List of topics

independence and Wronskian

Abel

constant coefficient equations

nonhomogeneous equations and VoP

damped oscillators

Laplace and friends

Quick linear algebra refresh

For any **homogeneous linear** ODE, the solution set forms a vector space: sums and scalar multiples of solutions are also solutions (superposition).

For a second-order equation, we expect two degrees of freedom, so the solution space should be 2-dimensional. Two solutions y_1, y_2 form a **basis** when they are independent and span all solutions. Some people call such a basis a **fundamental set** of solutions. In this case, every solution can be written uniquely as

$$y = c_1 y_1 + c_2 y_2.$$

The **Wronskian** $W = y_1 y_2' - y_1' y_2$ checks independence between two functions y_1 and y_2.

If you can't remember crap like me: Wronskian

Think of physics and how a system is characterized by the position y and the momentum y'. We use the state vector

$$\begin{bmatrix} y \\ y' \end{bmatrix}$$

to represent this state. And the Wronskian between two "states" y_1, y_2 is just the determinant of the matrix formed by the two state vectors:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

(Hopefully you remember how to take the determinant...)

Abel's theorem

For two functions, the Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

taking a nonzero value at one point gives independence. For solutions of the same equation $y'' + p(x)y' + q(x)y = 0$, Abel's identity says

$$W(x) = W(x_0) \exp\left(- \int_{x_0}^{x} p(t)\, \mathrm{d}t \right).$$

B/c the exponential is nonvanishing, if the Wronskian vanishes once, it vanishes everywhere and the solutions are dependent; if nonzero once, the pair stays independent.

Method? Trick? Test at a convenient point to find out.

Try it: Abel's identity without solving the ODE

Two solutions of $xy'' - 2y' + 5xy = 0$ are defined on $x > 0$.
Suppose their Wronskian satisfies $W(1) = -3$.
Find $W(x)$ and determine whether the solutions are independent.

Solution: Abel's identity without solving the ODE

First put the equation into standard form

$$y'' - \frac{2}{x}y' + 5y = 0$$

We see that $p(x) = -2/x$, so

$$W(x) = -3\exp\left(-\int_1^x \frac{-2}{t}\,\mathrm{d}t\right) = -3e^{2\ln x} = -3x^2.$$

This is nonzero for $x > 0$, so the solutions are independent on that domain.

Finding a second linearly independent solution

If we're given only one solution y_1 of

$$y'' + p(t)y' + q(t)y = 0.$$

We can use both Abel's identity and the definition of Wronskian to cook up a second solution that's linearly independent from y_1.

1. **Use Abel:** $W(y_1, y_2)(t) = C \exp\left(-\int p(t)\,\mathrm{d}t\right), \quad C \neq 0.$
2. **Wronskian definition:** $W(y_1, y_2)(t) = y_1 y_2' - y_1' y_2$
3. **Set equal and find y_2:** Divide by y_1^2 (where $y_1 \neq 0$) to get

$$\left(\frac{y_2}{y_1}\right)' = \frac{Ce^{-\int p(t)\,\mathrm{d}t}}{y_1^2}.$$

Integrate and solve for y_2.

Constant coefficient equations

Your equation is $ay'' + by' + cy = 0$. You notice that $y = e^{rx}$ is an r-eigenfunction of $\frac{\mathrm{d}}{\mathrm{d}x}$, so you plug it into the equation to find the characteristic polynomial $ar^2 + br + c = 0$.

roots	fundamental solutions	general solution
distinct real r_1, r_2	$e^{r_1 x}, e^{r_2 x}$	$c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$\alpha \pm i\beta$	$e^{\alpha x}\cos\beta x,\ \ e^{\alpha x}\sin\beta x$	$e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$
repeated real r	e^{rx}, xe^{rx}	$(c_1 + c_2 x)e^{rx}$

You can kinda tell that the multiplicity of the root matches with the power of the monomial multiplying the exponential... The formula generalizes to higher order/ multiplicity, and it's the same deal for repeated complex roots.

Try it: constant coefficient equation eyeballing practice

Suppose the characteristic polynomial of some constant coefficient equation of unknown order has roots (listed with multiplicity)

$$-2+4i, \quad -2-4i, \quad -3, \quad -3, \quad -3, \quad 0, \quad 0.$$

(a) What is the order of L?

Try it: constant coefficient equation eyeballing practice

Suppose the characteristic polynomial of some constant coefficient equation of unknown order has roots (listed with multiplicity)

$$-2+4i, \quad -2-4i, \quad -3, \quad -3, \quad -3, \quad 0, \quad 0.$$

(a) What is the order of L?

