15-317 Lecture 23: (Ordered) Proofs as (Concurrent) Programs

- Ordered Metatheory (briefly)
- Subsingleton logic
- Concurrent Programs
- Extending back to ordered logic

Metatheory

- Identity $\dfrac{}{A \Vdash A}\ id_A$ is admissible

- Cut $\dfrac{\Omega \Vdash A \qquad \Omega_L\ A\ \Omega_R \Vdash C}{\Omega_L\ \Omega\ \Omega_R \Vdash C}\ cut_A$ is admissible

Subsingleton Logic

$$\frac{A\ \Omega \Vdash B}{\Omega \Vdash A \setminus B}\ \setminus R \longrightarrow \frac{A \Vdash B}{\cdot \Vdash A \setminus B}\ \setminus R^*$$

- Context contains at most one prop.

$$\frac{\omega \Vdash A}{\omega \Vdash A \oplus B}\ \oplus R_1 \qquad \frac{A \Vdash C \qquad B \Vdash C}{A \oplus B \Vdash C}\ \oplus L$$

$$\frac{\omega \Vvdash A \qquad \omega \Vvdash B}{\omega \Vvdash A \,\&\, B}\ \&R \qquad\qquad \frac{A \Vvdash C}{A \,\&\, B \Vvdash C}\ \&L_1$$

$$\frac{}{\cdot \Vvdash 1}\ 1R \qquad\qquad \frac{\cdot \Vvdash C}{1 \Vvdash C}\ 1L$$

$$\frac{}{\omega \Vvdash \top}\ \top R \qquad\qquad \frac{}{0 \Vvdash C}\ 0L$$

$$\frac{}{A \Vvdash A}\ id \qquad\qquad \frac{\omega \Vvdash A \qquad A \Vvdash C}{\omega \Vvdash C}\ cut_A$$

Programs as Proofs

choice of logic

Substructural prop. $\longleftrightarrow$ session types

proof $\longleftrightarrow$ concurrent programs

cut reduction $\longleftrightarrow$ computation

choice of <u>presentation</u> of the logic

$$\omega \Vdash A \quad \longleftrightarrow \quad \underbrace{\omega \Vdash P : A}_{}$$

P is a process using ω and providing A

$$\frac{}{A \Vvdash\ \leftrightarrow\ : A}\ \mathrm{id}_A \qquad \frac{\omega \Vvdash P : A \qquad A \Vvdash Q : C}{\omega \Vvdash (P \mid Q) : C}\ \mathrm{cut}$$

$$\omega \Vvdash P : A \oplus B \qquad\qquad A \oplus B \Vvdash Q : C$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\omega \Vvdash P' : A}}{\omega \Vvdash (R.\mathrm{inl} ; P') : A \oplus B}\ \oplus R_1 \qquad \dfrac{\dfrac{\mathcal{E}_1}{A \Vvdash Q_1 : C} \qquad \dfrac{\mathcal{E}_2}{B \Vvdash Q_2 : C}}{A \oplus B \Vvdash \mathrm{caseL}(\mathrm{inl} \Rightarrow Q_1 \mid \mathrm{inr} \Rightarrow Q_2) : C}\ \oplus L}{\omega \Vvdash ((R.\mathrm{inl} ; P') \mid (\mathrm{caseL}(\ldots))) : C}\ \mathrm{cut}_{A \oplus B}$$

$$\omega \Vvdash ((R.\mathrm{inl} ; P') \mid (\mathrm{caseL}(\ldots))) : C \longrightarrow_R P' \mid Q_1$$

$$\frac{\omega \Vvdash P_1 : A \qquad \omega \Vvdash P_2 : B}{\omega \Vvdash \mathrm{caseR}\,(\mathrm{inl} \Rightarrow P_1 \mid \mathrm{inr} \Rightarrow P_2) : A \,\&\, B}\ \&R$$

$$\frac{A \Vvdash Q' : C}{A \,\&\, B \Vvdash (\mathrm{L.inl} \,;\, Q') : C}\ \&L_1$$

$$\frac{}{\cdot \Vvdash \mathrm{closeR} : \mathbf{1}}\ \mathbf{1}R$$

$$\frac{\cdot \Vvdash Q' : C}{\mathbf{1} \Vvdash (\mathrm{waitL} \,;\, Q') : C}\ \mathbf{1}L$$

