**Constructive Logic (15-317), Spring 2022**
**Recitation 7: Classical Logic and Theorem Proving (2022-03-02)**
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# 1 Classical Logic

In lecture, we discussed a judgemental formulation of natural deduction for classical logic, founded on the logical judgements $A\ \mathit{true}$ and $A\ \mathit{false}$.

## 1.1 Classical Logic Rules

**Conjunction**

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\ \wedge T \qquad \frac{A \wedge B\ \mathit{false}}{A\ \mathit{false}}\ \wedge F_1 \qquad \frac{A \wedge B\ \mathit{false}}{B\ \mathit{false}}\ \wedge F_2$$

**Disjunction**

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee T_1 \qquad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee T_2 \qquad \frac{A\ \mathit{false} \quad B\ \mathit{false}}{A \vee B\ \mathit{true}}\ \vee F$$

**Implication**

$$\frac{\begin{array}{c}[A\ \mathit{true}]_u \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ \supset I^u \qquad \frac{A\ \mathit{true} \quad B\ \mathit{false}}{A \supset B\ \mathit{false}}\ \supset F$$

**Truth and Falsehood**

$$\frac{}{\top\ \mathit{true}}\ \top T \qquad \frac{}{\bot\ \mathit{false}}\ \bot F$$

**Negation**

$$\frac{A\ \mathit{false}}{\neg A\ \mathit{true}}\ \neg T \qquad \frac{A\ \mathit{true}}{\neg A\ \mathit{false}}\ \neg F$$

**Contradiction**

$$\frac{A\ \mathit{true} \quad A\ \mathit{false}}{\#}\ \# \qquad \frac{\begin{array}{c}[A\ \mathit{false}]_u \\ \vdots \\ \#\end{array}}{A\ \mathit{true}}\ T\#^u \qquad \frac{\begin{array}{c}[A\ \mathit{true}]_u \\ \vdots \\ \#\end{array}}{A\ \mathit{false}}\ F\#^u$$

## 1.2 Example Proofs

Provide derivations of the following judgements in Classical Logic.

**Task 1.** $(\neg B \supset \neg A) \supset (A \supset B)\ \mathit{true}$

**Task 2.** $(\neg A \vee A) \supset (\neg\neg A \supset A)\ \mathit{true}$

## 2 All the sequent calculi

We have seen in lecture four different sequent calculi, each improving on the previous for automatic (and, let's be honest, manual) proof search.

### 2.1 Sequent calculus

First there was sequent calculus, which can be obtained quite straightforwardly from the natural deduction calculus with verification judgments.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C} \supset L$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_2$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \Longrightarrow P} \, init \qquad \frac{}{\Gamma \Longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \Longrightarrow C} \bot L$$

### 2.2 Restricted sequent calculus

We quickly realize that the sequent calculus above can't be good for proof search, as it keeps a copy of every formula potentially wasting memory and increasing the search space. So we notice we can restrict it and, in the end, the only formula we actually need to keep copies of are implications on the left.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P} \, init \qquad \frac{}{\Gamma \longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C} \bot L$$

### 2.3 Contraction-free sequent calculus (a.k.a. G4ip)

Still we have the problem of needing to keep implications on the left around. By analyzing what might happen on the left side of an implication more carefully, we can come up with a calculus where this implicit contraction of implications no longer occurs. This is perfect for proof search and it gives directly a decision procedure for propositional intuitionistic logic (which is good anyway, since this is indeed a decidable fragment).

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \qquad \frac{P \in \Gamma \quad \Gamma, B \longrightarrow C}{\Gamma, P \supset B \longrightarrow C} P{\supset}L \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, \top \supset B \longrightarrow C} \top{\supset}L$$

$$\frac{\Gamma, D \supset E \supset B \longrightarrow C}{\Gamma, D \wedge E \supset B \longrightarrow C} \wedge{\supset}L \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \bot \supset B \longrightarrow C} \bot{\supset}L \qquad \frac{\Gamma, D \supset B, E \supset B \longrightarrow C}{\Gamma, D \vee E \supset B \longrightarrow C} \vee{\supset}L \qquad \frac{\Gamma, D, E \supset B \longrightarrow E \quad \Gamma, B \longrightarrow C}{\Gamma, (D \supset E) \supset B \longrightarrow C} \supset{\supset}L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P} \text{init} \qquad \frac{}{\Gamma \longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C} \bot L$$

$$\frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C} \top L$$

### 2.4 Exercises

In the lecture notes it is indicated that cut is admissible for the restricted calculus[1]. The proof is analogous to the one you have already seen, but since less formulas are kept around, some cases become simpler.

**Task 3.** Prove that if $\Gamma \longrightarrow A \supset B$ and $\Gamma, A \supset B \longrightarrow C$ then $\Gamma \longrightarrow C$ in the restricted sequent calculus (consider only the case where the cut formula is principal).

**Task 4.** Prove the following sequent in G4ip:

$$\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R$$

[1] Actually, cut is admissible for all the calculi listed here.