15-317 Lecture 8: Sequent Calculus

- HW3 due, HW4 out

- Sequents
- Sequent Calculus
- Connecting Sequent Calculus to Verifications
- Identity?

Why do we want a different proof system?

- To prove global soundness
- Make proof search easier

Sequents

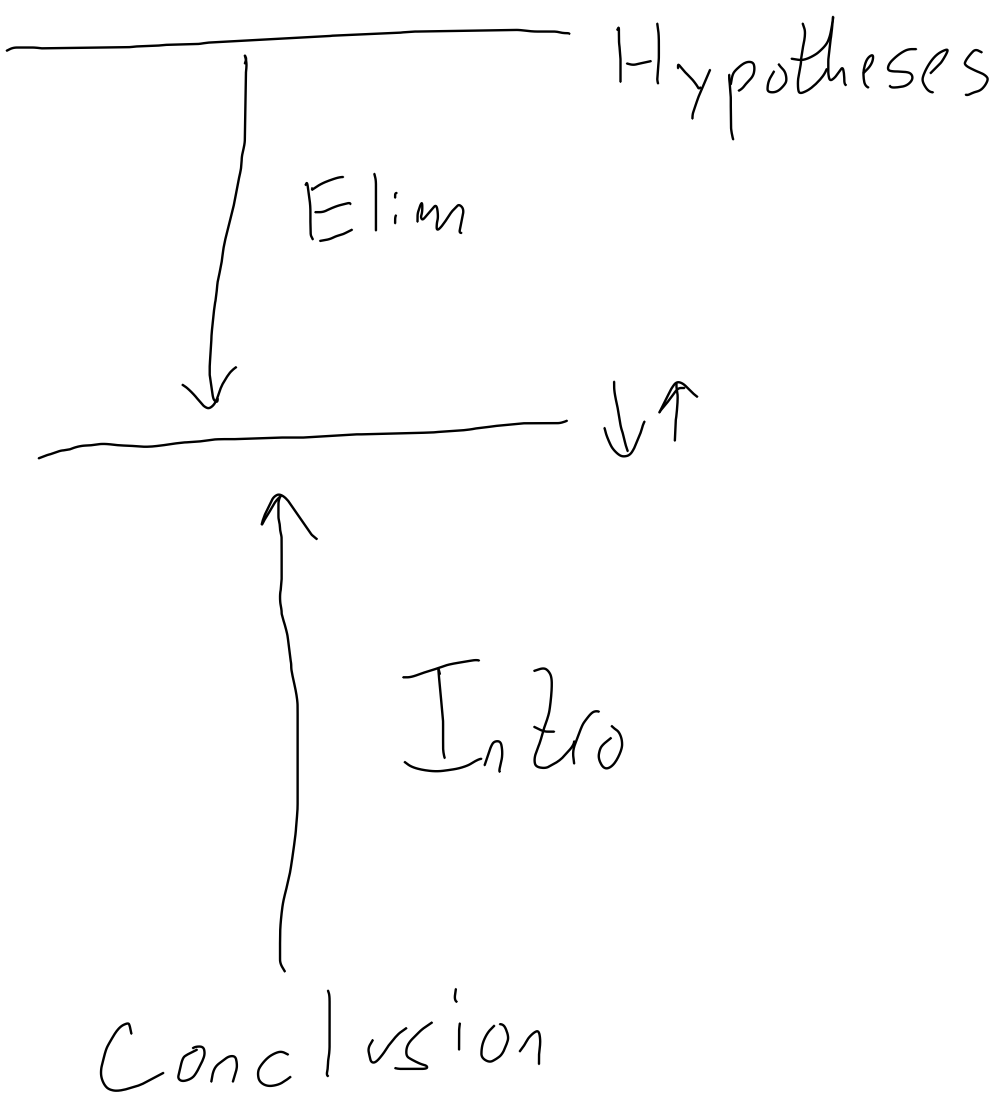
- The state of an ongoing verification looks like:

