Lecture Notes on Focusing

15-317: Constructive Logic
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1 Introduction

In this lecture we will finally put much of what we have learned on proof theory together, following the slogan *focusing* = *inversion* + *chaining*. Focusing has been developed by Andreoli [And92] using classical linear logic, but it has proved to be a remarkably robust concept (see, for example, Liang and Miller [LM09]). We will follow the formulation of Simmons [Sim14], which includes particularly elegant proofs of the completeness of focusing using structural inductions.

2 Polarization

A key idea behind focusing is to limit nondeterminism by sequencing inferences on connectives that have similar behaviors. One behavior is that of *inversion*, perhaps slightly misnamed. Andreoli calls such connectives *asynchronous*, which expresses that when we see such a connective we can always decompose it. *Synchronous* connectives, by contrast, are those that "may have to wait" until they can be decomposed, but once we have committed to one by *focusing* on it, we can continue to chain inferences on this one propositions and don't need to look elsewhere.

These concepts match perfectly in the sense that a connective that is *asynchronous* when it appears as a succedent will be *synchronous* as an antecedent. Intuitively, this derives from the nature of *harmony* between the right and the left rules as witnessed by cut reduction. An rule inferring and asynchronous proposition carries no information (the premise and conclusion are interderivable and therefore the rules does not gain or loose information), whle a rule inferring a synchronous proposition has to make a choice of some form. This choice is information which is "conveyed" to the asynchronous connective.

If we classify propositions by their behavior as *succedents*, then so-called *negative propositions* are asynchronous or, to say it differently, have invertible right rules. Conversely, *positive propositions* are asynchronous when they appear as *antecedents*, or, to say it differently, have invertible left rules. The so-called *shift operators* go back and forth between positive and negative propositions so that any proposition can be polarized. n

$$
\begin{array}{llcl}
\text{Neg. Props.} & A^-, B^- & ::= & A^+ \supset B^- \mid A^- \wedge B^- \mid \top \mid P^- \mid {\uparrow}A^+ \\
\text{Pos. Props.} & A^+, B^+ & ::= & A^+ \vee B^+ \mid \bot \mid A^+ \wedge B^+ \mid \top \mid P^+ \mid {\downarrow}A^-
\end{array}
$$

A few notes:

Conjunction and truth: Conjunction $A \wedge B$ and truth $\top$ appear as both positive and negative propositions. That's because There are invertible rules for conjunction both in the antecedent and the succedent. Really, it should be seen as an indication that there are two different conjunctions $A^- \wedge^- B^-$ and $A^+ \wedge B^+$ and two different truth constants $\top^-$ and $\top^+$ with different rules that happen to be *logically equivalent* even though they have different intrinsic properties, both from the perspective of proof search and the computational contents of proofs. For example, in a functional language, positive conjunction would correspond to an eager pairs, while negative conjunction corresponds to lazy pairs.

So, if we take proofs seriously as defining the meaning of propositions there should be two conjunctions, which are disambiguated in the polarized presentation of logic.

Atoms: Atoms may be viewed from one perspective as propositional variables, from another as "uninterpreted" propositions which means that only the logical assumptions we make about them imbue them with meaning. Each can be independently assiged an arbitrary polarity, as long as all occurrences of an atom are given the same polarity.

Quantifiers: The universal quantifier is negative since its right rule is invertible, while the existential quantifier is positive. We do not treat them formally to avoid the syntactic complication of introducing terms, parameters, their types, and the relevant typing judgments.

3 Inversion

Inversion decomposes all asynchronous connectives until we reach a sequent where all proposition in the sequent are either atoms or synchronous. In order for inversion to proceed deterministically, first decompose asynchronous connectives in the succedent and then in the antecedent. We use an *ordered context* Ω^+ (as in Lecture 12) consisting of all positive propositions.

Stable succedent	ρ	$::=$	$A^+ \mid P^-$
Stable antecedents	Γ	$::=$	$\cdot \mid \Gamma, A^- \mid \Gamma, P^+$
Right inversion	$\Gamma\,;\,\Omega^+ \xrightarrow{R} A^-$		
Left inversion	$\Gamma\,;\,\Omega^+ \xrightarrow{L} \rho$		
Stable sequent	$\Gamma\,;\,\rho$		

The rules are summarized in Figure 1

4 Chaining

Once inversion has completed, we have to focus on a single proposition, either a positive succedent or a negative antecedents, and then chain together inference on the proposition in focus. In particular, no other propositions are considered, and only one proposition can be in focus. This gives us two new forms of judgments.

