

## 15-317 Lecture 3: Harmony

- All proof rules can be found on course site, under Logics

- Reuse/non-use of hypotheses

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- How do we tell what rules are right?

- Local soundness ( $\Rightarrow_R$ )

- Local completeness ( $\Rightarrow_E$ )

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- Substitution in hypotheticals

Hypothesis reuse

$$\frac{\frac{A \text{ true}^u}{\vdots} \quad B \text{ true}}{A \supset B \text{ true}} \supset I^u$$

- Can use  $u$   $\infty$  times, e.g.

$$\frac{\frac{T \text{ true}}{T I}}{A \supset T \text{ true}} \supset I^u$$

$$\frac{A \text{ true}^u}{A \supset A \text{ true}} \supset I^u$$

- Can use  $u$  multiple times

$$\frac{\frac{A \text{ true}^u \quad A \text{ true}^u}{A \wedge A \text{ true}} \quad A I}{A \supset (A \wedge A) \text{ true}} \supset I^u$$

Scope

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A true  
⋮  
B true

u

Can use u anywhere  
in this :

B true

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A > B true

$\supset I^u$

Can only use v in this premise

A true  
⋮  
C true

v

B true  
⋮  
C true

w

Only use w in this premise

$\vee E^{v,w}$

A  $\vee$  B true

---

C true

A true  
⋮  
C true

v

A true  
⋮  
C true

w

---

A  $\vee$  A

C

$\vee E^{v,w}$

What can go wrong when defining rules?  
 What rules are correct?

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$$\frac{A \text{ true}}{A \vee B \text{ true}} \vee I_1 \qquad \frac{B \text{ true}}{A \vee B \text{ true}} \vee I_2$$

$$\frac{A \vee B \text{ true}}{A \text{ true}} \vee E_1?$$

$$\frac{\frac{\frac{}{T \text{ true}} TI}{A \vee T \text{ true}} \vee I_2}{A \text{ true}} \vee E_1$$

Elim rule is too strong, in that it allows concluding information we don't get from intro

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

$$\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_1$$

(no  $\wedge E_2$ )

$$\frac{A \wedge B \text{ true}}{A \text{ true} \quad B \text{ true}} \wedge E_1$$

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

Elim rules are too weak,  $A \wedge B$  is not enough to prove itself

# Harmony

- Local soundness
  - Elim rules not too strong; immediately
  - Whenever we intro then, elim a connective, we can get a simpler proof w/o that intro.
  - Elim only gets out what is put in by intro
- Local completeness
  - Elim rules not too weak.
  - $\frac{A}{A}$  can be proven w/o just directly using A.
  - $\frac{A \quad B}{A \quad B}$  can be proven using OE/OI

# Other conditions on rules

- A rule should only deal with one connective
  - Can cause circularity in definitions

# Some notation

- $\left. \begin{array}{l} \mathcal{D} \\ A \text{ true} \end{array} \right\} \mathcal{D} \text{ is a derivation of } A \text{ true}$
- $\left. \begin{array}{l} A \text{ true} \\ \vdots \\ E \\ B \text{ true} \end{array} \right\} E \text{ is a hyp. derivation of } \begin{array}{l} A \text{ true} \\ \vdots \\ B \text{ true} \end{array}$
- $\mathcal{D} \xrightarrow{\text{reduces to}} \mathcal{D}'$  (as in local soundness)
- $A \text{ true} \xRightarrow{R} A \text{ true}$
- $\mathcal{D} \xrightarrow{\text{expands to}} \mathcal{D}'$
- $A \text{ true} \xRightarrow{E} A \text{ true}$

# Conjunction

## Local Soundness

$$\frac{\frac{\mathcal{D} \quad \mathcal{E}}{A \text{ true} \quad B \text{ true}} \wedge I}{\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_1} \Rightarrow_R \mathcal{D} \quad A \text{ true}$$

$$\frac{\frac{\mathcal{D} \quad \mathcal{E}}{A \text{ true} \quad B \text{ true}} \wedge I}{\frac{A \wedge B \text{ true}}{B \text{ true}} \wedge E_2} \Rightarrow_R \mathcal{E} \quad B \text{ true}$$

## Local Completeness

$$\mathcal{D} \quad A \wedge B \text{ true} \Rightarrow_E \frac{\frac{\mathcal{D} \quad \mathcal{B}}{A \wedge B \text{ true}} \wedge E_1 \quad \frac{\mathcal{B}}{A \wedge B \text{ true}} \wedge E_2}{\frac{A \text{ true} \quad B \text{ true}}{\wedge I}}$$