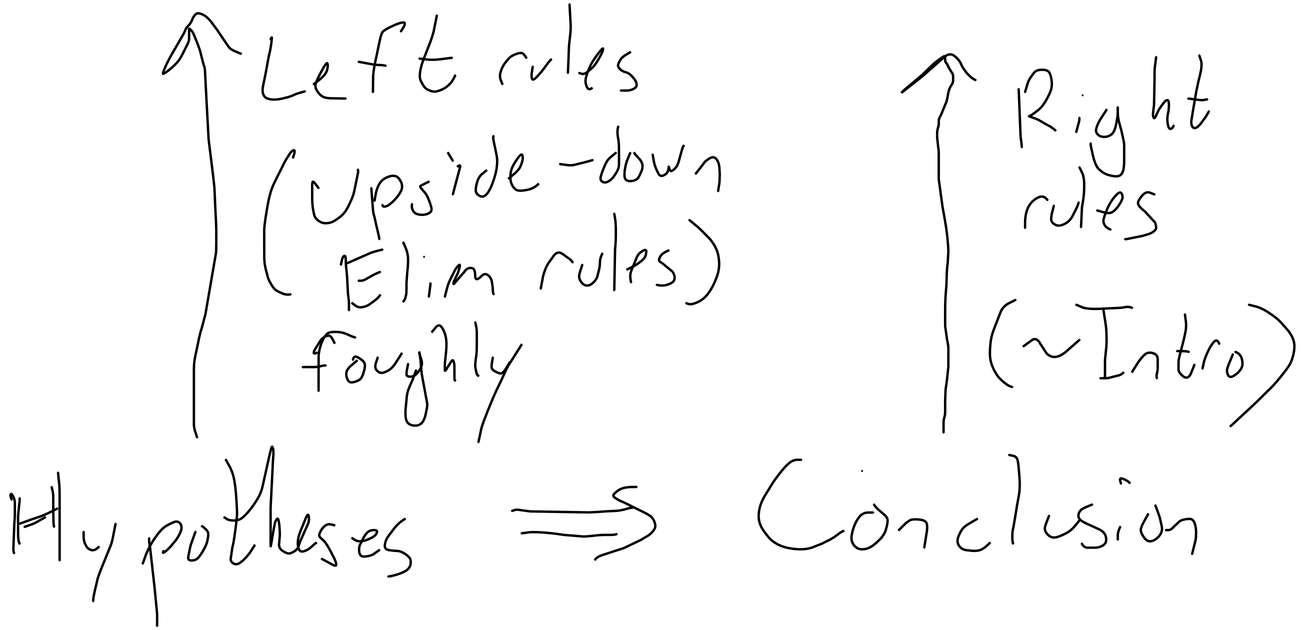
$$A_1 \downarrow \quad \ldots \quad A_n \downarrow$$
$$\vdots$$
$$C \uparrow$$

- Sequents represent this state

- $\underbrace{\underbrace{A_1\ \text{left}, \ldots, A_n\ \text{left}}_{\text{antecedents}} \Longrightarrow \underbrace{C\ \text{right}}_{\text{succedent}}}_{\text{sequent}}$

- Usually omit "left" and "right"


Hypotheses
Elim
Intro
Conclusion


Left rules
(upside-down
Elim rules)
roughly
Right
rules
(~Intro)
Hypotheses ⟹ Conclusion

Conjunction

$$\frac{A\uparrow \qquad B\uparrow}{A\wedge B\uparrow}\wedge I \longrightarrow \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\wedge B}\wedge R$$

$$\frac{A\wedge B\downarrow}{A\downarrow}\wedge E_1 \longrightarrow \frac{\Gamma, A\wedge B, A \Rightarrow C}{\Gamma, A\wedge B \Rightarrow C}\wedge L_1$$

$$\frac{\Gamma, A\wedge B, A, B \Rightarrow C}{\Gamma, A\wedge B \Rightarrow C}\wedge L?$$

$$\frac{\Gamma, A\wedge B, B \Rightarrow C}{\Gamma, A\wedge B \Rightarrow C}\wedge L_2$$

Implication

$$\dfrac{\dfrac{}{A\downarrow}u \\ \vdots \\ B\uparrow}{A \supset B\uparrow}\supset I^u \longrightarrow \dfrac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\supset R$$

$$\left[\dfrac{\Gamma, u{:}A \Longrightarrow B}{}\supset R^u\right]$$

$$\dfrac{A \supset B\downarrow \quad A\uparrow}{B\downarrow}\supset E \longrightarrow \dfrac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\supset L$$

$$\left[\dfrac{\Gamma, A \supset B \Longrightarrow A}{\Gamma, A \supset B \Longrightarrow B}\supset L?\right]$$

Disjunction

$$\frac{A\uparrow}{A\vee B\uparrow}\vee I_1 \longrightarrow \frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A\vee B}\vee R_1$$

$$\frac{A\vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}^{u}\\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{v}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\vee E^{u,v} \longrightarrow \frac{\Gamma, A\vee B, A \Longrightarrow C \quad \Gamma, A\vee B, B \Longrightarrow C}{\Gamma, A\vee B \Longrightarrow C}\vee L$$

Truth/Falsity

$$\frac{}{T\uparrow}\,TI \longrightarrow \frac{}{\Gamma \Rightarrow T}\,TR \qquad \text{no } TL$$

$$\frac{F\downarrow}{C\uparrow}\,FE \longrightarrow \frac{}{\Gamma, F \Rightarrow C}\,FL$$

no FR

Identity

$$\frac{P\downarrow}{P\uparrow}\,{\downarrow\uparrow} \longrightarrow \frac{}{\Gamma, P \Longrightarrow P}\,id_P$$

