15-317 Lecture 3: Harmony

- All proof rules can be found on course site, under Logics
- Reuse/non-use of hypotheses

- How do we tell what rules are right?
- Local soundness ($\Rightarrow_R$)
- Local completeness ($\Rightarrow_E$)

- Substitution in hypotheticals

Hypothesis reuse

$$\cfrac{\begin{array}{c}\overline{A\ true}^{\,u}\\ \vdots \\ B\ true\end{array}}{A \supset B\ true}\supset I^{u}$$

- Can use u $\emptyset$ times, e.g.

$$\cfrac{\cfrac{}{\top\ true}\,\top I}{A \supset \top\ true}\supset I^{u}$$

$$\cfrac{\overline{A\ true}^{\,u}}{A \supset A\ true}\supset I^{u}$$

- Can use u multiple times

$$\cfrac{\cfrac{\overline{A\ true}^{\,u} \quad \overline{A\ true}^{\,u}}{A \wedge A\ true}\wedge I}{A \supset (A \wedge A)\ true}\supset I^{u}$$

Scope

$$\dfrac{\begin{array}{c}\overline{A\ \text{true}}^{\,u}\\ \vdots \\ B\ \text{true}\end{array}}{A \supset B\ \text{true}}\supset I^{u}$$

Can use u anywhere in this $\vdots$

Can only use v in this premise

Only use w in this premise

$$\dfrac{A \vee B\ \text{true} \qquad \begin{array}{c}\overline{A\ \text{true}}^{\,v}\\ \vdots \\ C\ \text{true}\end{array} \qquad \begin{array}{c}\overline{B\ \text{true}}^{\,w}\\ \vdots \\ C\ \text{true}\end{array}}{C\ \text{true}}\vee E^{v,w}$$

$$\dfrac{A \vee A \qquad \begin{array}{c}\overline{A}^{\,v}\\ \vdots \\ C\end{array} \qquad \begin{array}{c}\overline{A}^{\,w}\\ \vdots \\ C\end{array}}{C}\vee E^{v,w}$$

What can go wrong when defining rules?
what rules are correct?

$$\frac{A\ \mathrm{true}}{A \lor B\ \mathrm{true}}\ \lor I_1 \qquad \frac{B\ \mathrm{true}}{A \lor B\ \mathrm{true}}\ \lor I_2$$

$$\frac{A \lor B\ \mathrm{true}}{A\ \mathrm{true}}\ \lor E_1 ?$$

Elim rule is too strong, in that it allows concluding information we don't get from intro

$$\cfrac{\cfrac{\cfrac{}{\top\ \mathrm{true}}\ \top I}{A \lor \top\ \mathrm{true}}\ \lor I_2}{A\ \mathrm{true}}\ \lor E_1$$

$$\frac{A\ \mathrm{true} \qquad B\ \mathrm{true}}{A \land B\ \mathrm{true}}\ \land I \qquad \frac{A \land B\ \mathrm{true}}{A\ \mathrm{true}}\ \land E_1$$

(no $\land E_2$)

$$\begin{array}{c} A \land B\ \mathrm{true} \\ \vdots \end{array}$$

$$\cfrac{\cfrac{A \land B\ \mathrm{true}}{A\ \mathrm{true}}\ \land E_1 \qquad B\ \mathrm{true}}{A \land B\ \mathrm{true}}\ \land I$$

Elim rules are too weak, $A \land B$ is not enough to prove itself

Harmony

- Local soundness
  - Elim rules not too strong
  - Whenever we intro then immediately elim a connective, we can get a simpler proof w/o that intro.
  - Elim only gets out what is put in by intro
- Local completeness
  - Elim rules not too weak.
  - $\begin{matrix} A \\ \vdots \\ A \end{matrix}$ can be proven w/o just directly using A.
  - $\begin{matrix} A \circ B \\ \vdots \\ A \circ B \end{matrix}$ can be proven using $\circ E / \circ I$

Other conditions on rules

- A rule should only deal with one connective
  - Can cause circularity in definitions

Some notation

- $\left.\begin{matrix} \mathcal{D} \\ A \text{ true} \end{matrix}\right\}$ $\mathcal{D}$ is a derivation of A true
- $\left.\begin{matrix} A \text{ true} \\ \vdots\ \mathcal{E} \\ B \text{ true} \end{matrix}\right\}$ $\mathcal{E}$ is a hyp. derivation of $\begin{matrix} A \text{ true} \\ \vdots \\ B \text{ true} \end{matrix}$
- $\begin{matrix} \mathcal{D} \\ A \text{ true} \end{matrix} \overset{\text{reduces to}}{\Longrightarrow_R} \begin{matrix} \mathcal{D}' \\ A \text{ true} \end{matrix}$ (as in local soundness)
- $\begin{matrix} \mathcal{D} \\ A \text{ true} \end{matrix} \overset{\text{expands to}}{\Longrightarrow_E} \begin{matrix} \mathcal{D}' \\ A \text{ true} \end{matrix}$

