Constructive Logic (15-317), Spring 2022
Recitation 2: Harmony (2022-01-26)
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1 Harmony

Proof-theoretic harmony is a necessary, but not sufficient, condition for the well-behavedness of a logic; harmony ensures that the connectives are *locally* well-behaved, and is closely related to the critical cases of cut and identity elimination which we may discuss later on. Therefore, when designing or extending a logic, checking harmony is a first step.

From the verificationist standpoint, a connective is *harmonious* if its elimination rules are neither too strong nor too weak in relation to its introduction rules. The first condition is called *local soundness* and the second condition is called *local completeness*. The content of the soundness condition is a method to reduce or simplify proofs (by demonstrating there is a more direct proof to the detour), which is shown via *local reduction(s)*. The content of completeness is a method to expand any arbitrary proof into a canonical proof (i.e. one that ends in an introduction rule), which is shown via *local expansion(s)*. Later on in this course, we shall justify local reductions and local expansions in terms of reduction and expansion in programming.

1.1 Conjunction

Local soundness for conjunction is witnessed by the following two reduction rules:

$$\dfrac{\dfrac{\boxed{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix}} \quad \begin{matrix}\mathcal{E}\\ B\ \mathit{true}\end{matrix}}{A \wedge B\ \mathit{true}}\wedge\mathsf{I}}{A\ \mathit{true}}\wedge\mathsf{E}_1 \quad \longrightarrow_{\mathsf{R}} \quad \boxed{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix}}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix} \quad \boxed{\begin{matrix}\mathcal{E}\\ B\ \mathit{true}\end{matrix}}}{A \wedge B\ \mathit{true}}\wedge\mathsf{I}}{B\ \mathit{true}}\wedge\mathsf{E}_2 \quad \longrightarrow_{\mathsf{R}} \quad \boxed{\begin{matrix}\mathcal{E}\\ B\ \mathit{true}\end{matrix}}$$

Local completeness is witnessed by the following expansion rule:

$$\boxed{\begin{matrix}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{matrix}} \quad \longrightarrow_{\mathsf{E}} \quad \dfrac{\dfrac{\boxed{\begin{matrix}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{matrix}}}{A\ \mathit{true}}\wedge\mathsf{E}_1 \quad \dfrac{\boxed{\begin{matrix}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{matrix}}}{B\ \mathit{true}}\wedge\mathsf{E}_2}{A \wedge B\ \mathit{true}}\wedge\mathsf{I}$$

1.2 Disjunction

Local soundness:

$$\dfrac{\dfrac{\boxed{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix}}}{A \vee B\ \mathit{true}}\vee\mathsf{I}_1 \quad \boxed{\begin{matrix}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{matrix}} \quad \begin{matrix}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{matrix}}{C\ \mathit{true}}\vee\mathsf{E}^{u,v} \quad \longrightarrow_{\mathsf{R}} \quad \boxed{\begin{matrix}\dfrac{\boxed{\mathcal{D}}}{A\ \mathit{true}}u\\ \mathcal{E}\\ C\ \mathit{true}\end{matrix}}$$

$$\dfrac{\dfrac{\boxed{\begin{matrix}\mathcal{D}\\ B\ \mathit{true}\end{matrix}}}{A \vee B\ \mathit{true}}\vee\mathsf{I}_2 \quad \begin{matrix}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{matrix} \quad \boxed{\begin{matrix}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{matrix}}}{C\ \mathit{true}}\vee\mathsf{E}^{u,v} \quad \longrightarrow_{\mathsf{R}} \quad \boxed{\begin{matrix}\dfrac{\boxed{\mathcal{D}}}{B\ \mathit{true}}v\\ \mathcal{F}\\ C\ \mathit{true}\end{matrix}}$$

Local completeness:

$$\boxed{\begin{matrix}\mathcal{D}\\ A \vee B\ \mathit{true}\end{matrix}} \longrightarrow_{\mathsf{E}} \dfrac{\boxed{\begin{matrix}\mathcal{D}\\ A \vee B\ \mathit{true}\end{matrix}} \quad \dfrac{\overline{A\ \mathit{true}}^{\,u}}{A \vee B\ \mathit{true}}\vee\mathsf{I}_1 \quad \dfrac{\overline{B\ \mathit{true}}^{\,v}}{A \vee B\ \mathit{true}}\vee\mathsf{I}_2}{A \vee B\ \mathit{true}}\vee\mathsf{E}^{u,v}$$

1.3 Implication

Local soundness:

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^u \qquad \boxed{\begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array}}}{B}\supset\mathsf{E} \;\longrightarrow_{\mathsf{R}}\; \boxed{\begin{array}{c}\dfrac{\boxed{\mathcal{E}}}{A\ \mathit{true}}\,u\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}$$

Local completeness:

$$\boxed{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array}} \;\longrightarrow_{\mathsf{E}}\; \dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array}} \qquad \overline{A\ \mathit{true}}^{\,u}}{B\ \mathit{true}}\supset\mathsf{E}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^u$$

Task 1. Show that $\top$ and $\bot$ are harmonious by showing the local reduction(s) and local expansion(s).

