

1 Modal Logic

$\Box A$: A is necessarily true (i.e., A true in every reachable world).

$\Diamond A$: A is possibly true (i.e., A true in some reachable world).

1.1 Rules

Hypothesis

$$\frac{A \text{ true} \in \Gamma}{\Delta; \Gamma \vdash A \text{ true}} \text{ hyp} \quad \frac{A \text{ valid} \in \Delta}{\Delta; \Gamma \vdash A \text{ true}} \text{ hypv}$$

Conjunction

$$\frac{\Delta; \Gamma \vdash A \text{ true} \quad \Delta; \Gamma \vdash B \text{ true}}{\Delta; \Gamma \vdash A \wedge B \text{ true}} \wedge I \quad \frac{\Delta; \Gamma \vdash A \wedge B \text{ true}}{\Delta; \Gamma \vdash A \text{ true}} \wedge E_1 \quad \frac{\Delta; \Gamma \vdash A \wedge B \text{ true}}{\Delta; \Gamma \vdash B \text{ true}} \wedge E_2$$

Disjunction

$$\frac{\Delta; \Gamma \vdash A \text{ true}}{\Delta; \Gamma \vdash A \vee B \text{ true}} \vee I_1 \quad \frac{\Delta; \Gamma \vdash B \text{ true}}{\Delta; \Gamma \vdash A \vee B \text{ true}} \vee I_2$$

$$\frac{\Delta; \Gamma \vdash A \vee B \text{ true} \quad \Delta; \Gamma, A \text{ true} \vdash C \text{ true} \quad \Delta; \Gamma, B \text{ true} \vdash C \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} \vee E$$

Implication

$$\frac{\Delta; \Gamma, A \text{ true} \vdash B \text{ true}}{\Delta; \Gamma \vdash A \supset B \text{ true}} \supset I \quad \frac{\Delta; \Gamma \vdash A \supset B \text{ true} \quad \Delta; \Gamma \vdash A \text{ true}}{\Delta; \Gamma \vdash B \text{ true}} \supset E$$

Truth and Falsehood

$$\frac{}{\Delta; \Gamma \vdash T \text{ true}} TI \quad \frac{\Delta; \Gamma \vdash F \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} FE$$

Necessity

$$\frac{\Delta; \cdot \vdash A \text{ true}}{\Delta; \Gamma \vdash \Box A \text{ true}} \Box I \quad \frac{\Delta; \Gamma \vdash \Box A \text{ true} \quad \Delta, A \text{ valid}; \Gamma \vdash C \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} \Box E$$

$$\frac{\Delta; \Gamma \vdash \Box A \text{ true} \quad \Delta, A \text{ valid}; \Gamma \vdash C \text{ poss}}{\Delta; \Gamma \vdash C \text{ poss}} \Box E_p$$

Possibility

$$\frac{\Delta; \Gamma \vdash A \text{ poss}}{\Delta; \Gamma \vdash \Diamond A \text{ true}} \Diamond I \quad \frac{\Delta; \Gamma \vdash A \text{ true}}{\Delta; \Gamma \vdash A \text{ poss}} \text{ here} \quad \frac{\Delta; \Gamma \vdash \Diamond A \text{ true} \quad \Delta; A \text{ true} \vdash A \text{ poss} C}{\Delta; \Gamma \vdash C \text{ poss}} \Diamond E$$

1.2 Example Proofs

Provide derivations of the following judgements in Modal Logic.

Task 1. $\cdot \vdash \Box A \supset \Box \Box A$ true

$$\text{Solution 1: } \frac{\frac{\frac{\frac{}{\Box A \text{ true} \vdash \Box A \text{ true}}{\Box A \text{ true} \vdash \Box A \text{ true}}{\text{hyp}}}{\Box A \text{ true} \vdash \Box \Box A \text{ true}}{\Box I}}{\Box A \supset \Box \Box A \text{ true}}{\supset I}}{\Box E}$$

