Carefully read the problem instructions.

1. Give a real general solution for each of the following equations.

(a) $y'' - y = \dfrac{1}{1+e^t}$

The characteristic equation is

$$r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm 1,$$

so

$$y_h(t) = C_1 e^t + C_2 e^{-t}.$$

With $y_1 = e^t$, $y_2 = e^{-t}$, and

$$W(t) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2,$$

We use variation of parameters to get a particular solution

$$\begin{aligned} y_p(t) &= -y_1(t) \int \frac{y_2(t) g(t)}{W(t)} \, \mathrm{d}t + y_2(t) \int \frac{y_1(t) g(t)}{W(t)} \, \mathrm{d}t \\ &= \frac{e^t}{2} \int \frac{e^{-t}}{1+e^t} \, \mathrm{d}t - \frac{e^{-t}}{2} \int \frac{e^t}{1+e^t} \, \mathrm{d}t. \end{aligned}$$

Integrate by parts each of the integrals,

$$\int \frac{e^{-t}}{1+e^t} \, \mathrm{d}t = -e^{-t} + \ln(1+e^{-t}),$$

and

$$\int \frac{e^t}{1+e^t} \, \mathrm{d}t = \ln(1+e^t).$$

We get one particular solution

$$y_p(t) = \frac{1}{2} \left[-1 + e^t \ln(1+e^{-t}) - e^{-t} \ln(1+e^t) \right].$$

The general solution is therefore

$$\boxed{y(t) = C_1 e^t + C_2 e^{-t} + \frac{1}{2} \left[-1 + e^t \ln(1+e^{-t}) - e^{-t} \ln(1+e^t) \right].}$$

(b) $v'''' + 18v'' + 81v = 0$

The characteristic equation is

$$r^4 + 18r^2 + 81 = 0 \quad \Rightarrow \quad (r^2+9)^2 = 0.$$

The roots $r = \pm 3i$ each have multiplicity two. So the general solution is

$$\boxed{v(t) = (C_1 + C_2 t)\cos(3t) + (C_3 + C_4 t)\sin(3t).}$$

2. The functions t and t^2 are two solutions of the homogeneous equation

$$t^2 y'' - 2ty' + 2y = 0 \quad \text{over } t > 0.$$

(a) Compute their Wronskian.

The Wronskian is

$$W(t, t^2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2.$$

(b) Solve the IVP

$$t^2 y'' - 2ty' + 2y = \frac{t^3}{t^2+4}, \quad y(1) = y'(1) = 0, \quad \text{over } t > 0.$$

First divide everything by t^2 to put the equation into standard form:

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{t}{t^2+4}.$$

From the first part, we know that the two solutions $y_1(t) = t$ and $y_2(t) = t^2$ are independent whenever $t > 0$, and their Wronskian is $W(t) = t^2$.

Variation of parameters:

$$y_p(t) = y_1(t)\int \underbrace{\left(-\frac{y_2(t)g(t)}{W(t)}\right)}_{u_1'(t)}\,\mathrm{d}t + y_2(t)\int \underbrace{\frac{y_1(t)g(t)}{W(t)}}_{u_2'(t)}\,\mathrm{d}t$$

$$\begin{aligned}
&= -t\int \frac{t^2\cdot\frac{t}{t^2+4}}{t^2}\,\mathrm{d}t + t^2\int \frac{t\cdot\frac{t}{t^2+4}}{t^2}\,\mathrm{d}t \\
&= -t\int \frac{t}{t^2+4}\,\mathrm{d}t + t^2\int \frac{1}{t^2+4}\,\mathrm{d}t \\
&= -\frac{t}{2}\ln(t^2+4) + \frac{t^2}{2}\arctan\frac{t}{2}.
\end{aligned}$$

Thus the general solution is

$$y(t) = C_1 t + C_2 t^2 - \frac{t}{2}\ln(t^2+4) + \frac{t^2}{2}\arctan\frac{t}{2}.$$

Applying the IC, $y(1) = 0$ gives

$$C_1 + C_2 = \frac{1}{2}\ln 5 - \frac{1}{2}\arctan\frac{1}{2}.$$

Differentiating,

$$y'(t) = C_1 + 2C_2 t - \frac{1}{2}\ln(t^2+4) - \frac{t^2}{t^2+4} + t\arctan\frac{t}{2} + \frac{t^2}{t^2+4}.$$

The rational terms cancel, so $y'(1) = 0$ gives

$$C_1 + 2C_2 = \frac{1}{2}\ln 5 - \arctan\frac{1}{2}.$$

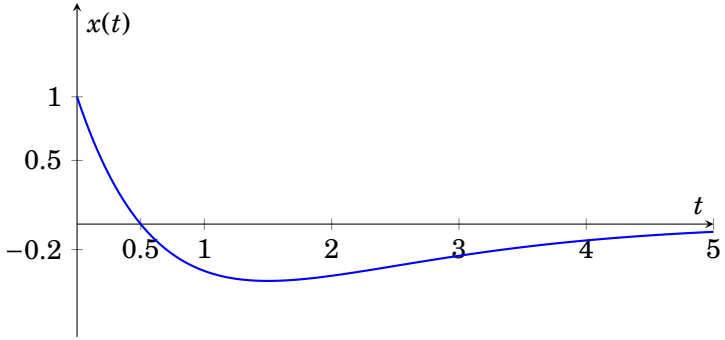
Together, the two IC equations tell us

$$C_2 = -\frac{1}{2}\arctan\frac{1}{2}, \qquad C_1 = \frac{1}{2}\ln 5.$$

And we conclude that the solution of the IVP is

$$\boxed{y(t) = \frac{t}{2}\ln\frac{5}{t^2+4} + \frac{t^2}{2}\left(\arctan\frac{t}{2} - \arctan\frac{1}{2}\right), \quad t > 0.}$$

3. Suppose the graph of the displacement of a damped harmonic oscillator as a function of time is given below.



Based on the graph, can you determine whether the system is overdamped, critically damped, or underdamped? Explain.

