15-317 Lecture 21: Linear Logic

- State change / Facts as resources
- Rules of Linear Logic
- Metatheory
  - No weakening or contraction (in general)
  - Identity and cut still hold (in modified form)
- Other substructural logic?

$2H_2 + O_2 \longrightarrow 2H_2O$

$H_2 \underbrace{+} O_2 \longrightarrow H_2O_2$

$2H_2O_2 \underbrace{+} Pt \underbrace{\longrightarrow} 2H_2O + O_2 + Pt$

$\otimes$ $\multimap$

\tensor \lolli

$$\frac{\Delta_A \Vdash A \qquad \Delta_B \Vdash B}{\Delta_A, \Delta_B \Vdash A \otimes B}\ \otimes R$$

$$\frac{\Delta, A, B \Vdash C}{\Delta, A \otimes B \Vdash C}\ \otimes L$$

$$\frac{\Delta, A \Vdash B}{\Delta \Vdash A \multimap B} \multimap R$$

$$\frac{\Delta_A \Vdash A \qquad \Delta_B, B \Vdash C}{\Delta_A, \Delta_B, A \multimap B \Vdash C} \multimap L$$

$$(H_2 \otimes H_2 \otimes O_2) \multimap (H_2O \otimes H_2O)$$

$$(H_2 \otimes O_2) \multimap (H_2O_2)$$

$$(H_2 \otimes H_2 \otimes O_2) \multimap (H_2O_2 \otimes H_2)$$

\oplus

$$(H_2 \otimes H_2 \otimes O_2) \multimap (H_2O_2 \otimes H_2) \oplus (H_2O \otimes H_2O)$$

"internal choice"

$$(H_2 \otimes H_2 \otimes O_2) \multimap (H_2O_2 \otimes H_2) \,\&\, (H_2O \otimes H_2O)$$

"with"

"external choice"

$$\frac{\Delta \Vdash A}{\Delta \Vdash A \oplus B}\;\oplus R_1 \qquad \frac{\Delta \Vdash B}{\Delta \Vdash A \oplus B}\;\oplus R_2$$

$$\frac{\Delta, A \Vdash C \qquad \Delta, B \Vdash C}{\Delta, A \oplus B \Vdash C}\;\oplus L \qquad \frac{\Delta \Vdash A \qquad \Delta \Vdash B}{\Delta \Vdash A \& B}\;\& R$$

$$\frac{\Delta, A \Vdash C}{\Delta, A \& B \Vdash C}\;\& L_1 \qquad \frac{\Delta, B \Vdash C}{\Delta, A \& B \Vdash C}\;\& L_2$$

$$\dfrac{}{\underbrace{\Gamma, A \longrightarrow A}}\ \text{id (structural)}$$

$$\dfrac{}{A \Vdash A}\ \text{id (linear)}$$

$$\dfrac{\dfrac{\overset{\times}{A, B} \overset{?}{\Vdash} A}{A \otimes B \overset{?}{\Vdash} A}\ \otimes L \qquad A \otimes B \overset{?}{\Vdash} B}{A \otimes B \overset{?}{\Vdash} A \& B}\ \& R$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{A \Vdash A}\,id
    \qquad
    \dfrac{
      \dfrac{}{B \Vdash B}\,id
    }{B \Vdash (A \oplus B)}\,\oplus R_2
  }{A, B \Vdash A \otimes (A \oplus B)}\,\otimes R
  \qquad\qquad\qquad\qquad
}{
  \dfrac{A, B \Vdash \ldots}{(A \otimes B) \Vdash (A \otimes (A \oplus B)) \,\&\, (B \otimes (A \oplus B))}\,\otimes L
}\,\& R
$$

$$\frac{}{\cdot \Vdash 1}\ 1R \qquad \frac{\Delta \Vdash C}{\Delta, 1 \Vdash C}\ 1L$$

$\rightarrow A \otimes 1 \equiv 1 \otimes A \equiv A$

$$\frac{}{\Delta \Vdash \top}\ \top R \qquad \text{No } \top L$$

$\rightarrow \top \& A \equiv A \& \top \equiv A$

$$\cancel{\frac{\qquad}{\qquad}}\ 0R \qquad \frac{}{\Delta, 0 \vdash C}\ 0L$$

$$0 \oplus A \equiv A \oplus 0 \equiv A$$

Can give $\bot$ $\parr$ in classical LL.

