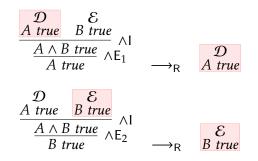
# 1 Harmony

Proof-theoretic harmony is a necessary, but not sufficient, condition for the well-behavedness of a logic; harmony ensures that the connectives are *locally* well-behaved, and is closely related to the critical cases of cut and identity elimination which we may discuss later on. Therefore, when designing or extending a logic, checking harmony is a first step.

From the verificationist standpoint, a connective is *harmonious* if its elimination rules are neither too strong nor too weak in relation to its introduction rules. The first condition is called *local soundness* and the second condition is called *local completeness*. The content of the soundness condition is a method to reduce or simplify proofs (by demonstrating there is a more direct proof to the detour), which is shown via *local reduction(s)*. The content of completeness is a method to expand any arbitrary proof into a canonical proof (i.e. one that ends in an introduction rule), which is shown via *local expansion(s)*. Later on in this course, we shall justify local reductions and local expansions in terms of reduction and expansion in programming.

# 1.1 Conjunction

Local soundness for conjunction is witnessed by the following two reduction rules:

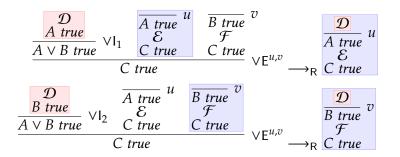


Local completeness is witnessed by the following expansion rule:

$$\begin{array}{cccc}
 & \mathcal{D} \\
 & \underline{A \land B \ true} \\
 & \underline{A \land$$

## 1.2 Disjunction

Local soundness:



Local completeness:

$$\underbrace{\begin{array}{c} \mathcal{D} \\ A \lor B \ true} \\ A \lor B \ true \end{array}}_{\mathsf{E}} \xrightarrow{\mathsf{D}}_{\mathsf{E}} \underbrace{\begin{array}{c} \overline{A \ vB \ true} \\ A \lor B \ true \end{array}}_{\mathsf{E}} \underbrace{\begin{array}{c} \overline{A \ vB \ true} \\ A \lor B \ true \end{array}}_{\mathsf{E}} \underbrace{\begin{array}{c} \overline{A \ vB \ true} \\ \nabla \mathsf{I}_1 \\ A \lor B \ true \end{array}}_{\mathsf{E}} \underbrace{\begin{array}{c} \overline{B \ true} \\ \overline{A \lor B \ true} \\ \nabla \mathsf{E}^{u,v} \end{array}}_{\mathsf{E}^{u,v}} \xrightarrow{\mathsf{VI}_2}_{\mathsf{E}^{u,v}}$$

#### 1.3 Implication

Local soundness:

Local completeness:

$$\begin{array}{cccc}
\mathcal{D} & & \overline{A \supset B \ true} & \overline{A \ true} & u \\
\xrightarrow{\mathcal{D}} & & \overline{B \ true} & \overline{A \ true} & \neg E \\
\xrightarrow{\mathcal{D}} & & \overline{A \supset B \ true} & \neg I^{u}
\end{array}$$

**Task 1.** Show that  $\top$  and  $\perp$  are harmonious by showing the local reduction(s) and local expansion(s).

#### **1.4 Experiment: Alternative Implication**

What if we replaced the  $\supset$ E rule with the following elimination rule <sup>1</sup>:

$$\frac{A \supset B \text{ true } A \text{ true } C \text{ true }}{C \text{ true }} U$$

Task 2. Can we show local soundness and completeness for this version of the implication connective?

### 1.5 Harmonious or Not Harmonious, That's a Question

**Task 3.** Consider a connective  $\ltimes$  defined by the following rules:

$$\frac{A \text{ true }}{A \text{ true }} u$$

$$\frac{A \text{ true } B \text{ true }}{A \ltimes B \text{ true }} \ltimes I^{u} \qquad \frac{A \ltimes B \text{ true }}{B \text{ true }} \ltimes E$$

- 1. Is this connective locally sound? If so, provide the local reduction; if not, briefly explain why.
- 2. Is this connective locally complete? If so, provide the local expansion; if not, briefly explain why.

**Task 4.** Consider a connective ₩, whose introduction rule is defined as follow.

$$\frac{\overline{B \ true}}{\vdots}^{u}$$

$$\frac{A \ true \ C \ true}{\bigstar(A, B, C) \ true} \checkmark^{u}$$

Come up with a set of zero or more elimination rule(s) for this connective that are harmonious with the introduction rule. Show that they are harmonious.

<sup>&</sup>lt;sup>1</sup>This kind of rule is sometimes called a nuisance rule.

#### 1.6 You've Got to Know Context

Let's revisit the natural deduction in context notation. Try to prove the following judgement.

$$A \wedge B$$
 true  $\vdash A \wedge B$  true

One simple derivation we could write is

$$\overline{A \land B \ true \vdash A \land B \ true} \ \mathsf{Hyp}$$

On the other hand, if you like troubles, you could write the following derivation, which is equally correct:

$$\frac{\overline{A \land B \ true \vdash A \land B \ true}}{\underline{A \land B \ true \vdash A \ true}} \overset{\text{Hyp}}{\land \text{E1}} \xrightarrow{\overline{A \land B \ true \vdash A \land B \ true}} \overset{\text{Hyp}}{A \land B \ true \vdash B \ true} \overset{\text{Hyp}}{\land \text{E2}} \overset{\text{Hyp}}{\land \text{E3}}$$

What we can observe that, this is exactly corresponding to the local expansion we have written for  $\wedge$ . In fact, a local expansion, is nothing but a proof of *A* true  $\vdash$  *A* true, except for that we don't use Hyp rule immediately.

**Task 5.** Write two derivation for the following judgement, one use Hyp immediately, one not. Observe how they correspond to the local expansions of  $\lor$ .

$$A \lor B$$
 true  $\vdash A \lor B$  true