Right focus	$\Gamma \longrightarrow [A^+]$
Left focus	$\Gamma, [A^-] \longrightarrow \rho$

The rules can be found in Figure 2. Some remarks:

Atoms: Much of the power of focusing comes from the fact that left focus $[P^-]$ *fails* unless the succedent is also P^-. Dually, right focus $[P^+]$ *fails* unless P^+ is one of the antecedents. Note also that it is not possible to focus on a positive atom in the antecedent or a negative atom in the succedent.

Shifts: In contrast, $\uparrow L$ and $\downarrow R$ just lose focus and return to the appropriate inversion judgment.

$$\frac{\Gamma\,;\Omega^+ \cdot A^+ \xrightarrow{R} B^-}{\Gamma\,;\Omega^+ \xrightarrow{R} A^+ \supset B^-}\ {\supset}R$$

$$\frac{\Gamma\,;\Omega^+ \xrightarrow{R} A^- \quad \Gamma\,;\Omega^+ \xrightarrow{R} B^-}{\Gamma\,;\Omega^+ \xrightarrow{R} A^- \wedge B^-}\ {\wedge}R \qquad \frac{}{\Gamma\,;\Omega^+ \xrightarrow{R} \top}\ {\top}R$$

$$\frac{\Gamma\,;\Omega^+ \xrightarrow{L} P^-}{\Gamma\,;\Omega^+ \xrightarrow{R} P^-}\ pR \qquad \frac{\Gamma\,;\Omega^+ \xrightarrow{L} A^+}{\Gamma\,;\Omega^+ \xrightarrow{R} {\downarrow}A^+}\ {\uparrow}R$$

$$\frac{\Gamma\,;\Omega^+ \cdot A^+ \xrightarrow{L} \rho \quad \Gamma\,;\Omega^+ \cdot B^+ \xrightarrow{L} \rho}{\Gamma\,;\Omega^+ \cdot A^+ \vee B^+ \xrightarrow{L} \rho}\ {\vee}L \qquad \frac{}{\Gamma\,;\Omega^+ \cdot \bot \xrightarrow{L} \rho}\ {\bot}L$$

$$\frac{\Gamma\,;\Omega^+ \cdot A^+ \cdot B^+ \xrightarrow{L} \rho}{\Gamma\,;\Omega^+ \cdot A^+ \wedge B^+ \xrightarrow{L} \rho}\ {\wedge}L \qquad \frac{\Gamma\,;\Omega^+ \xrightarrow{L} \rho}{\Gamma\,;\Omega^+ \cdot \top \xrightarrow{L} \rho}\ {\top}L$$

$$\frac{\Gamma, P^+\,;\Omega^+ \xrightarrow{L} \rho}{\Gamma\,;\Omega^+ \cdot P^+ \xrightarrow{L} \rho}\ pL \qquad \frac{\Gamma, A^-\,;\Omega^+ \xrightarrow{L} \rho}{\Gamma\,;\Omega^+ \cdot {\downarrow}A^- \xrightarrow{L} \rho}\ {\downarrow}L$$

$$\frac{\Gamma \longrightarrow \rho}{\Gamma\,;\cdot \xrightarrow{L} \rho}\ \mathsf{stable}$$

Figure 1: Inversion phase of focusing

$$\frac{\Gamma \longrightarrow [A^+]}{\Gamma \longrightarrow A^+}\ \mathsf{focus}R \qquad \frac{\Gamma, [A^-] \longrightarrow \rho}{\Gamma, A^- \longrightarrow \rho}\ \mathsf{focus}L$$

$$\frac{\Gamma \longrightarrow [A^+]}{\Gamma \longrightarrow [A^+ \vee B^+]}\ \vee R_1 \quad \frac{\Gamma \longrightarrow [B^+]}{\Gamma \longrightarrow [A^+ \vee B^+]}\ \vee R_2 \qquad \text{no right rule for } [\bot]$$

$$\frac{\Gamma \longrightarrow [A^+] \quad \Gamma \longrightarrow [B^+]}{\Gamma \longrightarrow [A^+ \wedge B^+]}\ \wedge R \qquad \frac{}{\Gamma \longrightarrow [\top]}\ \top R$$

$$\frac{}{\Gamma, P^+ \longrightarrow [P^+]}\ \mathsf{id}^+ \qquad \frac{\Gamma\ ;\ \cdot \xrightarrow{R} A^-}{\Gamma \longrightarrow [{\downarrow}A^-]}\ {\downarrow}R$$

