15317 Lecture 10: Cut

- HW4 due, HW5 out
- Midterm 1 Tuesday 22 Feb.

- Proving cut
- Applications of cut

Theorem (cut)/ If $\underset{\mathcal{D}}{\Gamma \Longrightarrow A}$ and $\underset{\mathcal{E}}{\Gamma, A \Longrightarrow C}$, then $\underset{\mathcal{F}}{\Gamma \Longrightarrow C}$

Proof/ Consider the last rule used in $\mathcal{D}/\mathcal{E}$

Case $\mathcal{D}$ ends with id:

$$\mathcal{D} = \frac{}{\Gamma', A \Longrightarrow A}\,\text{id} \quad , \quad \overset{\mathcal{E}}{\Gamma', A, A \Longrightarrow C}$$

$$\left(\begin{array}{c} \mathcal{E} \\ \Gamma', A, A \Longrightarrow C \\ \hdashline \Gamma', A \Longrightarrow C \end{array} \; C \right.$$

Case $\mathcal{E}$ ends in id:

- id_A

- id_C

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \qquad \mathcal{E} = \dfrac{}{\Gamma, A \Longrightarrow A}\, id_A \qquad \begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ \Gamma', C \Longrightarrow A\end{array} \qquad \mathcal{E} = \dfrac{}{\Gamma', C, A \Longrightarrow C}\, id_C \qquad \dfrac{}{\Gamma', C \Longrightarrow C}\, id_C$$

Case $\mathcal{D}$ ends in $\wedge L_1$:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma', B \wedge D, B \Rightarrow A}}{\Gamma', B \wedge D \Rightarrow A}\,\wedge L_1 \qquad \overset{\mathcal{E}}{\Gamma', B \wedge D, A \Rightarrow C}$$

$$\mathcal{F} = \dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma', B \wedge D, B \Rightarrow A} \qquad \dfrac{\overset{\mathcal{E}}{\Gamma', B \wedge D, A \Rightarrow C}}{\Gamma', B \wedge D, B, A \Rightarrow C}\,W}{\Gamma', B \wedge D, B \Rightarrow C}\,\text{i.h. } (\mathcal{D}_1 < \mathcal{D})}{\Gamma', B \wedge D \Rightarrow C}\,\wedge L_1$$

Case $\mathcal{E}$ ends in $\wedge L_1$:

- A is the principal formula $B \wedge D$ of $\wedge L_1$] delayed to later
- A is a side formula of $\wedge L_1$]

$$\begin{array}{c} \mathcal{D} \\ \Gamma', B \wedge D \Rightarrow A \end{array} \qquad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma', B \wedge D, B, A \Rightarrow C\end{array}}{\Gamma', B \wedge D, A \Rightarrow C}\,\wedge L_1$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ \Gamma', B \wedge D \Rightarrow A\end{array}}{\Gamma', B \wedge D, B \Rightarrow A}\,W \qquad \begin{array}{c}\mathcal{E}_1 \\ \Gamma', B \wedge D, B, A \Rightarrow C\end{array}}{\Gamma', B \wedge D, B \Rightarrow C}\,\text{i.h. } (\mathcal{E}_1 < \mathcal{E})}{\Gamma', B \wedge D \Rightarrow C}\,\wedge L_1$$

Ind. Hyp.: Given $\begin{matrix}\mathcal{D} \\ \Gamma \Rightarrow A\end{matrix}$ and $\begin{matrix}\mathcal{E} \\ \Gamma, A \Rightarrow C\end{matrix}$, if $\mathcal{D}_1 < \mathcal{D}$

and $\mathcal{E}_1 \leq \mathcal{E}$, with $\begin{matrix}\mathcal{D}_1 \\ \Gamma' \Rightarrow A\end{matrix}$ and $\begin{matrix}\mathcal{E}_1 \\ \Gamma', A \Rightarrow D\end{matrix}$, then

$\Gamma' \Rightarrow D$.