- For proofs today, we will use

$$\frac{}{\Gamma, A \Longrightarrow A}\,id_A \quad \text{for arbitrary } A$$

- This rule needs to be proven/derived.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vee B, A \Rightarrow A}\,id}{A \vee B, A \Rightarrow B \vee A}\,\vee R_2 \qquad \dfrac{\dfrac{}{A \vee B, B \Rightarrow B}\,id}{A \vee B, B \Rightarrow B \vee A}\,\vee R_1}{A \vee B \Rightarrow B \vee A}\,\vee L}{\Rightarrow (A \vee B) \supset (B \vee A)}\,\supset R$$

Verifications to Sequents

- Goal $A\uparrow$ iff $\Longrightarrow A$
- Proof by induction on the derivations of $A\uparrow$ or $\Longrightarrow A$.
- I.H.

$$\begin{array}{c} A_1\downarrow, \ldots, A_n\downarrow \\ \vdots\ \mathcal{D} \\ A\uparrow \end{array} \quad \text{iff} \quad \begin{array}{c} \mathcal{E} \\ A_1, \ldots, A_n \Longrightarrow A \end{array}$$

Case on last rule used in $\mathcal{D}/\mathcal{E}$

Case $\mathcal{D}$ ends in $\supset I$

$$\begin{array}{c} \overline{A_1\downarrow}, \ldots, \overline{A_n\downarrow} \qquad \overline{A\downarrow}^{\,u} \\ \vdots\ \mathcal{D}_1 \\ B\uparrow \end{array}$$

$$\mathcal{D} = \dfrac{B\uparrow}{A\supset B\uparrow}\supset I^u \quad \Longrightarrow \quad \dfrac{\overset{\text{i.h.}(\mathcal{D}_1)}{A_1, \ldots, A_n, A \Longrightarrow B}}{A_1, \ldots, A_n \Longrightarrow A\supset B}\supset R$$

Case $\mathcal{D}$ ends in $\vee E$

$$A_1\downarrow, \ldots, A_n\downarrow \qquad \begin{array}{c} \overline{A\downarrow}^{\,u} \\ \vdots\ \mathcal{D}_1 \end{array} \qquad \begin{array}{c} \overline{B\downarrow}^{\,v} \\ \vdots\ \mathcal{D}_2 \end{array}$$

$$\mathcal{D} = \dfrac{A\vee B\downarrow \qquad C\uparrow \qquad C\uparrow}{C\uparrow}\vee E^{u,v} \quad \Longrightarrow \quad \dfrac{\overset{\text{i.h.}(\mathcal{D}_1)}{A_1, \ldots, A_n, A \Longrightarrow C} \qquad \overset{\text{i.h.}(\mathcal{D}_2)}{A_1, \ldots, A_n, B \Longrightarrow C}}{A_1, \ldots \underline{A\vee B} \ldots, A_n \Longrightarrow C}\vee L$$

Case $\mathcal{E}$ ends in $\wedge R$

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma \Rightarrow A} \qquad \overset{\mathcal{E}_2}{\Gamma \Rightarrow B}}{\underset{A_1, \ldots, A_n}{\Gamma} \Rightarrow A \wedge B}\,\wedge R \quad \Longrightarrow \quad \begin{array}{c} A_1\downarrow, \ldots, A_n\downarrow \\ \dfrac{\overset{i.h.(\mathcal{E}_1)}{A\uparrow} \qquad \overset{i.h.(\mathcal{E}_2)}{B\uparrow}}{A \wedge B \uparrow}\,\wedge I \end{array}$$

Case $\mathcal{E}$ ends in FL

$$\mathcal{E} = \dfrac{}{\underbrace{\Gamma, F}_{A_1, \ldots, A_n} \Rightarrow C}\,FL \quad \Longrightarrow \quad \begin{array}{c} A_1\downarrow, \ldots F\downarrow \ldots, A_n\downarrow \\ \\ \dfrac{F\downarrow}{C\uparrow}\,FE \end{array}$$

Revisiting Soundness/completeness

- Soundness: If $A\uparrow$ and $\begin{array}{c} A\downarrow \\ \vdots \\ C\uparrow \end{array}$ then $C\uparrow$

If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$, then $\Gamma \Longrightarrow C$

- Completeness: $\begin{array}{c} A\downarrow \\ \vdots \\ A\uparrow \end{array}$ forall A

$$\overline{\Gamma, A \Longrightarrow A}\ id_A$$ is admissible for all A

Identity Theorem

$$\frac{\quad}{\Gamma, A \Rightarrow A}\ id_A$$ is admissible

- A rule is <u>derivable</u> if there is some derivation $\mathcal{D}$ of it (from premises to conclusion)

- A rule is <u>admissible</u> if every instance of the rule has some derivation.

