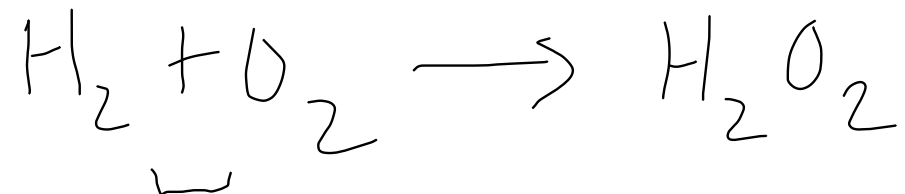
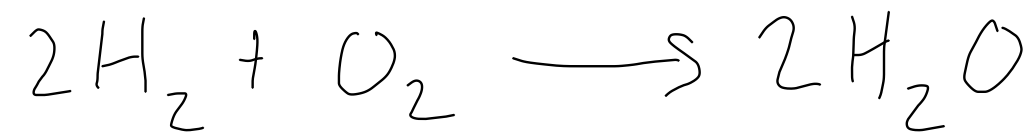


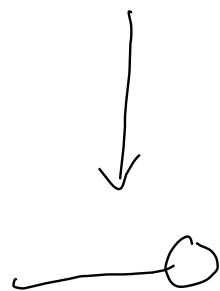
# 15-317 Lecture 21 : Linear Logic

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- State change / facts as resources
- Rules of Linear Logic
- Metatheory
  - No weakening or contraction (in general)
  - Identity and cut still hold (in modified form)
- Other substructural logic?



Tensor



Null

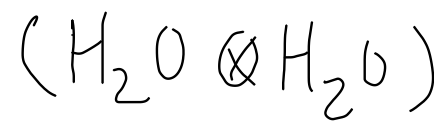
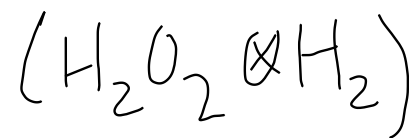
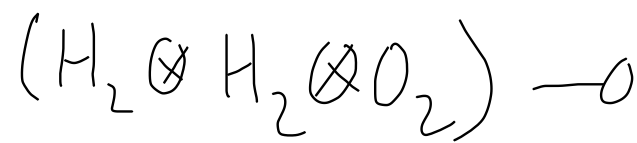
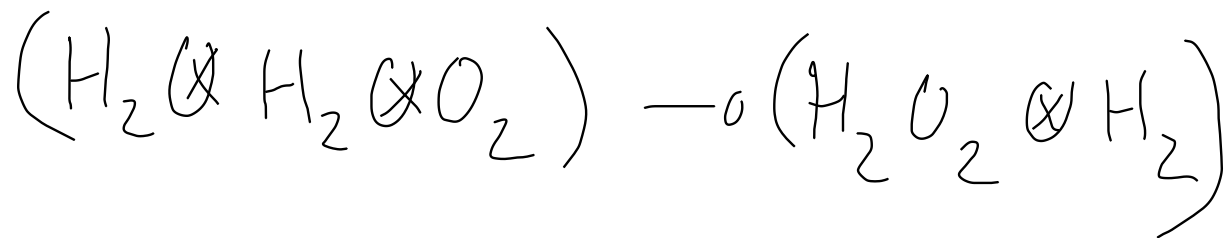
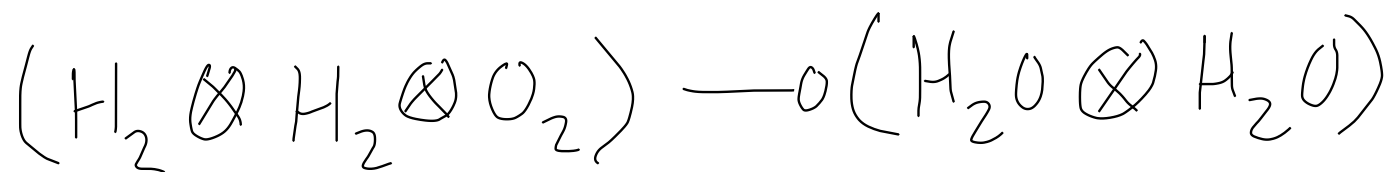
$$\frac{\Delta_A \Vdash A \quad \Delta_B \Vdash B}{\Delta \Vdash A \otimes B} \otimes R$$

