List of topics

Differential Equations Basics

Slope Fields and Equilibria

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DE and IVP terminology

A **differential equation** relates an **unknown function** to one or more of its derivatives.

A DE can have many solutions (general solutions) due to integration constants. An **initial condition** (IC) selects a particular solution by fixing these constants.

Together, a DE and an IC make up an **initial value problem** (IVP). A **solution** to an IVP is a function that satisfies both the equation and the initial condition.

Try This: Identify the Parts of an IVP

Consider

$$\frac{\mathrm{d}^2 r}{\mathrm{d}y^2} + r^2 \frac{\mathrm{d}r}{\mathrm{d}y} - r = y, \qquad r(1) = 2.$$

1. What are the independent and dependent variables?
2. What is the unknown function?
3. What is the initial condition? Does the function $r(y) = y^2$ satisfy the initial condition?

Classifying a Differential Equation

- **ODE or PDE:** an ODE has derivatives with respect to one independent variable; a PDE has partial derivatives with respect to two or more independent variables.
- **Order:** the highest derivative that appears.
- **Autonomous:** the independent variable does not appear explicitly.
- **Linear:** terms involving the dependent variable appear only to the first power and are not multiplied together

  Caution: derivatives are linear!

Try This: Classify this Equation

Consider

$$\frac{\partial x}{\partial t} + 2xs = t + s.$$

Decide whether the equation is:

- an ODE or a PDE;
- first order or higher order;
- autonomous or non-autonomous;
- linear or nonlinear.

Try This: Classify this Equation

Consider

$$\frac{\partial x}{\partial t} + 2xs = t + s.$$

Decide whether the equation is:

- an ODE or a PDE; ODE (don't be fooled by ∂)
- first order or higher order; 1st order
- autonomous or non-autonomous; non-autonomous
- linear or nonlinear. linear

Slope Fields

For

$$\frac{dy}{dx} = f(x, y),$$

the **slope field** draws a short line segment of slope $f(x, y)$ at each point (x, y). (It's the gradient field of y if you know multivariable calculus.)

A solution curve (graph of the solution) is tangent to those line segments at every point it passes through.

Try This: Read a Slope Field Algebraically

Consider

$$\frac{dy}{dx} = x - y.$$

- Where are the slope segments horizontal?
- At $(0, 1)$, is a solution increasing or decreasing?
- At $(2, 1)$, is a solution increasing or decreasing?

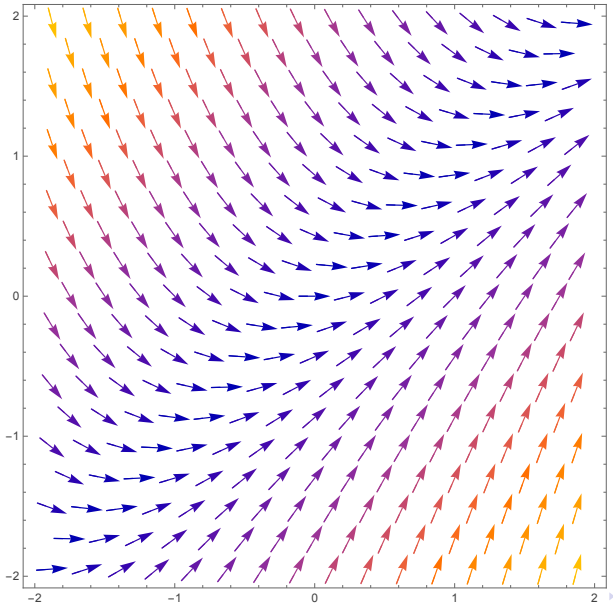
Try This: Read a Slope Field Algebraically

Consider

$$\frac{dy}{dx} = x - y.$$

- Where are the slope segments horizontal?
  On the diagonal, when $x = y$
- At $(0, 1)$, is a solution increasing or decreasing?
  slope = -1 $\Rightarrow$ solution decreasing
- At $(2, 1)$, is a solution increasing or decreasing?
  slope = 1 $\Rightarrow$ solution increasing

$y' = x - y$



Equilibrium Solutions

For an autonomous equation

$$\frac{dy}{dx} = f(y),$$

an equilibrium solution is a constant solution $y(x) = c$.

To find equilibria, solve

$$f(c) = 0.$$

Try This: Find the Equilibria

Consider

$$\frac{dy}{dx} = (y+2)y(y-1)(y-3).$$

1. Find all equilibrium solutions.
2. Check the signs of $\frac{dy}{dx}$ between the equilibria.

Stability

For an equilibrium $y = c$:

- **Stable:** nearby solutions approach c as $x \to \infty$.
- **Unstable:** some nearby solutions do not approach c as $x \to \infty$.
- **Semi-stable:** solutions approach c from one side but not the other.

Check signs of slope $\frac{dy}{dx}$ on either side of each equilibrium to determine the behavior.

Try This: Determine Stability

Suppose

$$\frac{dy}{dx} = 0.$$

Classify the equilibrium $y = 1$.

Separable Equations

A first-order ODE is **separable** when it can be written as

$$\frac{dy}{dx} = f(x)g(y).$$

In this case, separate the x,y-terms

$$\frac{1}{g(y)}\, dy = f(x)\, dx.$$

then integrate and solve for y.

Caution: Dividing by $g(y)$ assumes $g(y) \neq 0$; check the zeros for constant solutions first.

Which of the following are separable?