Solution 1: There is no local reduction for $\top$.

Local completeness of $\top$ can be witnessed by

$$\boxed{\begin{array}{c}\mathcal{D}\\ \top\ \mathit{true}\end{array}} \quad\longrightarrow_{\mathsf{E}}\quad \dfrac{}{\top\ \mathit{true}}\top\mathsf{I}$$

There is no local reduction for $\bot$.

Local completeness of $\bot$ can be witnessed by

$$\boxed{\begin{array}{c}\mathcal{D}\\ \bot\ \mathit{true}\end{array}} \longrightarrow_{\mathsf{E}} \dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ \bot\ \mathit{true}\end{array}}}{\bot\ \mathit{true}}\bot\mathsf{E}$$

We say that $\top$ is a degenerative connective of $\wedge$, and that $\bot$ is a degenerative connective of $\vee$. It is actually interesting to observe that local reductions and local expansions demonstrate this as well. If we think $\wedge$ as *binary* conjunction and $\top$ as *nullary* conjunction, we will notice that binary conjunction has exactly *two* elimination rules, and nullary conjunction has exactly *zero* elimination rules. As such, binary conjunction has two local reductions, and nullary conjunction has zero local reduction. In local expansion, binary conjunction makes use of two copy of the given derivation, and nullary conjunction makes use of zero copy.

Similarly, we can view $\vee$ as binary disjunction, and $\bot$ as nullary disjunction. Again, binary disjunction has two introduction rules, and nullary disjunction has zero introduction rule. Binary disjunction has two local reductions, nullary disjunction has zero.

This idea can easily generalize to higher conjunction and disjunction as well. We can define trinary conjunction and disjunction connectives that follow exactly this pattern.

1.4 Experiment: Alternative Implication

What if we replaced the $\supset\mathsf{E}$ rule with the following elimination rule [1]:

$$\dfrac{A \supset B\ \mathit{true} \qquad A\ \mathit{true} \qquad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\supset\mathsf{E}^u$$

Task 2. Can we show local soundness and completeness for this version of the implication connective?

[1] This kind of rule is sometimes called a nuisance rule.

Solution 2:

$$\cfrac{\cfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,v}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^{v} \qquad \begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array} \qquad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\supset\mathsf{E}^{u} \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\mathcal{E}\\ \cfrac{A\ \mathit{true}}{A\ \mathit{true}}\,v\\ \mathcal{D}\\ \cfrac{B\ \mathit{true}}{B\ \mathit{true}}\,u\\ \mathcal{F}\\ C\ \mathit{true}\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array} \longrightarrow_{\mathsf{E}} \quad \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array} \qquad \overline{A\ \mathit{true}}^{\,u} \qquad \overline{B\ \mathit{true}}^{\,v}}{B\ \mathit{true}}\supset\mathsf{E}^{v}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^{u}$$

1.5 Harmonious or Not Harmonious, That's a Question

Task 3. Consider a connective $\ltimes$ defined by the following rules:

$$\cfrac{A\ \mathit{true} \qquad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ B\ \mathit{true}\end{array}}{A \ltimes B\ \mathit{true}}\ltimes I^{u} \qquad\qquad \cfrac{A \ltimes B\ \mathit{true}}{B\ \mathit{true}}\ltimes E$$

1. Is this connective locally sound? If so, provide the local reduction; if not, briefly explain why.
2. Is this connective locally complete? If so, provide the local expansion; if not, briefly explain why.

Solution 3:

1. Yes. This can be demonstrated via a local reduction.

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \qquad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ B\ \mathit{true}\end{array}}{A \ltimes B\ \mathit{true}}\ltimes I^{u}}{B\ \mathit{true}}\ltimes E \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\mathcal{D}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ B\ \mathit{true}\end{array}$$

2. No. In order to give a local expansion, we need to be able to extract enough information from a derivation of $A \ltimes B\ \mathit{true}$ using elimination rules to reconstruct it with introduction rules. The elimination rule $\ltimes E$ only gives us $B\ \mathit{true}$, whereas we also need $A\ \mathit{true}$ to apply the introduction rule $\ltimes I$.

