Constructive Logic (15-317), Spring 2022
Recitation 1: Dcheck and Natural Deduction (2022-01-19)
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1 Propositions and Judgements

When working in logic, we make a distinction between the *object language* and the *metalanguage*. The object language consists of the formal objects that we are studying (in this case, propositions are in the object language), while the metalanguage is used to make statements about things in the object language (like how judgements can describe properties of propositions).

As an example, consider a judgement A prop, with the intuitive meaning "A is a proposition". Note that this judgment is entirely separate from A true — a false proposition is still a proposition, after all.

How should this judgement be defined? The case for $A \wedge B$ is fairly simple:

$$\frac{A \text{ prop} \quad B \text{ prop}}{A \wedge B \text{ prop}} \wedge F$$

$A \wedge B$ is a proposition when both A and B are propositions. That is, $\wedge$ allows us to build two propositions, A and B, into a larger proposition $A \wedge B$.

Exercise 1. Give a rule that lets you prove the conclusion $A \supset B$ prop. What about T prop?

Exercise 2. Prove that $(A \supset B) \supset A$ prop, assuming that A prop and B prop. You can think of this as giving a proof for the hypothetical judgement $\begin{array}{c} A \text{ prop} \quad B \text{ prop} \\ \vdots \\ (A \supset B) \supset A \text{ prop} \end{array}$. Can you prove that $(A \supset B) \supset A$ true? What about $\begin{array}{c} A \text{ prop} \quad B \text{ prop} \\ \vdots \\ (A \supset B) \supset A \text{ true} \end{array}$?

2 Dcheck

Dcheck is a tool for parsing and checking derivations, and will be used to autograde your proof-writing assignments. Documentation and examples for Dcheck can be found on the course website. Below, we will look at a few examples of Dcheck derivations.

Example 1. We will first look at a very simple example:

$$\dfrac{\dfrac{}{A\ \mathsf{true}}\,u}{A \supset A\ \mathsf{true}}\supset I^u$$

This proof can be represented in Dcheck as follows:

```
system ND
deriv a_first_derivation =
    A => A true
    by ImpI(u)
    >>
    A true
    by u
```

The first line, `system ND`, tells Dcheck that we are using the Natural Deduction proof system.

The second line starts a derivation or proof, using the `deriv` keyword. We name this derivation `a_first_derivation`, and it begins after the equals sign.

Dcheck proofs go in the opposite direction to the proofs we write on the board or on paper. As such, the first line of the derivation proper, `A => A true`, is just the goal $A \supset A$ true that we are trying to prove, using => for implication. Each subgoal in a derivation is followed by the `by` keyword and the name of the rule we want to apply. In this case, we want to use the $\supset I^u$, or implication introduction rule, shortened as `ImpI(u)`, with u indicating the name we give to the hypothesis introduced by the rule.

We separate each rule from its premises with the token >>. Here, the $\supset I^u$ rule only has a single premise, whose goal in this case is A true. We can justify this with the hypothesis u, finishing the proof.

Example 2. Now, we'll look at a slightly more involved example:

$$\dfrac{\dfrac{\dfrac{}{A\ \mathsf{true}}\,u \quad \dfrac{}{A\ \mathsf{true}}\,u}{A \wedge A\ \mathsf{true}}\wedge I}{A \supset (A \wedge A)\ \mathsf{true}}\supset I^u$$

In Dcheck:

```
system ND
deriv a_second_derivation =
    A => (A /\ A) true
    by ImpI(u)
    >>
    A /\ A true
    by AndI
    >>
        {
        A true
        by u
        }

        {
        A true
        by u
        }
```

The main new thing to observe is that when a rule has more than one premise, like $\wedge I$ (`AndI`), each premise is enclosed in curly braces.

3 Hypothetical Judgements

We have seen two ways to work with hypothetical judgements, where we want to prove a judgement under some conditions. The first approach uses *floating hypotheses*, as in the following $\supset I^u$ rule:

$$\dfrac{\begin{array}{c}\overline{A \text{ true}}\ u \\ \vdots \\ B \text{ true}\end{array}}{A \supset B \text{ true}} \supset I^u$$

The section $\begin{array}{c}\overline{A \text{ true}}\ u \\ \vdots \\ B \text{ true}\end{array}$ should be filled in with a proof of B true that is allowed to use A true as an assumption, or hypothesis.

We can also instead make hypotheses explicit in the judgements that we are proving, using the judgement $A_1 \text{ true}, A_2 \text{ true}, \ldots, A_n \text{ true} \vdash A \text{ true}$, in which we will often abbreviate the collection $A_1 \text{ true}, \ldots, A_n \text{ true}$ as Γ. We call Γ a *context*, and might read $\Gamma \vdash A$ true as "A is true in (or with) context Γ". For this to make sense, we need a way to use hypotheses in Γ, just like we use the hypothesis u above. To do so, we use the *hyp* rule, shown below in a few different forms:

$$\dfrac{A \text{ true} \in \Gamma}{\Gamma \vdash A \text{ true}}\ hyp_1 \qquad\qquad \dfrac{}{\Gamma, A \text{ true} \vdash A \text{ true}}\ hyp_2$$

In the first rule, we write, somewhat informally (since we haven't defined $\in$) "A true $\in \Gamma$". This can be read as saying that if one of the hypotheses in Γ is that "A true", we are entitled to conclude that $\Gamma \vdash A$ true. The second rule is similar in concept, but assumes that A true is at the end of Γ. We generally assume that Γ is a set, so it can freely be reordered, and in this case, the two rules are equivalent.[1]

Making use of contexts, we might imagine a rule something like this:

$$\dfrac{A \text{ true} \vdash B \text{ true}}{\vdash A \supset B \text{ true}} \supset I^*$$

This is a perfectly good rule, and is a counterpart to the $\supset I^u$ rule we examined above. However, we usually want to be able to apply rules midway through a proof, where we may already have some hypotheses. This version of the rule can only be used to prove judgements of the form $\vdash A \supset B$ true, saying that $A \supset B$ is true with no hypotheses. By allowing extra hypotheses in this rule, we get a more standard version: [2]

$$\dfrac{\Gamma, A \text{ true} \vdash B \text{ true}}{\Gamma \vdash A \supset B \text{ true}} \supset I$$

Exercise 3. Give a version of the $\wedge I$ rule using contexts. What about the $\wedge E1$ and $\wedge E2$ rules?

[1] We will see systems where this is not the case near the end of the course, and will have to work more precisely with our rules then.

[2] You may notice that $\vdash$ seems to be serving the same purpose as $\supset$ — they are in fact related in much the same way that the proposition "A" is related to the judgement "A true". While $\supset$ expresses implication in the object language, being a formal symbol that we are reasoning about, $\vdash$ expresses a similar concept in the metalanguage. We might say that $\supset$ is a form of $\vdash$ *internal to* the logic, or that $\supset$ *internalizes* $\vdash$ into the logic. An important distinction, however, is that $\supset$ takes a proposition on each side to form a new proposition, while $\vdash$ (at least for now) only makes sense when we give it a list of judgements A_i true on the left and a judgement A true on the right, producing a full judgement. Exercise: Do you see anything in the object language that the metalanguage's "," between A_i true judgements in Γ corresponds to?