"par"

$\Delta \Vdash A \longrightarrow \Gamma ; \Delta \Vdash A$

Δ: resources

Γ: unrestricted resources

replicable resources

$$\frac{\Gamma, A ; \Delta, A \Vdash C}{\Gamma, A ; \Delta \Vdash C}\ \text{copy}$$

$!A$: of course A

replicable A

$$\frac{\Gamma ; \cdot \Vdash A}{\Gamma ; \cdot \Vdash !A}\ !R$$

$$\frac{\Gamma, A ; \Delta \Vdash C}{\Gamma ; \Delta, !A \Vdash C}\ !L$$

$A \supset B \qquad ((!A) \multimap B)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$A, B; A \Vdash A$}
\RightLabel{copy}
\UnaryInfC{$A, B; \cdot \Vdash A$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$A, B; B \Vdash B$}
\RightLabel{copy}
\UnaryInfC{$A, B; \cdot \Vdash B$}
\RightLabel{$\& R$}
\BinaryInfC{$A, B; \cdot \Vdash A \& B$}
\RightLabel{$! R$}
\UnaryInfC{$A, B; \cdot \Vdash !(A \& B)$}
\RightLabel{$! L^2$}
\UnaryInfC{$\cdot ; !A, !B \Vdash !(A \& B)$}
\RightLabel{$\otimes L$}
\UnaryInfC{$\cdot ; !A \otimes !B \Vdash !(A \& B)$}
\end{prooftree}
$$

Metatheory

- Weakening

$$\frac{\Gamma \,;\, \Delta \Vdash C}{\Gamma, A \,;\, \Delta \Vdash C}\, w$$

- contraction

$$\frac{\Gamma, A, A \,;\, \Delta \Vdash C}{\Gamma, A \,;\, \Delta \Vdash C}\, c$$

- identity Any proof of $A \Vdash A$ need only use id at atomic propositions

$$\dfrac{\dfrac{\dfrac{}{A \Vdash A}\,\text{i.h.}(A)}{A \,\&\, B \Vdash A}\,\&L_1 \qquad \dfrac{\dfrac{}{B \Vdash B}\,\text{i.h.}(B)}{A \,\&\, B \Vdash B}\,\&L_2}{A \,\&\, B \Vdash A \,\&\, B}\,\&R$$

Cut

$$\dfrac{\overset{\mathcal{D}}{\Gamma ; \Delta_1 \Vdash A} \qquad \overset{\mathcal{E}}{\Gamma ; \Delta_2, A \Vdash C}}{\Gamma ; \Delta_1, \Delta_2 \Vdash C}\ \text{cut} \quad \Bigg\} \text{ is admissible}$$

Induction on $(A, \mathcal{D}, \mathcal{E})$.

Case $\oplus R_1 / \oplus L$

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma ; \Delta_2, A \Vdash C} \qquad \overset{\mathcal{E}_2}{\Gamma ; \Delta_2, B \Vdash C}}{\Gamma ; \Delta_2, A \oplus B \Vdash C}\ \oplus L$$

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Gamma ; \Delta_1 \Vdash A}}{\Gamma ; \Delta_1 \Vdash A \oplus B}\ \oplus R_1$$

$$\dfrac{\Gamma ; \Delta_1 \Vdash A \qquad \Gamma ; \Delta_2, A \Vdash C}{\Gamma ; \Delta_1, \Delta_2 \Vdash C}\ \text{i.h.}(A, \mathcal{D}', \mathcal{E}_1)$$

Case !R/!L

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma ; \cdot \Vdash A\end{array}}{\Gamma ; \cdot \Vdash\, !A}\,!R \qquad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A ; \Delta_2 \Vdash C\end{array}}{\Gamma ; \Delta_2, !A \Vdash C}\,!L$$

$$\dfrac{\Gamma ; \cdot \Vdash A \qquad \Gamma, A ; \Delta_2 \Vdash C}{\Gamma ; \Delta_2 \Vdash C}\ \text{i.h.}(A, \mathcal{D}_1, \mathcal{E}_1)$$

$$\frac{\Gamma;\cdot \Vdash A \qquad \Gamma, A; \Delta \Vdash C}{\Gamma; \Delta \Vdash C}\ \mathrm{cut}^{u}$$

- weakening only : Affine logic
- contruction only : Strict/Relevant Logic
- Removing exchange : Ordered logic

A lin

A str

$\Gamma; \cdot \Vdash A$ str

$\Gamma; \Delta \Vdash A$ str $\times$