15-317 Lecture 20: Modal Logic

- Different modes of truth
- Modal operators
- Modal axioms
- S4, judgmentally
- Relating axioms to rules

Modal operators

- Is A always true?
- Is A ever true?
  - Was A always true?
  - Will A always be true?

$\Box A$

$\Diamond A$

$\Box A \supset A$

$A \supset \Diamond A$

$\Diamond A \nvdash \Box A$

$\Box(A \supset B) \supset \Box A \supset \Box B$

Common Modal Axioms

- N If $\vdash A$ then $\vdash \Box A$
- K $\Box(A \supset B) \supset \Box A \supset \Box B$
- T $\Box A \supset A$
- D $\Box A \supset \Diamond A$
- 4 $\Box A \supset \Box\Box A$
- B $A \supset \Box\Diamond A$
- 5 $\Diamond A \supset \Box\Diamond A$

Always used

Sometimes used

Rules for S4

valid props. true props

$$\frac{\Delta \; ; \; \cdot \vdash A\ \text{true}}{\Delta \; ; \; \Gamma \vdash \underline{\Box A}\ \text{true}}\ \Box I \qquad \frac{\Delta ; \Gamma \vdash \Box A\ \text{true} \qquad \Delta, A\ \text{valid} ; \Gamma \vdash J}{\Delta ; \Gamma \vdash J}\ \Box E$$

$$\downarrow$$

$$\frac{\Box\Delta \vdash A\ \text{true}}{\Box\Delta, \Gamma \vdash \Box A\ \text{true}}\ \Box I^{*} \qquad \left(\frac{\Delta ; \Gamma \vdash \Box A\ \text{true}}{\Delta ; \Gamma \vdash \underline{A\ \text{valid}}}\ \Box E^{*} \right)$$

$$\dfrac{}{\Delta; \Gamma, A\ \mathrm{true} \vdash A\ \mathrm{true}}\ \mathrm{hyp} \qquad \dfrac{}{\Delta, A\ \mathrm{valid}; \Gamma \vdash A\ \mathrm{true}}\ \mathrm{hyp}V$$

$$\dfrac{\Delta; \Gamma \vdash A\ \mathrm{poss}}{\Delta; \Gamma \vdash \Diamond A\ \mathrm{true}}\ \Diamond I \qquad \dfrac{\Delta; \Gamma \vdash \Diamond A\ \mathrm{true} \qquad \Delta; A\ \mathrm{true} \vdash C\ \mathrm{poss}}{\Delta; \Gamma \vdash C\ \mathrm{poss}}\ \Diamond E$$

$$\dfrac{\Delta; \Gamma \vdash A\ \mathrm{true}}{\Delta; \Gamma \vdash A\ \mathrm{poss}}\ \mathrm{here}$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Delta; \cdot \vdash A\ true}}{\Delta; \Gamma \vdash \Box A\ true}\ \Box I \qquad \overset{\mathcal{E}}{\Delta, A\ valid; \Gamma \vdash J}}{\Delta; \Gamma \vdash J}\ \Box E$$

$$\Longrightarrow_R \qquad \overset{[\mathcal{D}/\ldots]\,\mathcal{E}}{\Delta; \Gamma \vdash J}$$

$$\left[\dfrac{}{\Delta; A\ valid; \Gamma \vdash A\ true}\ hypv\right]$$

Replace with

$$\left[\dfrac{\overset{\mathcal{D}}{\Delta; \cdot \vdash A\ true}}{\Delta; \Gamma \vdash A\ true}\ weakn^{*}\right]$$

$$\begin{array}{c}\mathcal{D}\\ \Delta; \Gamma \vdash \Box A \ true\end{array}$$

$$\Longrightarrow_E$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta; \Gamma \vdash \Box A \ true\end{array} \qquad \dfrac{\dfrac{}{\Delta, A\ valid; \cdot \vdash A\ true}\ hyp^{v}}{\Delta, A\ valid; \Gamma \vdash \Box A\ true}\ \Box I}{\Delta; \Gamma \vdash \Box A\ true}\ \Box E$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Delta; \Gamma \vdash A\ poss}}{\Delta; \Gamma \vdash \Diamond A\ true}\ \Diamond I \qquad \overset{\mathcal{E}}{\Delta; A\ true \vdash C\ poss}}{\Delta; \Gamma \vdash C\ poss}\ \Diamond E$$

$$\Rightarrow_R \qquad \dfrac{[\mathcal{D}/\ldots]\,\mathcal{E}}{\Delta; \Gamma \vdash C\ poss}$$

$$\left[\dfrac{\dfrac{}{\Delta; \Gamma, A\ true \vdash A\ true}\ hyp}{\Delta; \Gamma, A\ true \vdash A\ poss}\ here\right]$$

