15-317 Lecture 11: Classical Logic

- HW5 due Sat. 26 Feb
- HW6 out

- Proving A false
- Relating classical and constructive logic
- Classical logic and computation

$$\frac{A \text{ false} \qquad A \text{ ctrue}}{}$$

$$\frac{\Gamma;\Delta \vdash A \text{ ctrue} \qquad \Gamma;\Delta \vdash B \text{ ctrue}}{\Gamma;\Delta \vdash A \wedge B \text{ ctrue}} \wedge T$$

$$\frac{\Gamma;\Delta \vdash A \text{ false}}{\Gamma;\Delta \vdash A \wedge B \text{ false}} \wedge F_1$$

$$\frac{\Gamma;\Delta \vdash B \text{ false}}{\Gamma;\Delta \vdash A \wedge B \text{ false}} \wedge F_2$$

ctrue hyps. false hyps.

$$\frac{}{\Gamma;\Delta \vdash \top \ \mathit{true}}\ \top T \qquad \frac{}{\Gamma;\Delta \vdash \bot \ \mathit{false}}\ \bot F$$

$$\frac{\Gamma;\Delta \vdash A \ \mathit{true}}{\Gamma;\Delta \vdash A \vee B \ \mathit{true}}\ \vee T_1 \qquad \frac{\Gamma;\Delta \vdash B \ \mathit{true}}{\Gamma;\Delta \vdash A \vee B \ \mathit{true}}\ \vee T_2$$

$$\frac{\Gamma;\Delta \vdash A \ \mathit{false} \qquad \Gamma;\Delta \vdash B \ \mathit{false}}{\Gamma;\Delta \vdash A \vee B \ \mathit{false}}\ \vee F$$

$$\frac{\Gamma ; \Delta \vdash A \text{ false}}{\Gamma ; \Delta \vdash \neg A \text{ true}} \neg T \qquad \frac{\Gamma ; \Delta \vdash A \text{ true}}{\Gamma ; \Delta \vdash \neg A \text{ false}} \neg F$$

$$\frac{}{\Gamma , A \text{ true} ; \Delta \vdash A \text{ true}} \text{hyp}T \qquad \frac{}{\Gamma ; \Delta , A \text{ false} \vdash A \text{ false}} \text{hyp}F$$

Judgement $\#$, meaning contradiction

$$\frac{\Gamma ; \Delta \vdash A \text{ ctrue} \qquad \Gamma ; \Delta \vdash A \text{ false}}{\Gamma ; \Delta \vdash \#} \,\#I$$

$$\frac{\Gamma ; \Delta, A \text{ false} \vdash \#}{\Gamma ; \Delta \vdash A \text{ ctrue}} \,\#ET \qquad \frac{\Gamma, A \text{ ctrue} ; \Delta \vdash \#}{\Gamma ; \Delta \vdash A \text{ false}} \,\#EF$$

$$\frac{}{\cdot\,; A \vee \neg A, A \vdash A \text{ false}}\ \text{hyp}$$

$$\frac{}{\cdot\,; A \vee \neg A, A \vdash \neg A \text{ ctrue}}\ \neg T$$

$$\cdot\,; A \vee \neg A, A \vdash \neg A \text{ false}$$

$$\frac{}{\cdot\,; A \vee \neg A, A \vdash \#}\ \# I$$

$$\frac{}{\cdot\,; A \vee \neg A, A \vdash A \text{ false}}\ \text{hyp}$$

$$\frac{}{\cdot\,; A \vee \neg A, A \vdash A \text{ ctrue}}\ \# E$$

$$\frac{}{\cdot\,; A \vee \neg A, A \vdash \#}\ \# I$$

$$\frac{}{\cdot\,; A \vee \neg A \text{ false} \vdash A \text{ ctrue}}\ \# ET \qquad \cdot\,; A \vee \neg A \text{ false} \vdash A \text{ false}$$

$$\frac{}{\cdot\,; A \vee \neg A \text{ false} \vdash \#}\ \# I$$

$$\frac{}{\cdot\,; \cdot \vdash (A \vee \neg A) \text{ ctrue}}\ \# ET$$

Recovering E rules

$$\frac{\Gamma \vdash A \wedge B \ \mathit{true}}{\Gamma \vdash A \ \mathit{true}} \wedge E_1$$

$$\dfrac{\dfrac{\left\{\Gamma ; \Delta \vdash A \wedge B \ \mathit{ctrue}\right\}}{\Gamma ; \Delta, A \ \mathit{false} \vdash A \wedge B \ \mathit{ctrue}} W \qquad \dfrac{\dfrac{}{\Gamma ; \Delta, A \vdash A \ \mathit{false}} \mathit{hyp}}{\Gamma ; \Delta, A \ \mathit{false} \vdash A \wedge B \ \mathit{false}} \wedge F_1}{\dfrac{\Gamma ; \Delta, A \ \mathit{false} \vdash \#}{\Gamma ; \Delta \vdash A \ \mathit{ctrue}} \# ET}$$

$$\dfrac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\,\vee E$$

$$\Gamma, A; \underline{\Delta} \vdash C\ \text{ctrue}$$

$$\dfrac{\Gamma; \Delta \vdash A \vee B\ \text{ctrue}}{\Gamma; \Delta, C \vdash A \vee B\ \text{ctrue}}\,W$$

$$\dfrac{\Gamma, A; \Delta \vdash C\ \text{ctrue}}{\Gamma, A; \Delta, C \vdash C\ \text{ctrue}}\,W \qquad \dfrac{}{\Gamma, A; \Delta, C \vdash C\ \text{false}}\,\text{hyp}$$