$$\Delta_A, \Delta_B \Vdash A \otimes B$$

$$\frac{\Delta, A, B \Vdash C}{\Delta, A \otimes B \Vdash C} \otimes L$$

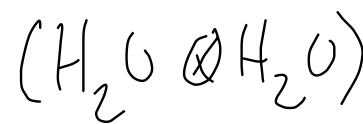
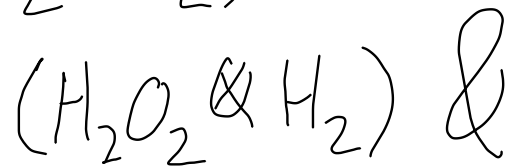
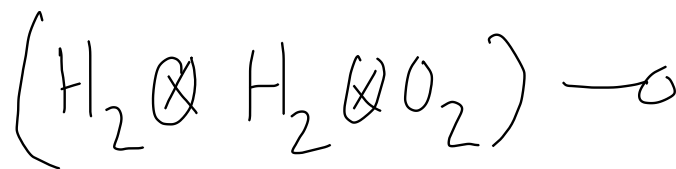
$$\frac{\Delta, A \Vdash B}{\Delta \Vdash A \multimap B} \multimap R$$

$$\frac{\Delta_A \Vdash A \quad \Delta_B \Vdash C}{\Delta_A, \Delta_B, A \multimap B \Vdash C} \multimap L$$



plus

"internal choice"



"with"

"external choice"

$$\frac{\Delta \Vdash A}{\Delta \Vdash A \oplus B} \oplus R_1$$

$$\frac{\Delta \Vdash B}{\Delta \Vdash A \oplus B} \oplus R_2$$

$$\frac{\Delta, A \Vdash C \quad \Delta, B \Vdash C}{\Delta, A \oplus B \Vdash C} \oplus L$$

$$\frac{\Delta \Vdash A \quad \Delta \Vdash B}{\Delta \Vdash A \& B} \& R$$

$$\frac{\Delta, A \Vdash C}{\Delta, A \& B \Vdash C} \& L_1$$

$$\frac{\Delta, B \Vdash C}{\Delta, A \& B \Vdash C} \& L_2$$

$$\frac{}{\Gamma, A \rightarrow A} \text{id (structural)}$$

$$\frac{}{A \Vdash A} \text{id (linear)}$$

$$\frac{A, B \Vdash^? A}{A \otimes B \Vdash^? A} \otimes L$$

$$\frac{A \otimes B \Vdash^? B}{A \otimes B \Vdash^? A \& B} \& R$$

$$A \otimes B \Vdash^? A \& B$$

$$\begin{array}{l}
 \frac{A \Vdash A \quad \text{id}}{\quad} \quad \frac{B \Vdash B \quad \text{id}}{\quad} \oplus R_2 \\
 \hline
 A, B \Vdash A \otimes (A \oplus B) \quad \otimes R \\
 \hline
 A, B \Vdash A \otimes (A \oplus B) \quad \otimes R
 \end{array}$$

$$\begin{array}{l}
 A, B \Vdash \dots \\
 \hline
 (A \otimes B) \Vdash (A \otimes (A \oplus B)) \& (B \otimes (A \oplus B)) \quad \otimes L
 \end{array}$$

---

$$\circ \mathbb{H} \perp \perp \quad \perp \mathbb{R}$$

---

$$\frac{\Delta \mathbb{H} \subset}{\Delta, \perp \mathbb{H} \subset} \perp \mathbb{L}$$

$$\rightarrow A \otimes \perp \equiv \perp \otimes A \equiv A$$

---

---

$$\Delta \mathbb{H} \top \quad \top \mathbb{R}$$

No  $\top \mathbb{L}$

$$\rightarrow \top \otimes A \equiv A \otimes \top \equiv A$$

~~\_\_\_\_\_~~ OR

\_\_\_\_\_  $\Delta, \emptyset \vdash C$  OL

$$\emptyset \oplus A \equiv A \oplus \emptyset \equiv A$$

---

Can give  $\perp$   $\delta$  in classical Lt.  
                  ↑  
                  "par"



$\Delta \Vdash A \longrightarrow$

$\Gamma; \Delta \Vdash A$   
resources  
unrestricted resources  
replicable resources

