $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$

Eigenpair: $(3, \mathbf{v})$ with $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Generalized eigenvector: $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so $(\mathbf{A} - 3\mathbf{I})\mathbf{w} = \mathbf{v}$.



iii. Negative eigenvalue in the original coordinates

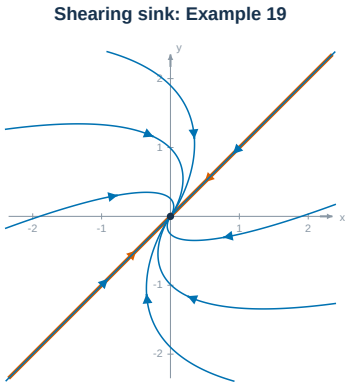
Eigenvalue test: $\lambda < 0$

These trajectories approach the origin and become tangent to the eigenline.

Example 19. $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$

Eigenpair: $(-2, \mathbf{v})$ with $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

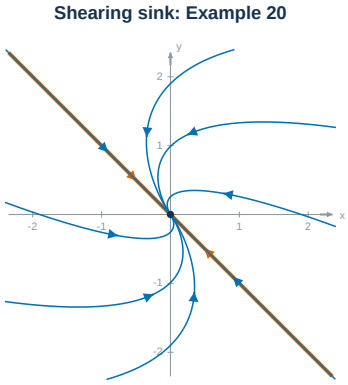
Generalized eigenvector: $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $(\mathbf{A} + 2\mathbf{I})\mathbf{w} = \mathbf{v}$.



Example 20. $\mathbf{A} = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$

Eigenpair: $(-2, \mathbf{v})$ with $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Generalized eigenvector: $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $(\mathbf{A} + 2\mathbf{I})\mathbf{w} = \mathbf{v}$.



iv. Zero eigenvalue: parallel shear

Eigenvalue test: $\lambda = 0, \quad \mathbf{A} \neq 0, \quad \mathbf{A}^2 = 0$

Because $\mathbf{A}^2 = 0$, the matrix exponential reduces to $e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A}$. For initial data $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = (\mathbf{I} + t\mathbf{A}) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t\mathbf{A} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The velocity $\mathbf{A} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is constant, so each non-equilibrium solution is a straight line. Since the range of $\mathbf{A}$ is its eigenline, these lines are parallel to the line of equilibria; initial points on that eigenline stay fixed.

Example 21. $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

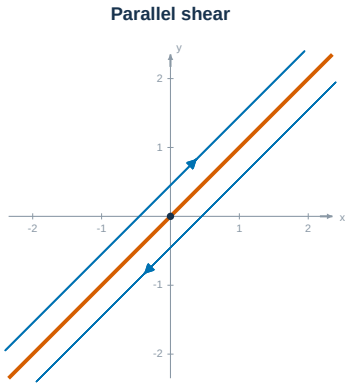
Eigenpair: $(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

For an initial point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$,

$$\mathbf{A}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = (y_0 - x_0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So if $y_0 > x_0$, the motion is up and right; if $y_0 < x_0$, it is down and left.

You can also test one point on each side of the equilibrium line, such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, if that's easier.



Example 22. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

Eigenpair: $(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

Here

$$\mathbf{A}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = (x_0 - y_0)\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so the motion is up and right when $x_0 > y_0$, and down and left when $x_0 < y_0$.