Conjunction

Local Soundness

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A\ true} \qquad \overset{\mathcal{E}}{B\ true}}{A \land B\ true}\land I}{A\ true}\land E_1 \quad \Longrightarrow_R \quad \overset{\mathcal{D}}{A\ true}$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A\ true} \qquad \overset{\mathcal{E}}{B\ true}}{A \land B\ true}\land I}{B\ true}\land E_2 \quad \Longrightarrow_R \quad \overset{\mathcal{E}}{B\ true}$$

Local Completeness

$$\overset{\mathcal{D}}{A \land B\ true} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\overset{\mathcal{D}}{A \land B\ true}}{A\ true}\land E_1 \qquad \dfrac{\overset{\mathcal{D}}{A \land B\ true}}{B\ true}\land E_2}{A \land B\ true}\land I$$

Substitution for hyp. judgements/derivations

- If $\begin{array}{c} \overline{A\ true}^{\,u} \\ \vdots\ \mathcal{E} \\ B\ true \end{array}$ and $\begin{array}{c} \mathcal{D} \\ A\ true \end{array}$, then

we can form $\begin{array}{c} \mathcal{D} \\ \overline{A\ true}^{\,u} \\ \vdots\ \mathcal{E} \\ B\ true \end{array}$ or $[\mathcal{D}/u]\mathcal{E}$,

substituting $\mathcal{D}$ for u in $\mathcal{E}$.

$$\mathcal{E} = \dfrac{\overline{A \wedge B\ true}^{\,u}}{A\ true}\wedge E_1 \qquad \mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ A\ true\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ B\ true\end{array}}{A \wedge B\ true}\wedge I$$

$$[\mathcal{D}/u]\mathcal{E} =$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1 \\ A\ true\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ B\ true\end{array}}{A \wedge B\ true}\wedge I}{A\ true}\wedge E_1$$

Implication

Local soundness

$$\cfrac{\cfrac{\begin{array}{c}\cfrac{}{A\ true}\,u\\ \vdots\,\mathcal{E}\\ B\ true\end{array}}{A \supset B\ true}\supset I^{u} \qquad \begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{B\ true}\supset E$$

$$\Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \overline{A\ true}\,u\\ \vdots\,\mathcal{E}\\ B\ true\end{array}$$

Local completeness

$$\begin{array}{c}\mathcal{D}\\ A \supset B\ true\end{array} \Longrightarrow_E \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ A \supset B\ true\end{array} \qquad \cfrac{}{A\ true}\,u}{B\ true}\supset E}{A \supset B\ true}\supset I^{u}$$

Disjunction

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{A \lor B\ true}\lor I_1 \qquad \begin{array}{c}\overline{A\ true}^{\,v}\\ \vdots\ \mathcal{E}\\ C\ true\end{array} \qquad \begin{array}{c}\overline{B\ true}^{\,w}\\ \vdots\ \mathcal{F}\\ C\ true\end{array}}{C\ true}\lor E^{v,w} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \overline{A\ true}^{\,v}\\ \vdots\ \mathcal{E}\\ C\ true\end{array}$$

- other reduction is similar

$$\begin{array}{c}\mathcal{D}\\ A \lor B\ true\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \lor B\ true\end{array} \qquad \dfrac{\overline{A\ true}^{\,v}}{A \lor B\ true}\lor I_1 \qquad \dfrac{\overline{B\ true}^{\,w}}{A \lor B\ true}\lor I_2}{A \lor B\ true}\lor E^{v,w}$$

Truth/Falsity

Local soundness:

- No way to intro then elim!
- Local soundness holds vacuously

Local completeness?

$$\begin{array}{c}\mathcal{D}\\ \top\ true\end{array} \Longrightarrow_E \frac{}{\top\ true}\,\top I$$

$$\begin{array}{c}\mathcal{D}\\ F\ true\end{array} \Longrightarrow_E \frac{\begin{array}{c}\mathcal{D}\\ F\ true\end{array}}{F\ true}\,FE$$

$$\frac{F\ true}{C\ true}\,FE$$