$\Gamma, A; \Delta, A \Vdash C$

copy

$\Gamma, A; \Delta \Vdash C$

! A : of course A  
replicable A

$\frac{\Gamma; \bullet \Vdash A}{\Gamma; \bullet \Vdash !A} !R$

$\frac{\Gamma, A; \Delta \Vdash C}{\Gamma; \Delta, !A \Vdash C} !L$

$$A \supset B \quad ((\neg A) \rightarrow B)$$

$$\frac{\frac{}{A, B'; A \Vdash A} \text{copy}}{A, B'; \Vdash A} \text{id}}$$

$$\frac{\frac{}{A, B'; B \Vdash B} \text{copy}}{A, B'; \Vdash B} \text{id}}{\quad} \& R$$

$$A, B'; \Vdash A \& B$$

$$\frac{}{A, B'; \Vdash \neg(A \& B)} \neg I$$

$$\frac{}{\neg A, \neg B \Vdash \neg(A \& B)} \neg L^2$$

$$\frac{}{\neg A \otimes \neg B \Vdash \neg(A \& B)} \otimes L$$

# Metatheory

- Weakening

$$\frac{\Gamma ; \Delta \Vdash C}{\Gamma, A ; \Delta \Vdash C} w$$

- contraction

$$\frac{\Gamma, A, A ; \Delta \Vdash C}{\Gamma, A ; \Delta \Vdash C} c$$

---

- identity

Any proof of  $A \Vdash A$  need only use id at atomic propositions

$\frac{A \vdash A \quad \&L_1}{A \& B \vdash A}$	$\frac{B \vdash B \quad \&L_2}{A \& B \vdash B}$
$\frac{A \& B \vdash A \quad A \& B \vdash B \quad \&R}{A \& B \vdash A \& B}$	

———— i.h. (A)

———— i.h. (B)

&R

cut  $\left. \begin{array}{l} \frac{\Gamma; \overset{\mathcal{D}}{\Delta_1} \Vdash A}{\Gamma; \Delta_1, \Delta_2 \Vdash C} \quad \frac{\Gamma; \overset{\mathcal{E}}{\Delta_2} \Vdash A \Vdash C}{\Gamma; \Delta_1, \Delta_2 \Vdash C} \text{ - cut} \end{array} \right\} \text{ is admissible}$

Induction on  $(A, \mathcal{D}, \mathcal{E})$ .  $\mathcal{E} = \frac{\Gamma; \overset{\mathcal{E}_1}{\Delta_2} \Vdash A \Vdash C \quad \Gamma; \overset{\mathcal{E}_2}{\Delta_2} \Vdash B \Vdash C}{\Gamma; \Delta_2, A \oplus B \Vdash C} \oplus L$

Case  $\oplus R, \oplus L$

$\mathcal{D} \frac{\Gamma; \overset{\mathcal{D}}{\Delta_1} \Vdash A}{\Gamma; \Delta_1 \Vdash A \oplus B} \oplus R,$

$\left( \frac{\Gamma; \Delta_1 \Vdash A}{\Gamma; \Delta_1, \Delta_2 \Vdash C} \quad \frac{\Gamma; \Delta_2 \Vdash A \Vdash C}{\Gamma; \Delta_1, \Delta_2 \Vdash C} \text{ - i.h. } (A, \mathcal{D}, \mathcal{E}_1) \right)$

Case  $\mathbb{R}/\mathbb{L}$

$$\mathcal{D} = \frac{\Gamma; \cdot \Vdash A}{\Gamma; \cdot \Vdash !A} \mathbb{R}$$

$$\mathcal{E} = \frac{\Gamma, A; \Delta_2 \Vdash C}{\Gamma; \Delta_2, !A \Vdash C} \mathbb{L}$$

$$\frac{\Gamma; \cdot \Vdash A \quad \Gamma, A; \Delta_2 \Vdash C}{\Gamma; \Delta_2 \Vdash C} \quad - \quad i.h.(A, \mathcal{D}, \mathcal{E})$$

$$\frac{\Gamma' \circ \text{H}A}{\Gamma; \Delta \text{H}L} \quad \frac{\Gamma, A; \Delta \text{H}L}{\Gamma; \Delta \text{H}L} \text{ cut}^u$$

- weakening only : Affine logic
- contraction only : Strict/Relevant Logic
- Removing exchange : Ordered logic

A lin

A str

$\Gamma; \cdot \Vdash A \text{ str}$

$\Gamma; \Delta \Vdash A \text{ str}$  X