

1 Quantifiers

Up to now, we have been vague about what, exactly, our atomic propositions A are representing. In order to discuss quantification, however, we need to be precise over what, exactly, we are quantifying over. We do this via a new judgment $t : \tau$, where τ is some to-be-defined type. Oftentimes, we are interested in some particular type, like the type of natural numbers or the type of Turing Machines, but the meaning of the \exists and \forall connectives are independent of this.

The rules for verifying these are as follows:

$$\begin{array}{c}
 \overline{a : A} \\
 \vdots \\
 A(a) \uparrow \\
 \hline
 \forall x : \tau. A(x) \uparrow \quad \forall I^a
 \end{array}
 \qquad
 \begin{array}{c}
 \forall x : \tau. A(x) \downarrow \quad t : \tau \\
 \hline
 A(t) \downarrow \quad \forall E
 \end{array}$$

$$\begin{array}{c}
 t : \tau \quad A(t) \uparrow \\
 \hline
 \exists x : \tau. A(x) \uparrow \quad \exists I
 \end{array}
 \qquad
 \begin{array}{c}
 \overline{a : \tau} \quad \overline{A(a) \downarrow}^u \\
 \vdots \\
 C \uparrow \\
 \hline
 \exists x : \tau. A(x) \downarrow \quad C \uparrow \quad \exists E^{a,u}
 \end{array}$$

By now, you should be comfortable with erasing the arrows to recover the rules defining these connectives for natural deduction. The intuition for these rules should be straightforward – to prove that some proposition $A(x)$ is true for all $x : \tau$, we should be able to derive $A(c)$ true for some arbitrary $c : \tau$. Similarly, we can introduce an existential by demonstrating some object satisfying the proposition.

Eliminating forall is similarly simple. To eliminate an existential, however, we must do a little more work. If we have $\exists x : \tau. A(x)$, then we may not assume anything else about the witness! It must be an object of type τ , and also that it satisfies $A(x)$, but any other properties must be abstracted out, to be replaced with an arbitrary object with the known properties.

2 Examples with quantifiers

Consider predicates $A(x)$ and $B(x)$ which depend on $x : \tau$.

Task 1. Show $\forall x : \tau. A(x) \wedge B(x) \supset \forall x : \tau. A(x) \wedge \forall x : \tau. B(x)$ true.

Solution 1:

$$\begin{array}{c}
 \frac{\overline{\forall x : \tau. A(x) \wedge B(x) \text{ true}}^p \quad \overline{u : \tau}}{A(u) \wedge B(u) \text{ true}} \forall E \quad \frac{\overline{\forall x : \tau. A(x) \wedge B(x) \text{ true}}^p \quad \overline{v : \tau}}{A(v) \wedge B(v) \text{ true}} \forall E \\
 \frac{A(u) \wedge B(u) \text{ true}}{A(u) \text{ true}} \wedge E_1 \quad \frac{A(v) \wedge B(v) \text{ true}}{A(v) \text{ true}} \wedge E_2 \\
 \frac{A(u) \text{ true}}{\forall x : \tau. A(x) \text{ true}} \forall I_u \quad \frac{A(v) \text{ true}}{\forall x : \tau. B(x) \text{ true}} \forall I_v \\
 \frac{\forall x : \tau. A(x) \wedge \forall x : \tau. B(x) \text{ true}}{\forall x : \tau. A(x) \wedge B(x) \supset \forall x : \tau. A(x) \wedge \forall x : \tau. B(x) \text{ true}} \supset I^p
 \end{array}$$

Next, let $A(x, y)$ be a formula with two variables $x : \tau$ and $y : \sigma$.

Task 2. Show that you can “swap” an existential and universal. Do a verification proof. To prove this, show that $\exists x : \tau. \forall y : \sigma. A(x, y) \supset \forall y : \sigma. \exists x : \tau. A(x, y)$.

3.1 All of the rules in one place!

Here are all of the Heyting arithmetic rules.

$$\begin{array}{c}
 \frac{}{0 : \text{nat}} \text{nat}I_0 \quad \frac{x : \text{nat}}{s x : \text{nat}} \text{nat}I_S \\
 \\
 \frac{x : \text{nat} \quad \overline{C(0) \text{ true}} \quad \overline{C(s y) \text{ true}} \quad \overline{C(y) \text{ true}}}{\overline{C(x) \text{ true}}} \text{nat}E^{y,u} \\
 \\
 \frac{}{0 = 0 \text{ true}} = I_{00} \quad \frac{x = y \text{ true}}{s x = s y \text{ true}} = I_{SS} \\
 \\
 \frac{0 = s x \text{ true}}{C \text{ true}} = E_{0S} \quad \frac{s x = 0 \text{ true}}{C \text{ true}} = E_{S0} \quad \frac{s x = s y \text{ true}}{x = y \text{ true}} = E_{SS} \\
 \\
 \frac{}{R(0, t_0, x.r.t_S) \Rightarrow_R t_0} \Rightarrow_R I_0 \quad \frac{}{R(s n, t_0, x.r.t_S) \Rightarrow_R [R(n, t_0, x.r.t_S)/r][n/x]t_S} \Rightarrow_R I_S \\
 \\
 \frac{A(x) \text{ true} \quad x \Rightarrow_R y}{A(y) \text{ true}} \Rightarrow_R E_1 \quad \frac{A(y) \text{ true} \quad x \Rightarrow_R y}{A(x) \text{ true}} \Rightarrow_R E_2
 \end{array}$$