# Substitution for hyp. judgements/derivations

- If  $\frac{A \text{ true}}{\vdots \mathcal{E}}$  and  $\mathcal{D}$  then  
 $B \text{ true}$   $A \text{ true},$

we can form  $\frac{\mathcal{D}}{A \text{ true}}^u$  or  $[\mathcal{D}/u] \mathcal{E}$ ,  
 $\vdots \mathcal{E}$   
 $B \text{ true}$

substituting  $\mathcal{D}$  for  $u$  in  $\mathcal{E}$ .

$$\mathcal{E} = \frac{\frac{A \wedge B \text{ true}}{}^u}{A \text{ true}} \wedge E_1$$

$$\mathcal{D} = \frac{\mathcal{D}_1 \quad \mathcal{D}_2}{A \text{ true} \quad B \text{ true}} \wedge I$$

↓

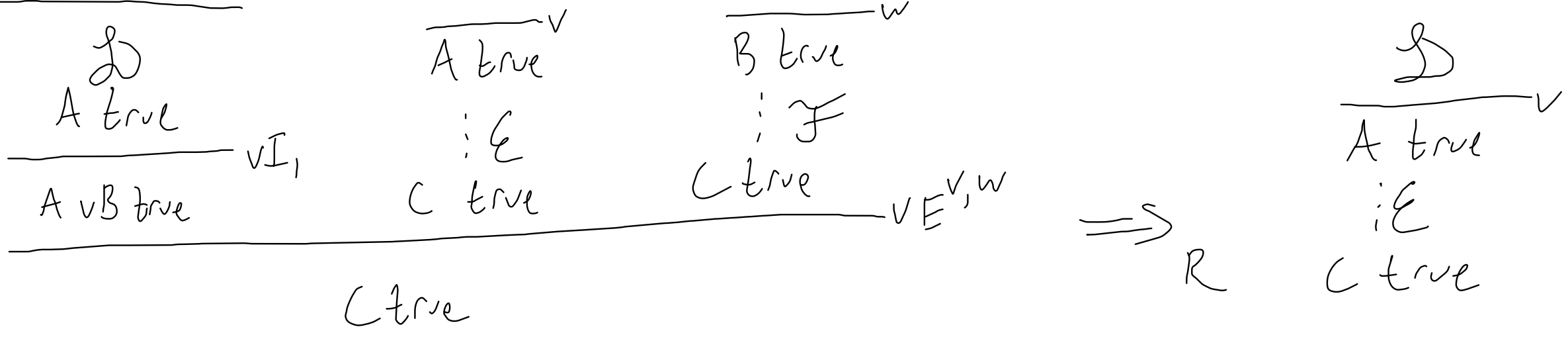
$$[\mathcal{D}/u] \mathcal{E} =$$

$$\frac{\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{A \text{ true} \quad B \text{ true}} \wedge I}{A \wedge B \text{ true}} \wedge E_1$$

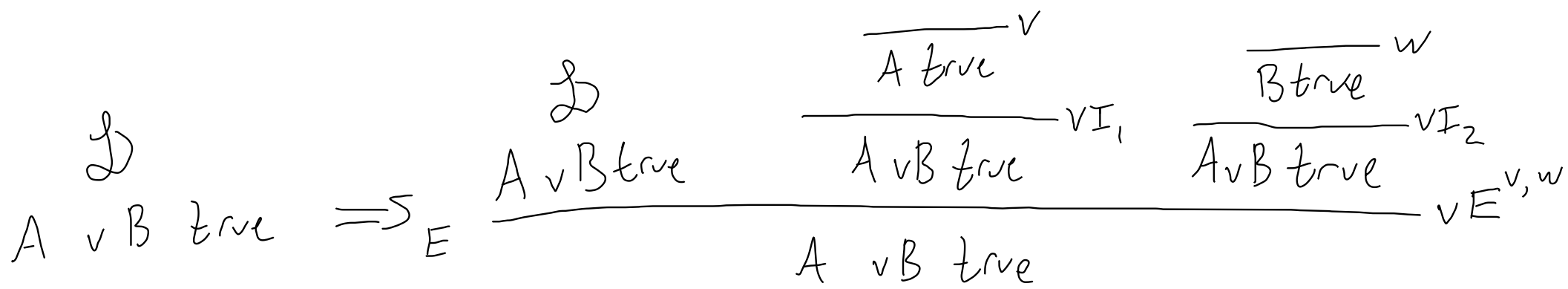




Disjunction



- other reduction is similar



# Truth/Falsity

Local soundness:

- No way to intro then elim!
- Local soundness holds vacuously

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Local completeness?

$$\begin{array}{l} \Im \\ T \text{ true} \Rightarrow E \quad \frac{\quad}{T \text{ true}} \text{TI} \\ \\ \Im \\ F \text{ true} \Rightarrow E \quad \frac{\begin{array}{c} \Im \\ F \text{ true} \end{array}}{F \text{ true}} \text{FE} \end{array}$$

$$\frac{F \text{ true}}{C \text{ true}} \text{FE}$$