(A) $\dfrac{dy}{dx} = \dfrac{y \sin x}{1 + y^2}$

(B) $\dfrac{dy}{dx} = \dfrac{x(1 + y^2)}{y}$

(C) $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

(D) $\dfrac{dy}{dx} = e^x \cos y$

Which of the following are separable?

(A) $\dfrac{dy}{dx} = \dfrac{y \sin x}{1 + y^2}$ yes

(B) $\dfrac{dy}{dx} = \dfrac{x(1 + y^2)}{y}$ yes

(C) $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$ no

(D) $\dfrac{dy}{dx} = e^x \cos y$ yes

First-Order Linear ODEs

Every first-order linear ODE can be put into a standard form:

$$\frac{dy}{dx} + f(x)y = g(x).$$

dependent stuff independent stuff

- If $g(x) = 0$, the equation is **homogeneous**.
- If $g(x) \neq 0$, the equation is **inhomogeneous**.

Try This: Recognize a Linear Equation

Which of the following has the form $\frac{dy}{dx} + f(x)y = g(x)$?

(a) $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2$ (b) $\dfrac{dy}{dx} + y^2 = 1$

For the linear equation, identify $f(x)$ and $g(x)$, and decide whether it is homogeneous.

Homogeneous Linear Equations

For

$$\frac{dy}{dx} + f(x)y = 0,$$

the general solution is

$$y(x) = Ce^{-\int f(x)\,dx}.$$

(No need to memorize! Derive using separation of variables, integrating factor, or variation of parameters.)

Caution: Keep track of the interval (of x) on which $f(x)$ and the solution are defined.

Try This: Homogeneous Linear Equation

Find the general solution of

$$\frac{dy}{dx} + \frac{2}{x}y = 0.$$

State the region where your solution is defined.

If I also impose the IC $y(1) = 2$, what's the region of validity?

Inhomogeneous Linear Equations

For

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x)$$

the two methods we covered are:

- integrating factor;
- variation of parameters.

Integrating Factor

For

$$\frac{dy}{dx} + f(x)y = g(x)$$

The integrating factor is a magical function $M(x)$ with

$$M'(x) = f(x)M(x).$$

Multiplying the equation by $M(x)$ turns LHS turns into a derivative, so we can integrate both sides and then solve for y:

$$(yM)' = y'M + M'y = y'M + fMy = (y' + fy)M = gM$$

Integrating factor, continued...

Caution: The whole equation is multiplied by the IF! That includes the g on the RHS, don't forget...

Formula of integrating factor

$$M(x) = e^{\int f(x)\,\mathrm{d}x}$$

(Just remember it's exponential something that spits out a copy of the coefficient f when being differentiated).

Try This: Integrating Factor

Solve

$$\frac{dz}{dt} + \frac{z}{2+t} = -\frac{\sin(t)}{2+t}.$$

1. Find an integrating factor.
2. Rewrite the left-hand side as one derivative.
3. Integrate and solve for z.

Variation of Parameters

Solve the **complementary equation** (homogeneous version of the equation)

$$\frac{dy}{dx} + f(x)y = 0,$$

and get the complementary solution $\widetilde{y}$.
Assume the solution of the original equation looks like

$$y(x) = u(x)\widetilde{y}(x).$$

for some unknown u, substitute this into the original equation to find $u(x)$. In the end, put u and $\widetilde{y}$ together to recover y.

Try This: Variation of Parameters

Suppose we want to solve

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\,y = \frac{1}{1+x^2}.$$

I hand you the complementary solution

$$\widetilde{y} = \frac{C}{1+x^2}.$$

Find the general solution.

Existence and Uniqueness Theorem

Consider the IVP

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

If f and f_y are continuous in a neighborhood of (x_0, y_0), then the IVP has a unique solution on some interval containing x_0.

Caution: This conclusion is local: it guarantees a solution near the initial point, but not necessarily a solution for every value of x.

Try This: Apply the Theorem

Consider

$$\frac{dy}{dx} = \frac{e^{x^2}\sin(y^3) + \sqrt{1+x^2}}{2 + x^2 + y^2}, \qquad y(7) = -2.$$

Decide whether the existence and uniqueness theorem guarantees a unique local solution to the IVP.

Try This: Converse?

For the IVP

$$\frac{dy}{dx} = |y|, \qquad y(0) = 0.$$

identify $f(x, y)$ and $f_y(x, y)$, then check their continuity around $(0, 0)$. Decide whether the existence and uniqueness theorem applies.
Is there a unique local solution to the IVP?

Upshot: The theorem gives a sufficient condition ($\Rightarrow$), not a necessary one ($\Leftarrow$).

Picard Iteration: successive approximations to the solution

For an IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0,$$

Start with the constant

$$y_0(x) = y_0,$$

and define recursively

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} \underbrace{f\big(t, y_n(t)\big)}_{\text{previous guy fed into slope}} \, \mathrm{d}t.$$

y_0: same constant coming from IC

If the existence and uniqueness theorem applies, $\lim_{n\to\infty} y_n(x)$ is a (local) solution to the IVP.

Example: First Picard Iterates for the Exponential

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, \qquad y(0) = 1.$$

Starting with $y_0(x) \equiv 1$:

$$y_1(x) = 1 + \int_0^x 1 \, \mathrm{d}t = 1 + x,$$

$$y_2(x) = 1 + \int_0^x (1 + t) \, \mathrm{d}t = 1 + x + \frac{x^2}{2},$$

$$y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) \mathrm{d}t = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$