Task 2. $\cdot \vdash \Box A \supset \Box \Diamond \Box A$ true

Solution 2:

$$\frac{\frac{\frac{\frac{\frac{\frac{}{A \text{ valid}; \cdot \vdash A \text{ true}}{A \text{ valid}; \cdot \vdash A \text{ true}}{\text{hypv}}}{A \text{ valid}; \cdot \vdash \Box A \text{ true}}{\Box I}}{A \text{ valid}; \cdot \vdash \Box A \text{ poss}}{\text{here}}}{A \text{ valid}; \cdot \vdash \Diamond \Box A \text{ true}}{\Diamond I}}{\Box I}}{\cdot \vdash \Box A \text{ true} \vdash \Box \Diamond \Box A \text{ true}}{\text{hyp}}}{\cdot \vdash \Box A \supset \Box \Diamond \Box A \text{ true}}{\supset I}$$

Task 3. $\cdot \vdash (\Box(A \supset B)) \supset \Box A \supset \Box B$ true

Solution 3:

$$\frac{\frac{\frac{\frac{\frac{}{\cdot \vdash \Box(A \supset B) \text{ true}, \Box A \text{ true} \vdash \Box(A \supset B) \text{ true}}{\cdot \vdash \Box(A \supset B) \text{ true}, \Box A \text{ true} \vdash \Box(A \supset B) \text{ true}}{\text{hyp}}}{\cdot \vdash \Box(A \supset B) \text{ true}, \Box A \text{ true} \vdash \Box B \text{ true}}{\supset I}}{\cdot \vdash \Box(A \supset B) \text{ true} \vdash \Box A \supset \Box B \text{ true}}{\supset I}}{\cdot \vdash (\Box(A \supset B)) \supset \Box A \supset \Box B \text{ true}}{\supset I}}{\frac{M \quad N}{A \supset B \text{ valid}; \Box(A \supset B) \text{ true}, \Box A \text{ true} \vdash \Box B \text{ true}}{\Box E}}{\Box E}$$

Where M =

$$\frac{}{A \supset B \text{ valid}; \Box A \supset B \text{ true}, \Box A \text{ true} \vdash \Box A \text{ true}}{\text{hyp}}$$

N =

$$\frac{\frac{\frac{\frac{}{A \supset B \text{ valid}, A \text{ valid}; \cdot \vdash A \supset B \text{ true}}{A \supset B \text{ valid}, A \text{ valid}; \cdot \vdash A \supset B \text{ true}}{\text{hypv}}}{A \supset B \text{ valid}, A \text{ valid}; \cdot \vdash \Box B \text{ true}}{\supset I}}{\frac{A \supset B \text{ valid}, A \text{ valid}; \Box(A \supset B) \text{ true}, \Box A \text{ true} \vdash \Box B \text{ true}}{\Box I}}{\supset E}$$

Task 4. $\therefore \vdash \Box(A \supset B) \supset \Diamond A \supset \Diamond B$ true

Solution 4:

$$\frac{\frac{\frac{\frac{\cdot; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash \Box(A \supset B) \text{ true}}{\cdot; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash B \text{ poss}}{\cdot; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash \Diamond B \text{ true}}{\cdot; \Box(A \supset B) \text{ true} \vdash \Diamond A \supset \Diamond B \text{ true}}}{\cdot; \vdash \Box(A \supset B) \supset \Diamond A \supset \Diamond B \text{ true}}}{M} \Box E_p$$

where M =

$$\frac{\frac{\frac{\frac{\cdot; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash \Diamond A \text{ true}}{A \supset B \text{ valid}; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash A \supset B \text{ true}}{A \supset B \text{ valid}; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash B \text{ true}}{A \supset B \text{ valid}; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash B \text{ poss}}}{A \supset B \text{ valid}; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash B \text{ poss}}}{A \supset B \text{ valid}; \Box(A \supset B) \text{ true}, \Diamond A \text{ true} \vdash B \text{ poss}} \Diamond E$$

X =

$$\frac{\cdot; \vdash A \supset B \text{ true}}{A \supset B \text{ valid}; A \text{ true} \vdash A \text{ true}} \text{hyp}$$