Task 4. Consider a connective $\maltese$, whose introduction rule is defined as follow.

$$\cfrac{A\ \mathit{true} \qquad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots\\ C\ \mathit{true}\end{array}}{\maltese(A, B, C)\ \mathit{true}}\maltese\mathsf{I}^{u}$$

Come up with a set of zero or more elimination rule(s) for this connective that are harmonious with the introduction rule. Show that they are harmonious.

Solution 4: We define the elimination rules as follow[2].

$$\cfrac{\maltese(A, B, C)\ \mathit{true}}{A\ \mathit{true}}\maltese\mathsf{E1}$$

$$\cfrac{\maltese(A, B, C)\ \mathit{true} \qquad B\ \mathit{true}}{C\ \mathit{true}}\maltese\mathsf{E2}$$

[2] The solution might not be unique.

We demonstrate that this connective is harmonious by showing that it satisfies local soundness and local completeness. Local soundness can be demonstrated via local reductions.

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}}{\maltese(A,B,C)\ \mathit{true}}\,\maltese\mathsf{I}^u}{A\ \mathit{true}}\,\maltese\mathsf{E1} \quad \longrightarrow_{\mathsf{R}} \quad \boxed{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \boxed{\begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}}}{\maltese(A,B,C)\ \mathit{true}}\,\maltese\mathsf{I}^u \quad \boxed{\begin{array}{c}\mathcal{F}\\ B\ \mathit{true}\end{array}}}{C\ \mathit{true}}\,\maltese\mathsf{E2} \quad \longrightarrow_{\mathsf{R}} \quad \boxed{\begin{array}{c}\boxed{\mathcal{F}}\\ \overline{B\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}}$$

Local completeness can be demonstrated via a local expansion.

$$\boxed{\begin{array}{c}\mathcal{D}\\ \maltese(A,B,C)\ \mathit{true}\end{array}} \longrightarrow_{\mathsf{E}} \dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ \maltese(A,B,C)\ \mathit{true}\end{array}}}{A\ \mathit{true}}\,\maltese\mathsf{E1} \quad \dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ \maltese(A,B,C)\ \mathit{true}\end{array}} \quad \overline{B\ \mathit{true}}^{\,u}}{C\ \mathit{true}}\,\maltese\mathsf{E2}}{\maltese(A,B,C)\ \mathit{true}}\,\maltese\mathsf{I}^u$$

1.6 You've Got to Know Context

Let's revisit the natural deduction in context notation. Try to prove the following judgement.

$$A \wedge B\ \mathit{true} \vdash A \wedge B\ \mathit{true}$$

One simple derivation we could write is

$$\dfrac{}{A \wedge B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\,\mathsf{Hyp}$$

On the other hand, if you like troubles, you could write the following derivation, which is equally correct:

$$\dfrac{\dfrac{\dfrac{}{A \wedge B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\,\mathsf{Hyp}}{A \wedge B\ \mathit{true} \vdash A\ \mathit{true}}\,\wedge\mathsf{E1} \quad \dfrac{\dfrac{}{A \wedge B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\,\mathsf{Hyp}}{A \wedge B\ \mathit{true} \vdash B\ \mathit{true}}\,\wedge\mathsf{E2}}{A \wedge B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\,\wedge\mathsf{I}$$

What we can observe that, this is exactly corresponding to the local expansion we have written for $\wedge$. In fact, a local expansion, is nothing but a proof of $A\ \mathit{true} \vdash A\ \mathit{true}$, except for that we don't use Hyp rule immediately.

Task 5. Write two derivation for the following judgement, one use Hyp immediately, one not. Observe how they correspond to the local expansions of $\vee$.

$$A \vee B\ \mathit{true} \vdash A \vee B\ \mathit{true}$$

Solution 5:

$$\dfrac{}{A \vee B\ \mathit{true} \vdash A \vee B\ \mathit{true}}\,\mathsf{Hyp}$$

$$\dfrac{\dfrac{}{A \vee B\ \mathit{true} \vdash A \vee B\ \mathit{true}}\,\mathsf{Hyp} \quad \dfrac{\dfrac{}{A \vee B\ \mathit{true}, A\ \mathit{true} \vdash A\ \mathit{true}}\,\mathsf{Hyp}}{A \vee B\ \mathit{true}, A\ \mathit{true} \vdash A \vee B\ \mathit{true}}\,\vee\mathsf{I1} \quad \dfrac{\dfrac{}{A \vee B\ \mathit{true}, B\ \mathit{true} \vdash B\ \mathit{true}}\,\mathsf{Hyp}}{A \vee B\ \mathit{true}, B\ \mathit{true} \vdash A \vee B\ \mathit{true}}\,\vee\mathsf{I2}}{A \vee B\ \mathit{true} \vdash A \vee B\ \mathit{true}}\,\vee\mathsf{E}$$