Solution. The characteristic polynomial has seven roots, counting multiplicities. Hence L is a $\boxed{\text{7th-order}}$ operator.

Try it: const. coeff. eqn eyeballing practice cont.

$$-2+4i, \quad -2-4i, \quad -3, \quad -3, \quad -3, \quad 0, \quad 0.$$

(b) Find a real general solution of the equation.

Try it: const. coeff. eqn eyeballing practice cont.

$$-2+4i, \quad -2-4i, \quad -3, \quad -3, \quad -3, \quad 0, \quad 0.$$

(b) Find a real general solution of the equation.

Use the root-to-solution correspondence:

- $-2 \pm 4i$:
$$e^{-2t}\cos(4t), \qquad e^{-2t}\sin(4t).$$
- -3, with multiplicity 3:
$$e^{-3t}, \qquad te^{-3t}, \qquad t^2e^{-3t}.$$
- 0, with multiplicity 2:
$$1, \qquad t.$$

Put constants everywhere to get the (real) general solution:

$$u(t) = e^{-2t}\big(C_1\cos(4t) + C_2\sin(4t)\big) + (C_3t^2 + C_4t + C_5)e^{-3t} + C_6t + C_7,$$

$$C_1, \ldots, C_7 \in \mathbb{R}.$$

Nonhomogeneous equations: structure

For $y'' + p(x)y' + q(x)y = g(x)$, a general solution looks like

$$y = y_h + y_p$$

where y_h solves the homogeneous equation (gives 0) and y_p is any particular solution (gives g).

Variation of parameters: if y_1, y_2 form a fundamental pair and $W = y_1y_2' - y_1'y_2$, you guess a particular solution $y_p = u_1y_1 + u_2y_2$ by putting unknown functions $u_1(x), u_2(x)$ in place of the constants, then

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}.$$

and you integrate to find out.

Caution: put things into **standard form** first!

If you can't remember crap like me: VoP as matrix equation

You're trying the equation on $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.
The unknowns' derivatives u_1', u_2' satisfy

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \begin{matrix} \leftarrow \text{imposed to make life easier} \\ \leftarrow \text{ forcing} \end{matrix}$$

You solve the equation by inverting the state matrix!

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g(t) \end{bmatrix} = \frac{1}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

this gives you the u_i' formulas. You'll still need to remember how to invert a 2x2..

Try it: normalize before using VoP

Use variation of parameters to solve

$$3y'' + 3y = 6\sin x.$$

What is the forcing $g(x)$ in standard form? Give the general solution.

Solution: normalize before using VoP

Divide by 3: $y'' + y = 2\sin x$, so $g(x) = 2\sin x$.
You eyeball the homegeneous solutions $y_1 = \cos x$, $y_2 = \sin x$, and that $W = 1$. So, plugging in the formulas:

$$u_1' = -2\sin^2 x, \qquad u_2' = 2\sin x \cos x.$$

Integrate to find $u_1 = -x + \sin x \cos x$ and $u_2 = \sin^2 x$. So then $y_p = u_1 \cos x + u_2 \sin x = -x \cos x$. Finally:

$$\boxed{y = c_1 \cos x + c_2 \sin x - x \cos x.}$$

The factor x appears because the forcing frequency matches the natural frequency.

Damped oscillator: classify by the discriminant

The mass-spring-damper model is

$$mx'' + bx' + kx = 0, \quad m > 0,\ b \geqslant 0,\ k > 0.$$

Use the discriminant $\Delta = b^2 - 4mk$ to classify the damping:

- $\Delta < 0$: underdamped; decaying oscillations.
- $\Delta = 0$: critically damped; fastest return without oscillation.
- $\Delta > 0$: overdamped; nonoscillatory return.

The exponential factor in the roots sets decay; the imaginary part sets angular frequency.

Try it: compare two physical systems

For unit mass, compare the systems

$$\text{A: } x'' + 2x' + 5x = 0, \qquad \text{B: } x'' + 6x' + 5x = 0.$$

Classify each response and state which one oscillates.

Solution: compare two physical systems

For A, $b^2 - 4mk = 4 - 20 = -16 < 0$: roots $-1 \pm 2i$, so it is underdamped and oscillates with envelope e^{-t}.

For B, the discriminant is $36 - 20 = 16 > 0$: roots $-1, -5$, so it is overdamped and does not oscillate. Both decay, but only A crosses equilibrium repeatedly.

Laplace transform: definition and existence

For $t \geqslant 0$,

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t.$$

The kernel is e^{-st}, kinda exponentially suppresses the behavior of f, and more so the larger s is.