There's no oscillation, which rules out the underdamped case. Either of the other two is possible but we can't tell from the graph.

4. Suppose $y_1(t) = t^2$ is a solution to the differential equation

$$ty'' - y' = 0, \qquad t > 0.$$

(a) Find the Wronskian $W(y_1, y_2)(t)$ of a fundamental set of solutions.

Dividing the equation by t to put it into standard form:

$$y'' - \frac{1}{t}y' = 0.$$

Abel's identity says

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right) = W(t_0)\exp\left(\int_{t_0}^{t} \frac{1}{s}\,\mathrm{d}t\right) = W(t_0)t.$$

You can take $W(t_0)$ to be any nonzero constant.

(b) Find a second linearly independent solution $y_2(t)$ of the differential equation.

By the definition of the Wronskian,

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

Using $y_1(t) = t^2$ and $W(t) = Ct$, we obtain

$$t^2 y_2' - 2t y_2 = Ct.$$

Dividing by t^2 gives

$$y_2' - \frac{2}{t} y_2 = \frac{C}{t}.$$

This is a first-order linear equation with integrating factor

$$\mu(t) = \exp\left(-\int \frac{2}{t}\,\mathrm{d}t\right) = t^{-2}.$$

Multiplying the equation by t^{-2} yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-2} y_2\right) = Ct^{-3}.$$

Integrating, we find

$$t^{-2} y_2 = -\frac{C}{2t^2} + D,$$

and hence

$$y_2 = -\frac{C}{2} + Dt^2.$$

Choosing $C = -2$ and $D = 0$, we obtain

$$\boxed{y_2(t) = 1.}$$

Since 1 and t^2 are linearly independent on $(0,\infty)$, the general solution is

$$\boxed{y(t) = C_1 t^2 + C_2.}$$

5. Compute the Laplace transform of $f(t) = te^{-2t}u(t-5)$ from its definition. (Here u is the unit step function.) Give the domain of definition.

Compute from the definition of Laplace transform:

$$\mathscr{L}[f](s) = \int_0^\infty e^{-st} te^{-2t} u(t-5)\,\mathrm{d}t = \int_5^\infty te^{-(s+2)t}\,\mathrm{d}t$$

This is divergent when $-(s+2) \geqslant 0 \Leftrightarrow s \leqslant -2$. Otherwise,

proceed and integrate by parts:

$$\begin{aligned}\int_5^\infty te^{-(s+2)t}\,\mathrm{d}t &= \int_5^\infty t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{e^{-(s+2)t}}{s+2}\right)\mathrm{d}t \\ &= -\frac{te^{-(s+2)t}}{s+2}\bigg|_5^\infty + \frac{1}{s+2}\int_5^\infty e^{-(s+2)t}\,\mathrm{d}t \\ &= \boxed{5\,\frac{e^{-5(s+2)}}{s+2} + \frac{e^{-5(s+2)}}{(s+2)^2}, \qquad s>-2}.\end{aligned}$$

6. Use Laplace transform to solve the initial value problem

$$x''+4x=\delta(t-3), \qquad x(0)=5, \quad x'(0)=0.$$

You may refer to the table on the last page.

Laplace transform the equation:

$$\mathscr{L}[x''](s)+4\mathscr{L}[x](s)=\mathscr{L}[\delta(t-3)](s).$$

Compute the ingredients:

$$\begin{aligned}\mathscr{L}[x](s) &= X(s)\\ \mathscr{L}[x'](s) &= s\mathscr{L}[x](s)-x(0)=sX(s)-5\\ \mathscr{L}[x''](s) &= s\mathscr{L}[x'](s)-x'(0)=s^2X(s)-5s-0\\ \mathscr{L}[\delta(t-3)](s) &= \int_0^\infty \delta(t-3)e^{-st}\,\mathrm{d}t = e^{-3s}\end{aligned}$$

So the Laplace transform of the IVP is

$$(s^2+4)X(s)-5s=e^{-3s}$$

And so

$$X(s)=\frac{5s+e^{-3s}}{s^2+4}$$

Taking inverse transform,

$$\begin{aligned} x(t) = \mathscr{L}^{-1}[X](t) &= \mathscr{L}^{-1}\left[\frac{5s + e^{-3s}}{s^2+4}\right] \\ &= 5\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] + \mathscr{L}^{-1}\left[e^{-3s}\right] * \mathscr{L}^{-1}\left[\frac{1}{s^2+4}\right] \\ &= 5\cos(2t) + \delta(t-3) * \left[\frac{1}{2}\sin(2t)\right] \\ &= \boxed{5\cos(2t) + \tfrac{1}{2}u(t-3)\sin(2(t-3))} \end{aligned}$$

where I

- used the previous calculation $\mathscr{L}[\delta(\tau-3)] = e^{-3s}$ to get the inverse transform of e^{-3s}; and
- computed

$$\delta(\tau-3) * g(\tau)(t) = \int_0^t \delta(\tau-3)g(t-\tau)\,\mathrm{d}\tau = g(t-3)u(t-3).$$

The unit step function is there because the delta measure contributes only at $\tau = 3$, and the integration domain $[0,t]$ contains 3 only when $t \geqslant 3$.

Function	**Laplace transform**
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$
te^{at}	$\frac{1}{(s-a)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$