(And likewise if $\mathcal{D}_1 \leq \mathcal{D}$ and $\mathcal{E}_1 < \mathcal{E}$)

$\mathcal{E}$ ends in $\wedge L_1$ ($A = B \wedge D$ principal):

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow D\end{array}}{\Gamma \Rightarrow B \wedge D}\,\wedge R$$

$$\mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, B\wedge D, B \Rightarrow C\end{array}}{\Gamma, B \wedge D \Rightarrow C}\,\wedge L_1$$

$$\mathcal{F} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow B\end{array} \qquad \underbrace{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow B\wedge D\end{array} \qquad \begin{array}{c}\mathcal{E}_1\\ \Gamma, B\wedge D, B \Rightarrow C\end{array}}{\Gamma, B \Rightarrow C}\,\text{i.h.}}_{\mathcal{F}_1}}{\Gamma \Rightarrow C}\,\text{i.h.}$$

$(\mathcal{E}_1 < \mathcal{E})$

$\mathcal{D}_1 < \mathcal{D}$

IH/ Given A, $\overset{\mathcal{D}}{\Gamma \Rightarrow A}$, and $\overset{\mathcal{E}}{\Gamma, A \Rightarrow C}$,

$\overset{\mathcal{D}_1}{\Gamma' \Rightarrow B}$ $\overset{\mathcal{E}_1}{\Gamma', B \Rightarrow D}$.

If $B < A$ or

$B = A$ and either $\mathcal{D}_1 < \mathcal{D}$ or $\mathcal{E}_1 < \mathcal{E}$, then

$\Gamma' \Rightarrow D$.

Case $A = B \supset D$, $\mathcal{D}$ ends in $\supset R$, $\mathcal{E}$ ends in $\supset L$

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma, B \Rightarrow D}}{\Gamma \Rightarrow B \supset D}\ \supset R$$

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma, B \supset D \Rightarrow B} \qquad \overset{\mathcal{E}_2}{\Gamma, B \supset D, D \Rightarrow C}}{\Gamma, B \supset D \Rightarrow C}\ \supset L$$

$$\mathcal{F}_1 = \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow B \supset D} \qquad \overset{\mathcal{E}_1}{\Gamma, B \supset D \Rightarrow B}}{\Gamma \Rightarrow B}\ \text{i.h.}\ (\mathcal{E}_1 < \mathcal{E}) \qquad \overset{\mathcal{D}_1}{\Gamma, B \Rightarrow D}}{\Gamma \Rightarrow D}\ \text{i.h.}\ (B < B \supset D)$$

$$\mathcal{F}_2 = \dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow B \supset D} \qquad \overset{\mathcal{E}_2}{\Gamma, B \supset D, D \Rightarrow C}}{\Gamma, D \Rightarrow C}\ \text{i.h.}\ (\mathcal{E}_2 < \mathcal{E})$$

$$\mathcal{F} = \dfrac{\overset{\mathcal{F}_1}{\Gamma \Rightarrow D} \qquad \overset{\mathcal{F}_2}{\Gamma, D \Rightarrow C}}{\Gamma \Rightarrow C}\ \text{i.h.}\ (D < B \supset D)$$

What cases do we need to cover?

- Identity cases ($\mathcal{D}$ or $\mathcal{E}$ ends in id) 2
- Principal cases (A is principal formula in $\mathcal{D}$ and $\mathcal{E}$) 1 per R/L rule pair(s)
- Commuting cases (A is aside formula in $\mathcal{D}$ or $\mathcal{E}$) 2 per L rule, 1 per R rule

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Notation/

Last rule in $\mathcal{D}$

Last rule in $\mathcal{E}$

- id/* or */id
- $\supset R / \supset L$, $\wedge R / \wedge L_1$, ..., $\vee R_1 / \vee L$
- $\wedge L_1 / *$, $* / \wedge L_1$, $* / \wedge R$, ...

Applications of cut

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$$\begin{tikzcd}
 & V \quad U \arrow[dl] \arrow[dr, shift left] & \\
ND \arrow[rr, "cut"] & & SC \arrow[ul, shift left]
\end{tikzcd}$$

- Global soundness (If $\begin{array}{c}\Gamma\downarrow\\ \vdots \\ A\uparrow\end{array}$ and $\begin{array}{c}\Gamma\downarrow \; A\downarrow\\ \vdots \\ C\uparrow\end{array}$ then $\begin{array}{c}\Gamma\downarrow\\ \vdots \\ C\uparrow\end{array}$)
- Identity justifies $\dfrac{A\downarrow}{A\uparrow}\downarrow\uparrow$
- Cut justifies $\dfrac{A\uparrow}{A\downarrow}$