3.2 Working with these ideas

Task 3. The judgmental form of the principle of induction can be used to show the following more traditional formulation that uses universal quantification:

$$\forall n : \text{nat}. C(0) \supset (\forall x : \text{nat}. C(x) \supset C(S x)) \supset C(n) \text{ true}.$$

Solution 3:

$$\frac{\frac{\frac{}{n : \text{nat}} \quad \overline{C(0)}^u}{\overline{C(n)}} \quad \frac{\frac{\overline{\forall x : \text{nat}. C(x) \supset C(S x)}^v \quad \overline{x : \text{nat}}}{\overline{C(x) \supset C(S x)}} \quad \forall E \quad \overline{C(x)}^w}{\overline{C(S x)}} \supset E}{\overline{C(n)}} \text{nat}E^{x,w}}{\overline{\forall x : \text{nat}. C(x) \supset C(S x)} \supset C(n)} \supset I^v}{\overline{\forall n : \text{nat}. C(0) \supset (\forall x : \text{nat}. C(x) \supset C(S x)) \supset C(n)}} \forall I^n$$

Task 4. (BONUS) Prove

$$\forall n : \text{nat}. R(n, 0, x.r.S (S r)) = R(n, n, x.r.S r) \text{ true}$$

You may assume for the purposes of this proof that $R(n, S y, x.r.S r) \Rightarrow_R R(S (R(n, y, x.r.S r)))$ (note that while they are equivalent, neither side actually reduces to the other).

Furthermore, *although this is not a rule*, assume you may step underneath successors, as if you have a rule $\Rightarrow_R I^*$ with the premise $x \Rightarrow_R y$ and conclusion $S x \Rightarrow_R S y$.

Solution 4: (BONUS) Let the following section be called X:

$$\frac{\frac{0 = 0 \text{ true}}{R(0, 0, x.r.S (S r)) \Rightarrow_R 0 \text{ true}} = I_{00} \quad \frac{}{R(0, 0, x.r.S (S r)) \Rightarrow_R 0 \text{ true}} \Rightarrow_R I_0}{\frac{}{R(0, 0, x.r.S (S r)) = 0 \text{ true}} \Rightarrow_R E_2} \frac{}{R(0, 0, x.r.S r) \Rightarrow_R 0 \text{ true}} \Rightarrow_R I_0}{\frac{}{R(0, 0, x.r.S (S r)) = R(0, 0, x.r.S r) \text{ true}} \Rightarrow_R E_2}$$

Let the following section be called Y:

$$\frac{\frac{\overline{R(y, 0, x.r.S(S r)) = R(y, y, x.r.S r) \text{ true}}^u}{S(R(y, 0, x.r.S(S r))) = S(R(y, y, x.r.S r)) \text{ true}} = I_{SS}}{S(S(R(y, 0, x.r.S(S r)))) = S(S(R(y, y, x.r.S r))) \text{ true}} = I_{SS} \quad \frac{\overline{R(S y, 0, x.r.S(S r)) \Rightarrow_R S(S(R(y, 0, x.r.S(S r)))) \text{ true}}}{R(S y, 0, x.r.S(S r)) = S(S(R(y, y, x.r.S r)))} \Rightarrow_R I_S \Rightarrow_R E_2$$

Let the following section be called Z:

$$\frac{\frac{\overline{R(S y, y, x.r.S r) \Rightarrow_R S(R(y, y, x.r.S r)) \text{ true}} \Rightarrow_R I_S}{Y \quad \frac{\overline{S(R(S y, y, x.r.S r)) \Rightarrow_R S(S(R(y, y, x.r.S r))) \text{ true}} \Rightarrow_R I^*}{R(S y, 0, x.r.S(S r)) = S(R(S y, y, x.r.S r)) \text{ true}} \Rightarrow_R E_2}}{\frac{\overline{R(S y, S y, x.r.S r) \Rightarrow_R S(R(S y, y, x.r.S r)) \text{ true}} \text{ given}}{R(S y, 0, x.r.S(S r)) = R(S y, S y, x.r.S r) \text{ true}} \Rightarrow_R E_2}$$

The full proof is then:

$$\frac{\frac{\overline{a : \text{nat}} \quad X \quad Z}{R(a, 0, x.r.S(S r)) = R(a, a, x.r.S r) \text{ true}} \text{ natE}^{y,u}}{\forall n : \text{nat}. R(n, 0, x.r.S(S r)) = R(n, n, x.r.S r) \text{ true}} \forall I^n$$