15-317 Lecture 12: Inversion

- HW 6 due Sat 5 Mar
- HW 7 out today

- Invertible rules
- Synchrony/Asynchrony
- An inversion calculus
- Contraction-free inversion calculus

Invertible rules

- Premises are equivalent to conclusion.

e.g. $\wedge R$

$$\dfrac{\Gamma \longrightarrow A \qquad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

$$\dfrac{\Gamma \longrightarrow A \wedge B \qquad \dfrac{\dfrac{}{\Gamma, A, B \longrightarrow A}\ \mathrm{id}}{\Gamma, A \wedge B \longrightarrow A}\ \wedge L}{\Gamma \longrightarrow A}\ \mathrm{cut}$$

Synchrony / Asynchrony

- A rule is Asynchronous if it can always be applied without affecting provability
- Synchronous otherwise

- A connective is right-(a)synchronous if its right rule is synchronous.
- Sim. for left

- write C^+ for a right-synchronous formula
- write A^- (Γ^-) for a left-synchronous formula (context)

Inversion calculus

- $\Gamma^- ; \Omega \xrightarrow{R} C$ (right inversion)
- $\Gamma^- ; \Omega \xrightarrow{L} C^+$ (left inversion)
- $\Gamma^- \xrightarrow{S} C^+$ (search)

- We think of Ω as ordered (like a stack)
- Can only access rightmost formula in Ω.
- Write $\Omega \cdot A$ instead of Ω, A

Right inversion

$$\frac{\Gamma^-; \Omega \xrightarrow{R} A \qquad \Gamma^-; \Omega \xrightarrow{R} B}{\Gamma^-; \Omega \xrightarrow{R} A \wedge B}\ \wedge R \qquad \frac{\Gamma^-; \Omega \cdot A \xrightarrow{R} B}{\Gamma^-; \Omega \xrightarrow{R} A \supset B}\ \supset R$$

$$\cancel{\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}\ \vee R_1} \qquad \frac{}{\Gamma^-; \Omega \xrightarrow{R} \top}\ \top R \qquad \frac{\Gamma^-; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega \xrightarrow{R} C^+}\ LR$$

Left inversion

$$\dfrac{}{\Gamma^-; \Omega \cdot F \xrightarrow{L} C^+}\ FL \qquad \dfrac{\Gamma^-; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot T \xrightarrow{L} C^+}\ TL$$

$$\dfrac{\Gamma^-; \Omega \cdot A \cdot B \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot (A \wedge B) \xrightarrow{L} C^+}\ \wedge L \qquad \dfrac{\Gamma^-; \Omega \cdot A \xrightarrow{L} C^+ \quad \Gamma^-; \Omega \cdot B \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot (A \vee B) \xrightarrow{L} C^+}\ \vee L$$

$$\dfrac{\Gamma^-, A^-; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot A^- \xrightarrow{L} C^+}\ \text{shift} \qquad \dfrac{\Gamma^- \xrightarrow{S} C^+}{\Gamma^-; \cdot \xrightarrow{L} C^+}\ SL$$

Search

- id, $\vee R_{1/2}$, $\supset L$

$$\frac{P \in \Gamma^-}{\Gamma^- \xrightarrow{S} P}\ id$$

$$\frac{\Gamma^-; \cdot \xrightarrow{R} A}{\Gamma^- \xrightarrow{S} A \vee B}\ \vee R_1 \qquad \frac{\Gamma^-; \cdot \xrightarrow{R} B}{\Gamma^- \xrightarrow{S} A \vee B}\ \vee R_2$$

$$\frac{\Gamma^-, A \supset B; \cdot \xrightarrow{R} A \qquad \Gamma^-; B \xrightarrow{L} C^+}{\Gamma^-, A \supset B \xrightarrow{S} C^+}\ \supset L$$

Metatheory

$$\Gamma^-; \Omega \xrightarrow{R} C$$

$$\Gamma^-; \Omega \xrightarrow{L} C^+$$

$$\Gamma^- \xrightarrow{S} C^+$$

$$\Gamma, \overline{\Omega} \longrightarrow C$$

$$\Gamma \longrightarrow C$$

$$\overline{\cdot} = \cdot$$

$$\overline{\Omega \cdot A} = \overline{\Omega}, A$$

Theorem (soundness of inversion) / If $\Gamma^-; \Omega \xrightarrow{R} C$, then $\Gamma, \overline{\Omega} \longrightarrow C$, if $\Gamma^-; \Omega \xrightarrow{L} C^+$, then $\Gamma^-, \overline{\Omega} \longrightarrow C$, and if $\Gamma^- \xrightarrow{S} C^+$, then $\Gamma \longrightarrow C$.

Theorem (completeness of inversion)

1. If $\Gamma^-, \overline{\Omega} \longrightarrow A$, then $\Gamma^-; \Omega \xrightarrow{R} A$

2. If $\Gamma^-, \overline{\Omega} \longrightarrow C^+$, then $\Gamma^-; \Omega \xrightarrow{L} C^+$

- No weakening in $\xrightarrow{R/L/S}$ (proof may get bigger)

e.g.

$$\dfrac{}{\Gamma^-; \Omega \cdot F \xrightarrow{L} C^+}\,FL \qquad \Gamma^-; \Omega \cdot F \cdot (A \vee B) \wedge (D \vee E) \xrightarrow{L} C^+$$

Contraction-free inversion

$$\frac{P \in \Gamma \qquad \Gamma, B \longrightarrow C}{\Gamma, P \supset B \longrightarrow C} \; P{\supset}L$$

invertible but synchronous

$$\frac{\Gamma, A_1, (A_2 \supset B) \longrightarrow A_2 \qquad \Gamma, B \longrightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \longrightarrow C} \; {\supset}{\supset}L$$

not invertible

$$\frac{\Gamma^-; \Omega \cdot A_1 \supset (A_2 \supset B) \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot (A_1 \wedge A_2) \supset B \xrightarrow{L} C^+} \wedge\supset L$$

$$\frac{\Gamma^-; \Omega \cdot B \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot \top \supset B \xrightarrow{L} C^+} \top\supset L$$

$$\frac{\Gamma^-; \Omega \cdot (A_1 \supset B) \cdot (A_2 \supset B) \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot (A_1 \vee A_2) \supset B \xrightarrow{L} C^+} \vee\supset L$$

$$\frac{\Gamma^-; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega \cdot F \supset B \xrightarrow{L} C^+} F\supset L$$

$$\frac{P \in \Gamma^- \qquad \Gamma^-; B \xrightarrow{L} C^+}{\Gamma^-, P \supset B \xrightarrow{S} C^+} P\supset L$$

$$\frac{\Gamma^-; A_1 \cdot (A_2 \supset B) \xrightarrow{R} A_2 \qquad \Gamma^-; B \xrightarrow{L} C^+}{\Gamma^-, (A_1 \supset A_2) \supset B \xrightarrow{S} C^+} \supset\supset L$$