Replace

$$\left[\dfrac{\overset{\mathcal{D}}{\Delta; \Gamma \vdash A\ poss}}{\Delta; \Gamma, A\ true \vdash A\ poss}\ weaken\right]$$

$$\overset{\mathcal{D}}{\Delta;\Gamma \vdash \Diamond A\ true}$$

$$\Longrightarrow_E \quad \cfrac{\cfrac{\overset{\mathcal{D}}{\Delta;\Gamma \vdash \Diamond A\ true} \qquad \cfrac{\cfrac{}{\Delta;\ A\ true \vdash A\ true}\,\text{hyp}}{\Delta;\ A\ true \vdash A\ poss}\,\text{here}}{\Delta;\Gamma \vdash A\ poss}\,\Diamond E}{\Delta;\Gamma \vdash \Diamond A\ true}\,\Diamond I$$

N If $\cdot \vdash A$ then $\cdot \vdash \Box A$

K $\Box(A \supset B) \supset \Box A \supset \Box B$

T $\Box A \supset A$

4 $\Box A \supset \Box\Box A$

T $\rightarrow$

$$
\dfrac{\dfrac{\dfrac{}{\cdot\,;\,\Box A\ \mathrm{true} \vdash \Box A\ \mathrm{true}}\,\mathrm{hyp} \qquad \dfrac{}{A\ \mathrm{valid}\,;\,\Box A\ \mathrm{true} \vdash A\ \mathrm{true}}\,\mathrm{hyp}^{v}}{\cdot\,;\,\Box A\ \mathrm{true} \vdash A\ \mathrm{true}}\,\Box E}{\cdot\,;\,\cdot \vdash \Box A \supset A\ \mathrm{true}}\,\supset I
$$

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    \dfrac{
      \dfrac{}{A\ \mathit{valid};\ \cdot \vdash A\ \mathit{true}}\,\mathit{hypv}
    }{A\ \mathit{valid};\ \cdot \vdash \square A\ \mathit{true}}\,\square I
  }{A\ \mathit{valid};\ \square A\ \mathit{true} \vdash \square\square A\ \mathit{true}}\,\square I
}{
  \dfrac{\cdot\,;\, \square A \vdash \square\square A\ \mathit{true}}{\cdot\,;\, \cdot \vdash \square A \supset \square\square A\ \mathit{true}}\,\supset I
}\,\square E
$$

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  \dfrac{}{\cdot\,;\ \Box(A\supset B), \Box A \vdash \Box(A\supset B)\ \mathit{true}}\ \mathrm{hyp}
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      \dfrac{
        \dfrac{}{A\supset B, A\,;\ \cdot \vdash A\supset B\ \mathit{true}}\ \mathrm{hypv}
        \qquad
        \dfrac{}{A\supset B, A\,;\ \cdot \vdash A\ \mathit{true}}\ \mathrm{hypv}
      }{A\supset B,\ A\,;\ \cdot \vdash B\ \mathit{true}}\ {\supset}E
    }{A\supset B, A\,;\ \Box(A\supset B), \Box A \vdash \Box B\ \mathit{true}}\ \Box I
  }{A\supset B\,;\ \Box(A\supset B), \Box A \vdash \Box B\ \mathit{true}}\ \Box E
}{\cdot\,;\ \Box(A\supset B),\ \Box A \vdash \Box B\ \mathit{true}}\ \Box E
$$

Usually : $\Diamond A := \neg \Box \neg A$

$\Diamond T \quad A \supset \Diamond A$

$\Diamond 4 \quad \Diamond \Diamond A \supset \Diamond A$

$\Diamond K \quad \Box (A \supset B) \supset \Diamond A \supset \Diamond B$

$$\dfrac{\dfrac{}{\vdash \Box(A\supset B)}\,\text{hyp} \qquad \dfrac{\dfrac{\dfrac{}{A\supset B;\Diamond A\vdash\Diamond A\ \text{true}}\,\text{hyp}\qquad \dfrac{\dfrac{\dfrac{}{A\supset B;A\vdash A\supset B\ \text{true}}\,\text{hypv}\quad \dfrac{}{A\supset B;A\vdash A\ \text{true}}\,\text{hyp}}{A\supset B;A\vdash B\ \text{true}}\,{\supset}E}{A\supset B;A\vdash B\ \text{poss}}\,\text{here}}{A\supset B;\Diamond A\vdash B\ \text{poss}}\,\Diamond E}{A\supset B;\Diamond A\vdash\Diamond B\ \text{true}}\,\Diamond I}{\cdot;\Box(A\supset B),\Diamond A\vdash\Diamond B\ \text{true}}\,\Box E$$