15-317 Lecture 9: Sequent Calculus II

- Midterm 1 in a week (22 Feb, 2022)
  - In class

- Recap of sequent calculus
- Disproof
- Optimizations of rules
- Classical sequent calculus (briefly)
- Relating sequent calculus to natural deduction?

Recap

- I/E rules $\Rightarrow$ L/R rules
- E rules became L rules (but parts flipped)

$$\frac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1 \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\ \vee L \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ \vee R_1$$

Disproof

- Rules work bottom-up
- Seq. calc. proofs satisfy the subformula property:
  The proof of a sequent $\Gamma \Longrightarrow A$ only references subformulas of either A or something in Γ

Theorem/ It is not the case that $\Longrightarrow F$.

Proof/ No Left rule applies, as left is empty
Identity does not apply
No Right rule applies (No FR)

Theorem / $\not\Longrightarrow A \vee \neg A$

Proof / Two rules apply ($\vee R_1$, $\vee R_2$)

Case $\vee R_1$:

We need to prove $\Longrightarrow A$

No rule applies.

Case $\vee R_2$:

Goal $\Longrightarrow A \supset F$

$\supset R$ is the only applicable rule

New Goal $A \Longrightarrow F$

No rule applies

Theorem/ If $\Rightarrow A \vee B$, then either $\Rightarrow A$ or $\Rightarrow B$

Proof/ The last rule of a proof of $\Rightarrow A \vee B$ must be $\vee R_i$.

Case $\vee R_1$:

In this case, $\Rightarrow A$ was used to get $\Rightarrow A \vee B$

Case $\vee R_2$:

$\Rightarrow B$ must have been used to get $\Rightarrow A \vee B$

Theorem/ $\not\Rightarrow (\neg\neg A) \supset A$

stuck

$\neg\neg A, A \Rightarrow F$

$\overline{\neg\neg A, A \Rightarrow \neg A}$ $\supset R$

$\overline{\qquad}$ FL

$\neg\neg A, A, F \Rightarrow F$

$\overline{\neg\neg A, A \Rightarrow F}$ $\supset L$

$\overline{\qquad}$ $\supset R$

$[\neg\neg A \Rightarrow \neg A$

$\overline{\qquad}$ FL

$\neg\neg A, F \Rightarrow A$

$\overline{\neg\neg A \Rightarrow A}$ $\supset L$

$\overline{\Rightarrow (\neg\neg A) \supset A}$ $\supset R$

why not

$$\frac{\neg\neg A, A, A \Rightarrow F}{\neg\neg A, A \Rightarrow \neg A} \supset R \quad ?$$

stuck

$$\frac{\neg\neg A, A \Rightarrow \neg A \qquad \dfrac{}{\neg\neg A, A, F \Rightarrow \neg A} FL}{\neg\neg A, A \Rightarrow \neg A} \supset L$$

Theorem/ If $\Gamma, A, A \Longrightarrow C$, then also $\Gamma, A \Longrightarrow C$.

Moreover, the proof has the same structure.

(Contraction)

Proof/ Choose one copy of A (A^1). Wherever the proof uses the other A (A^2), instead use A^1.

Theorem (Weakening)/ If $\Gamma \Rightarrow C$, then also $\Gamma, A \Rightarrow C$ with the same proof structure.

Proof/ Add A to every sequent in the proof of $\Gamma \Rightarrow C$ but never use it.

Rule optimizations

- Remove redundant premises or antecedents
- Potentially speed up proof search

optional

$$\frac{\Gamma, [A \lor B], A \Rightarrow C \qquad \Gamma, [A \lor B], B \Rightarrow C}{\Gamma, A \lor B \Rightarrow C} \lor L$$

$$\frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, [A \supset B], B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L$$

$$\frac{\Gamma, A \land B, A \Rightarrow C}{\Gamma, A \land B \Rightarrow C} \land L_1$$

$$\frac{\Gamma, [A \land B], A, B \Rightarrow C}{\Gamma, A \land B \Rightarrow C} \land L$$

Classical seq. calc.

- From $\Gamma \Longrightarrow A$ with one succedent to

  $\Gamma \overset{CL}{\Longrightarrow} \Delta$ with multiple succedents

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\underline{A} \overset{CL}{\Longrightarrow} A \vee \neg A, \underline{A}, \neg A, F}\,id}{\overset{CL}{\Longrightarrow} A \vee \neg A, A, \neg A}\,\supset R}{\overset{CL}{\Longrightarrow} A \vee \neg A, A}\,\vee R_2}{\overset{CL}{\Longrightarrow} A \vee \neg A}\,\vee R_1
$$

Relating ND and SC

Goal: If $\Gamma \vdash C$ then $\Gamma \Longrightarrow C$

$$\begin{array}{c} \Gamma \\ \vdots\ \mathcal{D} \\ C \end{array}$$

Proof/ By induction on $\mathcal{D}$.

Case $\mathcal{D}$ ends in $\wedge I$

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ A\ \text{true}\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ B\ \text{true}\end{array}}{A \wedge B\ \text{true}} \wedge I \qquad \dfrac{\begin{array}{c}\text{i.h.}(\mathcal{D}_1) \\ \Gamma \Longrightarrow A\end{array} \qquad \begin{array}{c}\text{i.h.}(\mathcal{D}_2) \\ \Gamma \Longrightarrow B\end{array}}{\Gamma \Longrightarrow A \wedge B} \wedge R$$

Case $\wedge E_1$

$$\mathcal{D} = \dfrac{\begin{matrix}\Gamma \\ \vdots\ \mathcal{D}_1 \\ A \wedge B \text{ true}\end{matrix}}{A \text{ true}} \wedge E_1$$

$$\dfrac{\begin{matrix} i.h.(\mathcal{D}_1) \\ \Gamma \Longrightarrow A \wedge B \end{matrix} \qquad \dfrac{\overline{\Gamma, A \wedge B, A \Longrightarrow A}\ id}{\Gamma, A \wedge B \Longrightarrow A} \wedge L_1}{\Gamma \Longrightarrow A} \text{ cut}$$

Cut

$$\frac{\Gamma \Longrightarrow A \qquad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C}\ \text{cut}$$

- This is exactly global soundness!
- If the cut theorem holds, we can translate ND $\Longrightarrow$ SC. From SC $\Rightarrow$ VU, we can conclude the original form of global completeness.