Constructive Logic (15-317), Spring 2022
Recitation 5: Sequent Calculus (2022-02-15)
Clogic Staff

1 The Rules

Recall that left rules correspond to "upside down elimination rules" and that right rules correspond to introduction rules.

$$\frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_2 \qquad \frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ \vee L \qquad \frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2$$

No $\top L$. $\qquad \dfrac{}{\Gamma \Longrightarrow \top}\ \top R \qquad \dfrac{}{\Gamma, \bot \Longrightarrow C}\ \bot L \qquad$ No $\bot R$.

$$\frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ \supset L \qquad \frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ \supset R$$

$$\frac{}{\Gamma, A \Longrightarrow A}\ \mathsf{id}$$

2 Non-Provable Propositions

Sequent calculus represents a different presentation style for our proof calculi, without changing our set of provable propositions. To illustrate this, consider the following non-provable propositions, shown in sequent calculus. Note that the reverse implications of these statements are provable.

$(\neg A \supset \neg B) \supset (B \supset A)$:

$$\dfrac{\dfrac{\dfrac{X}{(\neg A \supset \neg B), B, A \Longrightarrow \bot}}{(\neg A \supset \neg B), B \Longrightarrow \neg A} \quad \dfrac{\dfrac{}{(\neg A \supset \neg B), B, \neg B \Longrightarrow B}\ \mathsf{id} \quad \dfrac{}{(\neg A \supset \neg B), B, \neg B, \bot \Longrightarrow A}\ \bot L}{(\neg A \supset \neg B), B, \neg B \Longrightarrow A}\ \supset L}{(\neg A \supset \neg B), B \Longrightarrow A}\ \supset L$$
$$\dfrac{(\neg A \supset \neg B), B \Longrightarrow A}{\cdot \Longrightarrow (\neg A \supset \neg B) \supset (B \supset A)}\ \supset R$$

$(A \supset B) \supset (\neg A \vee B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset B), A \Longrightarrow A}\ \mathsf{id} \quad \dfrac{X}{(A \supset B), B \Longrightarrow \bot}}{(A \supset B), A \Longrightarrow \bot}\ \supset L}{(A \supset B) \Longrightarrow \neg A}\ \supset R}{(A \supset B) \Longrightarrow (\neg A \vee B)}\ \vee R$$
$$\dfrac{(A \supset B) \Longrightarrow (\neg A \vee B)}{\cdot \Longrightarrow (A \supset B) \supset (\neg A \vee B)}\ \supset R$$

3 Some Example Proofs

Task 1. $\cdot \Longrightarrow A \supset A$

Task 2. $\cdot \Longrightarrow (A \wedge B) \supset (B \wedge A)$

Task 3. $\cdot \Longrightarrow (A \supset (B \wedge C)) \supset (A \supset B)$

Task 4. $\cdot \Longrightarrow (A \supset B \supset C) \supset (B \supset (A \supset C))$

Task 5. $\cdot \Longrightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))$