$$\frac{}{\omega \Vvdash \mathrm{caseR}() : \top}\ \top R$$

$$\frac{}{\mathbf{0} \Vvdash \mathrm{caseL}() : C}\ \mathbf{0}L$$

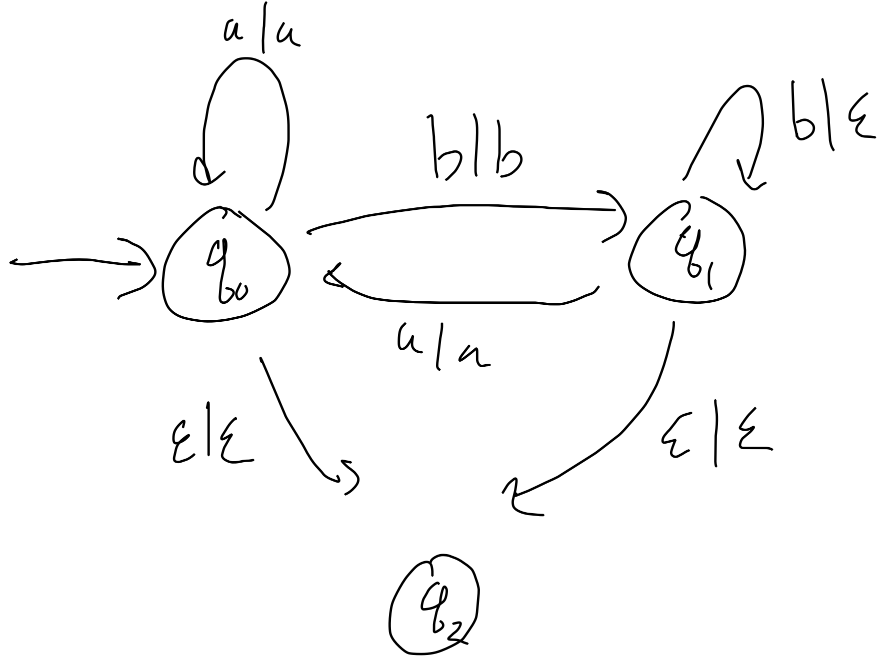
Running process configuration $P_1 \mid P_2 \mid P_3 \mid \dots \mid P_n$

$$\frac{(P \mid Q)}{P \mid Q}\,\text{cut} \qquad \frac{\longleftrightarrow}{\cdot}\,\text{id}$$

$$\frac{(R.l_k \,;\, P) \mid (\text{case}L\,(l_j \Rightarrow Q_j)_{j \in J})}{P \mid Q_k}\,\oplus$$

$$\frac{(\text{case}R(l_j \Rightarrow P_j)) \mid (L.l_k \,;\, Q)}{P_k \mid Q}\,\&$$

$$\frac{\text{close}R \mid (\text{wait}L \,;\, Q)}{Q}\,\mathbf{1}$$



$$\textit{string} \Vdash T : \textit{string}$$

$$\textit{string} = \oplus\{a : \textit{string},\ b : \textit{string},\ \$: 1\}$$

$$1 \Vdash \ulcorner babb \urcorner : \textit{string}$$

$$\ulcorner babb \urcorner = R.b\ ;\ R.a\ ;\ R.b\ ;\ R.b\ ;\ R.\$\ ;\ \leftrightarrow$$

$$\begin{aligned} Q_0 = \mathrm{case}\, L\ (&a \Rightarrow R.a\ ;\ Q_0 \\ \mid\ &b \Rightarrow R.b\ ;\ Q_1 \\ \mid\ &\$ \Rightarrow R.\$\ ;\ Q_2\ ) \end{aligned}$$

$$\begin{aligned} Q_1 = \mathrm{case}\, L\ (&a \Rightarrow R.a\ ;\ Q_0 \\ &b \Rightarrow Q_1 \\ &\$ \Rightarrow R.\$\ ;\ Q_2\ ) \end{aligned}$$

$$Q_2 = \leftrightarrow$$

$$T = Q_0$$

composing transducers

$$T_1 \parallel T_2$$

Extending to ordered Logic

$$\frac{\Omega_L\ A\ \Omega_R \Vdash C \qquad \Omega_L\ B\ \Omega_R \Vdash C}{\Omega_L\ A \oplus B\ \Omega_R \Vdash C}$$

$$\frac{\Omega \Vdash P :: (y : A_k)}{\Omega \Vdash (x.l_k(y) ; P) :: (x : \oplus_{j \in J} A_j)} \oplus R_k$$

$$\frac{\Omega_L (y_j : A_j)\ \Omega_R \Vdash Q_j :: (z : C) \quad (\text{for } j \in J)}{\Omega_L (x : \oplus_{j \in J} A_j) \Omega_R \Vdash (\text{case } x\ (l_j(y_j) \Rightarrow Q_j)) :: (z : C)} \oplus L$$

$$\frac{(x.l_k(y) ; P) \qquad (\text{case } x\ (l_j(y_j) \Rightarrow Q_j))}{P \qquad Q_k[y/y_k]}$$

$$\frac{\Omega \Vdash P :: (x : A) \qquad \Omega_L\, (\cancel{x} : A)\, \Omega_R \Vdash Q :: (z : C)}{\Omega_L\, \Omega\, \Omega_R \Vdash (x \leftarrow P \,;\, Q) :: (z : C)}\ \text{cut}$$

$$\frac{x \leftarrow P \,;\, Q}{P[a/x] \qquad Q[a/x]}\ (a \text{ fresh})$$

$$\frac{}{(x : A) \Vdash x \leftrightarrow y :: (y : A)}\ \text{id}$$

$$\left.\frac{x \leftrightarrow y}{x = y}\right\}\ \text{messy}$$

globally identify x, y