15-317 Lecture 2

- Homework 1 out today, due 27 Jan, 23:59.

- Hypothetical Judgements (again/more)
- More connectives
  - Disjunction/Or ($\vee$)
  - Falsity (F or $\perp$)

- Notational definition and derived rules
  - Equivalence ($\equiv$)
  - Negation ($\neg$)

- How do we justify proof rules?
  - What can go wrong?

Hypothetical Judgements

- A judgement with assumptions or hypotheses, written

$$\begin{array}{c} J_1 \ \dots \ J_n \\ \vdots \\ \hline J \end{array}$$

"If $J_1, \dots, J_n$ all hold, so does J.

- Substitution: given

$$\begin{array}{c} J_1, \ \dots \ J_n \\ \vdots \\ \hline J \end{array} \quad \text{and} \quad \begin{array}{c} K_1, \dots, K_m \\ \vdots \\ \hline J_i \end{array}$$

we can form

$$\begin{array}{c} J_1, \dots, J_{i-1}, K_1, \dots, K_m, J_{i+1}, \dots J_n \\ \vdots \\ \hline J \end{array}$$

Disjunction/Or ($\vee$)

- what should $\vee I$ look like?

$$\frac{A\ true}{A \vee B\ true}\ \vee I_1$$

$$\frac{B\ true}{A \vee B\ true}\ \vee I_2$$

- what about $\vee E$?

Not yet defined!

$$\cancel{\frac{A \vee B\ true \qquad \overbrace{B\ false}}{A\ true}\ \vee E?}$$

$$\frac{A \vee B\ true \qquad \begin{array}{c}\overline{A\ true}^{\,v}\\ \vdots\\ C\ true\end{array} \qquad \begin{array}{c}\overline{B\ true}^{\,w}\\ \vdots\\ C\ true\end{array}}{C\ true}\ \vee E^{v,w}$$

$$\frac{\Gamma \vdash A \vee B\ true \qquad \Gamma, A\ true \vdash C\ true \qquad \Gamma, B\ true \vdash C\ true}{\Gamma \vdash C\ true}\ \vee E$$

- How do we tell if a prop. A is true/provable?
- Find a proof.
- What if it's not provable?

$$\dfrac{\dfrac{\overline{A \lor B\ \mathrm{true}}^{\,u} \qquad \overline{A\ \mathrm{true}}^{\,v} \qquad \overline{A\ \mathrm{true}}^{\,v}}{A\ \mathrm{true}}\ \lor E^{v,w}}{(A \lor B) \supset A\ \mathrm{true}}\ \supset I^{u}$$

out of scope

$$\dfrac{\overline{A \lor B\ \mathrm{true}}^{\,u} \qquad \begin{matrix}\overline{A\ \mathrm{true}}^{\,v} \\ \vdots \\ A\ \mathrm{true}\end{matrix} \qquad \begin{matrix}\overline{B\ \mathrm{true}}^{\,w} \\ \vdots \\ A\ \mathrm{true}\end{matrix}}{A\ \mathrm{true}}$$

$$\dfrac{\overline{A \lor B\ \mathrm{true}}^{\,u} \qquad \begin{matrix}\overline{A\ \mathrm{true}}^{\,v} \\ \vdots \\ A\ \mathrm{true}\end{matrix}}{A\ \mathrm{true}}$$

Hyp. Deriv.

- A proof of

$$\begin{array}{c} J_1 \quad \dots \quad J_n \\ \vdots \\ \hline J \end{array}$$

is a hyp. deriv.

e.g.

$$\begin{array}{c} A\ true \\ \vdots \\ A\ true \end{array}$$

can be proven as $A\ true$

e.g.

$$\begin{array}{c} A \wedge B\ true \\ \vdots \\ A\ true \end{array}$$

can be proven as

$$\frac{A \wedge B\ true}{A\ true} \wedge E_1$$

e.g.

$$\begin{array}{c} (A \wedge B) \wedge C\ true \\ \vdots \\ A\ true \end{array}$$

can be proven as

$$\cfrac{\cfrac{(A \wedge B) \wedge C\ true}{A \wedge B\ true} \wedge E_1}{A\ true} \wedge E_1$$