$$\dfrac{\Gamma, A; \Delta, C \vdash C\ \text{ctrue} \qquad \Gamma, A; \Delta, C \vdash C\ \text{false}}{\Gamma, A; \Delta, C \vdash \#}\,\#I$$

$$\dfrac{\Gamma, A; \Delta, C \vdash \#}{\Gamma; \Delta, C \vdash A\ \text{false}}\,\#E$$

$$\dfrac{\Gamma; \Delta, C \vdash A\ \text{false} \qquad [\quad]}{\Gamma; \Delta, C \vdash A \vee B\ \text{false}}$$

$$\dfrac{\Gamma; \Delta, C \vdash A \vee B\ \text{ctrue} \qquad \Gamma; \Delta, C \vdash A \vee B\ \text{false}}{\Gamma; \Delta, C \vdash \#}\,\#I$$

$$\dfrac{\Gamma; \Delta, C \vdash \#}{\Gamma; \Delta \vdash C\ \text{ctrue}}\,\#E$$

Constructively $A \supset (\neg\neg A)$

Classically, $A \equiv \neg\neg A$

Theorem/ If A ctrue then $\neg\neg A$ true.

Stronger version/ If $\Gamma ; \Delta \vdash A$ ctrue then $\neg\neg\Gamma, \neg\Delta \vdash \neg\neg A$ true

and if $\Gamma ; \Delta \vdash A$ false then $\neg\neg\Gamma, \neg\Delta \vdash \neg A$ true

and if $\Gamma ; \Delta \vdash \#$ then $\neg\neg\Gamma, \neg\Delta \vdash F$ true

Computation

$$\frac{\Gamma; \Delta \vdash e_1 : A_1 \text{ ctrue} \quad \Gamma; \Delta \vdash e_2 : A_2 \text{ ctrue}}{\Gamma; \Delta \vdash \langle e_1, e_2 \rangle : A_1 \wedge A_2 \text{ true}}$$

$$\frac{\Gamma; \Delta \vdash \quad K : A_1 \text{ false}}{\Gamma; \Delta \vdash \quad K \circ \text{fst} : A_1 \wedge A_2 \text{ false}}$$

$$\frac{\Gamma; \Delta \vdash e : A_1 \text{ ctrue}}{\Gamma; \Delta \vdash \quad \text{inl } e : A_1 \vee A_2 \text{ ctrue}} \vee T_1$$

$$\frac{\Gamma; \Delta \vdash k_1 : A_1 \text{ false} \quad \Gamma; \Delta \vdash k_2 : A_2 \text{ false}}{\Gamma; \Delta \vdash \quad [k_1, k_2] : A_1 \vee A_2 \text{ false}} \vee F$$

$$\frac{\Gamma; \Delta \vdash e : A \qquad \Gamma; \Delta \vdash k : A}{\Gamma; \Delta \vdash k \lhd e : \#}$$

$$\frac{\Gamma; \Delta, u{:}A \vdash c : \#}{\Gamma; \Delta \vdash u{:}A.\,c : A \text{ ctrue}} \#ET^{u} \qquad \frac{\Gamma, u{:}A; \Delta \vdash c : \#}{\Gamma; \Delta \vdash u{:}A.\,c : A \text{ false}}$$

$$\frac{e_1 \to e_1'}{\langle e_1, e_2 \rangle \to \langle e_1', e_2 \rangle}$$

$$\frac{k_1 \to k_1'}{[k_1, k_2] \to [k_1', k_2]}$$

$$\ldots$$

$$\frac{e \to e'}{k \triangleleft e \to k \triangleleft e'} \qquad \frac{k \to k'}{k \triangleleft e \to k' \triangleleft e}$$

$$\frac{}{k \triangleleft (u : A).\, c \longrightarrow [k/u]\, c}$$

$$\frac{}{(u : A).\, c \triangleleft e \longrightarrow [e/u]\, c}$$

$$\frac{}{k \circ \mathsf{fst} \lhd \langle e_1, e_2 \rangle \longrightarrow k \lhd e_1}$$

$$\frac{}{k \circ \mathsf{snd} \lhd \langle e_1, e_2 \rangle \longrightarrow k \lhd e_2}$$

$$\frac{}{[k_1, k_2] \lhd \mathsf{inl}\ e \longrightarrow k_1 \lhd e}$$

$$\frac{}{[k_1, k_2] \lhd \mathsf{inr}\ e \longrightarrow k_2 \lhd e}$$

$$\underbrace{u : \overline{A \vee \neg A}\,.\; u \lhd \mathrm{inr}(\mathrm{not}(v{:}A.\; u \lhd \mathrm{inl}\, v))}_{e} : A \vee \neg A \ \mathrm{ctrue}$$

$$k \lhd e \longrightarrow \underset{[k_1, k_2]}{\overset{\downarrow}{k}} \lhd \mathrm{inr}(\mathrm{not}(v{:}A.\; k \lhd \mathrm{inl}\, v))$$

$$[k_1, \mathrm{not}\, e']$$

$$\longrightarrow \underset{\mathrm{not}\ e'}{\overset{\downarrow}{k_2}} \lhd \mathrm{not}(v{:}A.[k_1, k_2] \lhd \mathrm{inl}\, v)$$

$$\longrightarrow (v{:}A.\, [k_1, k_2] \lhd \mathrm{inl}\, v) \lhd e'$$

$$\longrightarrow [k_1, k_2] \lhd \mathrm{inl}\, e' \longrightarrow k_1 \lhd e'$$