For convergence, enough to ask: f is piecewise continuous on every finite interval, and doesn't grow faster than e^{-st}.

The (one-sided) convolution:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$$

Laplace transform rules

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1 F + c_2 G \qquad \text{(linearity)}$$
$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad \text{(translation in } s\text{)}$$
$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s) \qquad \text{(convolution theorem)}$$

Try it: ezpz rule application

Suppose $\mathcal{L}\{f(t)\} = F(s) = \dfrac{1}{s+2}$. Find the following

1. $\mathcal{L}\{e^{3t}f(t)\}$
2. $\mathcal{L}\{(f * f)(t)\}$

Solution: identify a shift and a convolution

A reduction in suppression becomes a translation in s:

$$\mathcal{L}\{e^{3t}f(t)\} = F(s-3) = \frac{1}{s-1}.$$

The convolution theorem gives

$$\mathcal{L}\{(f * f)(t)\} = F(s)^2 = \frac{1}{(s+2)^2}.$$

Partial fractions: match the factors and powers

If you look at the Laplace table you notice that other than the simple monomial guy, the highest order appearing in the rational functions is 2, so thankfully the partial fractions we need to deal with aren't too complicated...

Include a term for every power of each denominator factor:

$$\frac{\text{something}}{(s-a)^m} \quad \rightsquigarrow \quad \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}.$$

For an irreducible quadratic $q(s)$, make the numerator linear at each power:

$$\frac{\text{something else}}{q(s)^m} \quad \rightsquigarrow \quad \frac{B_1 s + C_1}{q(s)} + \cdots + \frac{B_m s + C_m}{q(s)^m}.$$

Example: partial fraction setup

$$\frac{1}{(s+1)^2(s^2+4)}$$

Example: partial fraction setup

$$\frac{1}{(s+1)^2(s^2+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4}.$$

then put the RHS over a common denominator, multiply things out and set the numerator equal to that of the LHS. Pick convenient values of s to find ABCD. It's just algebra. Tedious algebra.

Try it: invert a product by convolution

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

using the convolution theorem. Then check your result with partial fractions.

Solution: invert a product by convolution

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then by the convolution theorem,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau.$$

Applying this to the problem,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\tfrac{1}{(s+1)(s+2)}\right\}(t) &= \mathcal{L}^{-1}\left\{\tfrac{1}{s+1}\right\}(\tau) * \mathcal{L}^{-1}\left\{\tfrac{1}{s+2}\right\}(\tau)(t) \\ &= \int_0^t e^{-\tau}e^{-2(t-\tau)}\,\mathrm{d}\tau = e^{-2t}\int_0^t e^{\tau}\,\mathrm{d}\tau = e^{-t} - e^{-2t}.\end{aligned}$$

Partial fraction gives $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$. Same thing after inversion!

Derivatives and initial values

Suppose $F(s) = \mathcal{L}\{f(t)\}$, we've the derivative rule

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Applying the rule again gives

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s^2F(s) - sf(0) - f'(0).$$

See the initial values there? It's almost like this would be useful for IVPs...

Workflow: transform every term; substitute the initial data; solve the resulting algebra equation for $F(s)$; use partial fraction or transform rules to invert and recover $f(t)$.

Laplace method: a second-order IVP

Solve

$$y'' + 3y' + 2y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Laplace method: a second-order IVP

Solve

$$y'' + 3y' + 2y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Transforming the equation gives

$$(s^2 Y - s) + 3(sY - 1) + 2Y = 0, \qquad Y(s) = \frac{s+3}{(s+1)(s+2)}.$$

Partial fractions yield $Y = \dfrac{2}{s+1} - \dfrac{1}{s+2}$. Invert term by term:

$$\boxed{y(t) = 2e^{-t} - e^{-2t}.}$$

Exam checklist: (ChatGPT wrote this)

1. Homogeneous, constant coefficients? Solve the characteristic equation.
2. Justifying two solutions? Compute the Wronskian; use Abel when applicable.
3. Finding a second solution? Abel and Wronskian, divide by y_1^2 then integrate.
4. Forcing present? Write $y = y_h + y_p$; for VoP normalize first and track W and signs.
5. Oscillator? Use discriminant to classify its roots and motion.
6. Laplace definition? Use e^{-st} and check convergence in s.
7. Laplace for ODE? Transform every term, partial fraction/transformation rules and transform back.

The End

I have nothing more to say. Go do the practice problems, sleep well, and good luck.

- complimentary fake exam problems
- solutions $\uparrow$
- guy who wrote the ODE exams when I was a student