$\Gamma, A \Rightarrow A$ is derivable/provable for all A. (and all Γ)

Proof/ By induction on A | i.h.: For all Γ, For all $B < A$, $\Gamma, B \Rightarrow B$

subformula

Case $A = \top$

$$\frac{}{\Gamma, \top \Rightarrow \top}\ \top R$$

Case $A = F$

$$\frac{}{\Gamma, F \Rightarrow F}\ FL$$

Case $A = P$ (atomic)

$$\frac{}{\Gamma, P \Rightarrow P}\ id_P$$

Case $A = B \wedge C$

$$
\dfrac{
\dfrac{\overset{\text{i.h.}(B)}{\Gamma, B \wedge C, B \Longrightarrow B}}{\Gamma, B \wedge C \Longrightarrow B}\,\wedge L_1
\qquad
\dfrac{\overset{\text{i.h.}(C)}{\Gamma, B \wedge C, C \Longrightarrow C}}{\Gamma, B \wedge C \Longrightarrow C}\,\wedge L_2
}{
\Gamma, B \wedge C \Longrightarrow B \wedge C
}\,\wedge R
$$

Case $A = B \supset C$

$$
\dfrac{
\dfrac{
\overset{\text{i.h.}(B)}{\Gamma, B \supset C, B \Longrightarrow B}
\qquad
\overset{\text{i.h.}(C)}{\Gamma, B \supset C, B, C \Longrightarrow C}
}{
\Gamma, B \supset C, B \Longrightarrow C
}\,\supset L
}{
\Gamma, B \supset C \Longrightarrow B \supset C
}\,\supset R
$$

Case $A = B \vee C$

$$\dfrac{\dfrac{\overset{\text{i.h.}(B)}{\Gamma, B \vee C, B \Rightarrow B}}{\Gamma, B \vee C, B \Rightarrow B \vee C}\,\vee R_1 \qquad \dfrac{\overset{\text{i.h.}(C)}{\Gamma, B \vee C, C \Rightarrow C}}{\Gamma, B \vee C, C \Rightarrow B \vee C}\,\vee R_2}{\Gamma, B \vee C \Rightarrow B \vee C}\,\vee L$$

Addendum to Verifications $\longleftrightarrow$ Sequent calc.

Lemma/ If
$$\begin{array}{c} A_1\downarrow, \ldots, A_n\downarrow \\ \vdots\ \mathcal{D} \\ A\downarrow \end{array}$$
and $A_1, \ldots, A_n, A \Longrightarrow C$, then $A_1, \ldots, A_n \Longrightarrow C$.

Proof(ish)/ By induction on $\mathcal{D}$.

Case $\supset E$

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ A \supset B\downarrow\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ A\uparrow\end{array}}{B\downarrow} \longrightarrow$$

$$\dfrac{\dfrac{\dfrac{}{A_1, \ldots, A_n \Longrightarrow A}\text{"i.h.}(\mathcal{D}_2)\text{"}}{A_1, \ldots, A_n, A \supset B \Longrightarrow A}\text{weaken} \qquad \dfrac{A_1, \ldots, A_n, B \Longrightarrow C}{A_1, \ldots, A_n, A\supset B, B \Longrightarrow C}\text{weaken}}{\dfrac{A_1, \ldots, A_n, A \supset B \Longrightarrow C}{A_1, \ldots, A_n \Longrightarrow C}\text{i.h.}(\mathcal{D}_1)}$$

Case $\wedge E_1$ ($\wedge E_2$ is similar)

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ A \wedge B \downarrow\end{matrix}}{A \downarrow} \wedge E_1 \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{A_1, \ldots, A_n, A \Longrightarrow C}{A_1, \ldots, A_n, A \wedge B, A \Longrightarrow C} \text{weaken}}{A_1, \ldots, A_n, A \wedge B \Longrightarrow C} \wedge L_1}{A_1, \ldots, A_n \Longrightarrow C} \text{ i.h.}(\mathcal{D}_1)$$

- what is "weaken"?

$$\dfrac{\Gamma \Longrightarrow C}{\Gamma, A \Longrightarrow C} \text{weaken}$$

- If one can prove C from Γ, the additional hypothesis A can't hurt.

- "i.h. $(\mathcal{D}_2)$" from previous page?

- This is the i.h. for the $\begin{matrix} A_1\downarrow, \ldots, A_n\downarrow \\ \vdots \\ C\Uparrow \end{matrix}$ theorem on page 11. Technically, we should be proving both that theorem and this lemma at the same time / in the same induction.