Disjunction Examples

$$\dfrac{\dfrac{\overline{A\lor A\ \text{true}}^{\,u} \qquad \begin{array}{c}\overline{A\ \text{true}}^{\,v}\\ \vdots\\ A\ \text{true}\end{array} \qquad \begin{array}{c}\overline{A\ \text{true}}^{\,w\ (\text{not } v)}\\ \vdots\\ A\ \text{true}\end{array}}{A\ \text{true}}\lor E^{v,w}}{(A\lor A)\supset A\ \text{true}}\supset I^{u}$$

$$\dfrac{\begin{array}{c}\overline{A\supset B\ \text{true}}^{\,u}\\ \vdots\\ A\ \text{true}\end{array}}{(A\supset B)\supset A\ \text{true}}\supset I^{u}$$

stuck

$$\dfrac{\quad}{A\lor(A\supset B)\ \text{true}}\lor E$$

$$\dfrac{\dfrac{\overline{B\lor A\ \text{true}}^{\,u} \qquad \dfrac{\overline{B\ \text{true}}^{\,v}}{A\lor B\ \text{true}}\lor I_2 \qquad \dfrac{\overline{A\ \text{true}}^{\,w}}{A\lor B\ \text{true}}\lor I_1}{A\lor B\ \text{true}}\lor E^{v,w}}{(B\lor A)\supset(A\lor B)\ \text{true}}\supset I^{u}$$

$$\dfrac{\dfrac{}{\top\ \text{true}}\ \top I}{\top \vee A\ \text{true}}\ \vee I_1$$

Falsity (F)

- FI?

$$\cancel{\dfrac{\cdots}{F\ true}\ FI?}$$

No intro. rule

- FE? $\left[\begin{array}{l} F \text{ is } \emptyset\text{-ary } \vee \\ T \text{ is } \emptyset\text{-ary } \wedge \end{array}\right.$

$$\frac{F\ true}{C\ true}\ FE$$

compare

$$\dfrac{A \vee B\ true \quad \begin{array}{c}\overline{A\ true}^{\,v} \\ \vdots \\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}^{\,w} \\ \vdots \\ C\ true\end{array}}{C\ true}\ \vee E^{v,w}$$

n-ary rules

$$\frac{\bigwedge_{i=1}^{n} A_i \ \text{true}}{A_j \ \text{true}} \wedge E_j$$

$$\frac{A_1 \ \text{true} \quad \dots \quad A_n \ \text{true}}{\bigwedge_{i=1}^{n} A_i \ \text{true}} \wedge I$$

$$\frac{\bigvee_{i=1}^{n} A_i \ \text{true} \qquad \begin{array}{c} \overline{A_i \ \text{true}}^{\,a_i} \\ \vdots \\ C \ \text{true} \end{array} \ \text{for } i=1 \text{ to } n}{C \ \text{true}} \vee E$$

$$\frac{A_j \ \text{true}}{\bigvee_{i=1}^{n} A_i \ \text{true}} \vee I_j$$

Notational Definition

- Define new connectives from old ones.
- e.g. logical equivalence $\equiv$:

$$A \equiv B :\text{iff } (A \supset B) \wedge (B \supset A).$$

- e.g. negation $\neg$

$$\neg A :\overset{\text{iff}}{=} \underline{A \supset F}$$

Derived rules

$$\dfrac{\begin{array}{c}\overline{A\ true}^{\,u} \\ \vdots \\ F\ true\end{array}}{\neg A\ true}\ \neg I$$

$$\dfrac{\begin{array}{c}\overline{A\ true}^{\,u} \\ \vdots \\ F\ true\end{array}}{A \supset F\ true}\ \supset I^{u}$$

$$\frac{\neg A \text{ true} \qquad A \text{ true}}{F \text{ true}}\ \neg E$$

$$\frac{A \supset F \text{ true} \qquad A \text{ true}}{F \text{ true}}\ \supset E$$

$$\dfrac{\dfrac{\begin{array}{c}\overline{A \text{ true}}^{\,u} \\ \vdots \\ B \text{ true}\end{array}}{A \supset B \text{ true}}\ \supset I^{u} \qquad \dfrac{\begin{array}{c}\overline{B \text{ true}}^{\,v} \\ \vdots \\ A \text{ true}\end{array}}{B \supset A \text{ true}}\ \supset I^{v}}{(A \supset B) \wedge (B \supset A) \text{ true}}\ \wedge I$$

$$\dfrac{\begin{array}{c}\overline{A \text{ true}}^{\,u} \\ \vdots \\ B \text{ true}\end{array} \qquad \begin{array}{c}\overline{B \text{ true}}^{\,v} \\ \vdots \\ A \text{ true}\end{array}}{A \equiv B \text{ true}}\ \equiv I^{u,v}$$

$$\dfrac{\dfrac{(A \supset B) \wedge (B \supset A)\ true}{A \supset B\ true}\ \wedge E_1 \qquad A\ true}{B\ true}\ \supset E$$

$$\dfrac{A \equiv B\ true \qquad A\ true}{B\ true}\ \equiv E_1$$

Derived rules for $\equiv$ elimination

$$\dfrac{\dfrac{(A \supset B) \wedge (B \supset A)\ true}{B \supset A\ true}\ \wedge E_2 \qquad B\ true}{A\ true}\ \supset E$$

$$\dfrac{A \equiv B\ true \qquad B\ true}{A\ true}\ \equiv E_2$$