Welcome to 15-317 (Constructive Logic)

- Logistics
- Course outline
- Course goals

- Why constructive?

- Judgements
- Propositions (briefly)
- Proof Rules
- Conjunction, truth, implication

Course Outline

I : Proofs as Evidence of Truth

II : Proofs as Programs

III : Proof Search as Computation

IV : Other Logics

Goals

- Understanding Constructive Logic
- Apply Constructive Logic to CS

Constructivity

- Proofs give <u>effective</u> algorithms

Example/ There exist a, b irrational s.t. a^b is rational.

Consider $\sqrt{2}^{\sqrt{2}}$. <u>Is this rational?</u>

Case 1: Yes — take $a = b = \sqrt{2}$.

Case 2: No — take $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$.

Then $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational

If $\sqrt{2}^{\sqrt{2}}$ is rational or $\sqrt{2}^{\sqrt{2}}$ is irrational, then ....

- $\exists x.\ A(x)$ must have a witness — some specific x and a proof of $A(x)$
- A proof of $A \vee B$ is either a proof of A or of B
- $A \vee \neg A$ is not necessarily provable.
- A proof of $A \supset B$ is a function from proofs of A to proofs of B.

Example/ $\underbrace{\exists x\, (A(x) \supset \forall y.\, A(y))}$ $(: \underbrace{F \ \exists x.\, T})$

Proof (Classical)/

[If $\forall y\, A(y)$, then choose any y to be the desired x.

- [Otherwise $\exists x.\, \neg A(x)$, and this is desired x.

$\underbrace{\neg \forall y.\, A(y)}$

Judgements

- "Something we can know"
- "Something we can prove"
- Similar, but not the same as propositions.

- Separate object language (Formal objects we study, Propositions) from metalanguage

(how we talk about objects)

Main Judgement

A true or "A is true".

Other examples:

- A false
- A valid
- A true at time t
- $M : \tau$
- A prop

Proofs

- How do we define what a proposition means?

- Classically — when is the prop. true? (Truth tables)

- Constructively — the meaning of a proposition is defined by its set of proofs*

- What does $\underbrace{A \wedge B}$ mean?

Proof rules:

premises

$$\frac{A \text{ true} \qquad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

rule name

Conclusion

In general:

$$\frac{J_1 \quad \cdots \quad J_n}{J} [\text{name}]$$

$$\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_1 \qquad \frac{A \wedge B \text{ true}}{B \text{ true}} \wedge E_2$$

Truth, written $\top$

$$\frac{}{\top \text{ true}} \top I \qquad \text{No } \top E \text{ rule}$$

$$\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_1$$

Hypothetical Judgement

$$\begin{array}{c} J_1 \quad \ldots \quad J_n \\ \vdots \\ J \end{array}$$

"If $J_1, \ldots, J_n$ all hold, then so does J" or

"We can prove J from hypotheses $J_1, \ldots, J_n$."

Also written

$$\underbrace{J_1, \ldots, J_n \vdash J}$$

Implication

$$\dfrac{\begin{array}{c}\overline{A\ \text{true}}^{\,u} \\ \vdots \\ B\ \text{true}\end{array}}{(A \supset B)\ \text{true}} \supset I^{u}$$

$$\dfrac{\begin{array}{c}A\ \text{true} \\ \vdots \\ B\ \text{true}\end{array}}{A \supset B\ \text{true}}$$

Not valid

$$\dfrac{A\ \text{true} \qquad (A \supset B)\ \text{true}}{B\ \text{true}} \supset E$$

$$\cfrac{\cfrac{\cfrac{}{A\ true}\,u \qquad \cfrac{}{A\ true}\,u}{A \wedge A\ true}\,\wedge I}{(A \supset (A \wedge A))\ true}\,\supset I^{u}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A\ true}\,u \qquad \cfrac{}{B\ true}\,v}{A \wedge B\ true}\,\wedge I}{B \supset (A \wedge B)\ true}\,\supset I^{v}}{A \supset (B \supset (A \wedge B))\ true}\,\supset I^{u}$$