$$\frac{\Gamma \longrightarrow [A^+] \quad \Gamma, [B^-] \longrightarrow \rho}{\Gamma, [A^+ \supset B^-] \longrightarrow \rho}\ {\supset}L$$

$$\frac{\Gamma, [A^-] \longrightarrow \rho}{\Gamma, [A^- \wedge B^-] \longrightarrow \rho}\ \wedge L_1 \quad \frac{\Gamma, [B^-] \longrightarrow \rho}{\Gamma, [A^- \wedge B^-] \longrightarrow \rho}\ \wedge L_2 \qquad \text{no left rule for } [\top]$$

$$\frac{}{\Gamma, [P^-] \longrightarrow P^-}\ \mathsf{id}^- \qquad \frac{\Gamma\ ;\ A^+ \xrightarrow{L} \rho}{\Gamma, [{\uparrow}A^+] \longrightarrow \rho}\ {\uparrow}L$$

Figure 2: Chaining phase of focusing

5 Deriving Rules, Revisited

In this more general setting when compared to chaining, deriving inference rules is slightly more complex: one the chaining phase completes, we have to complete the subsequent inversion phase until we arrive once again at stable sequents. We show a simple example, for atoms a, b, and c.

$$a \wedge (a \supset (b \vee c)) \wedge (b \supset c) \supset c$$

First, we polarize the atoms. It looks as if a should be naturally positive (occurs only on the left-hand side of an implication or conjunction), which b and c are ambiguous. Let's make b positive and c negative. Then we polarize by inserting the minimal number of shifts.

$$a^+ \wedge {\downarrow}(a^+ \supset {\uparrow}(b^+ \vee {\downarrow}c^-)) \wedge (b^+ \supset c^-) \supset c^-$$

Overall, we have a negative proposition we start with

$$\cdot\,;\,\cdot \xrightarrow{R} a^+ \wedge {\downarrow}(a^+ \supset {\uparrow}(b^+ \vee {\downarrow}c^-)) \wedge (b^+ \supset c^-) \supset c^-$$

and apply inversion until we reach a stable sequent, namely

$$a^+, a^+ \supset {\uparrow}(b^+ \vee {\downarrow}c^-), b^+ \supset c^- \longrightarrow c^-$$

We can only focus on the second and third antecedent. We derive:

$$\dfrac{\dfrac{a^+ \in \Gamma}{\Gamma \longrightarrow [a^+]}\,\mathsf{id}^+R \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma\,;\, b^+ \vee {\downarrow}c^- \xrightarrow{L} \rho\end{array}}{\Gamma, [{\uparrow}(b^+ \vee {\downarrow}c^-)] \longrightarrow \rho}\,{\uparrow}L}{\Gamma, [a^+ \supset {\uparrow}(b^+ \vee {\downarrow}c^-)] \longrightarrow \rho}\,{\supset}L$$

We see that when we lost focus due to the shift we switched over to a left inversion phase which we now complete.

$$\dfrac{\dfrac{a^+ \in \Gamma}{\Gamma \longrightarrow [a^+]}\,\mathsf{id}^+R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, b^+ \longrightarrow \rho}{\Gamma, b^+\,;\,\cdot \xrightarrow{L} \rho}\,\mathsf{stable}}{\Gamma\,;\, b^+ \xrightarrow{L} \rho}\,pL \qquad \dfrac{\dfrac{\Gamma, c^- \longrightarrow \rho}{\Gamma, c^-\,;\,\cdot \xrightarrow{L} \rho}\,\mathsf{stable}}{\Gamma\,;\, {\downarrow}c^- \xrightarrow{L} \rho}\,{\downarrow}L}{\Gamma\,;\, b^+ \vee {\downarrow}c^- \xrightarrow{L} \rho}\,{\vee}L}{\Gamma, [{\uparrow}(b^+ \vee {\downarrow}c^-)] \longrightarrow \rho}\,{\uparrow}L}{\Gamma, [a^+ \supset {\uparrow}(b^+ \vee {\downarrow}c^-)] \longrightarrow \rho}\,{\supset}R$$

Summarizing this rule, we obtain

$$\dfrac{\Gamma, a^+, b^+ \longrightarrow \rho \quad \Gamma, a^+, c^- \longrightarrow \rho}{\Gamma, a^+ \longrightarrow \rho}\ R_1$$

This rules adds a negative atom c^- to the antecedents, so we need to derive another rule for it.

$$\dfrac{\dfrac{\rho = c^-}{\Gamma, c^-, [c^-] \longrightarrow \rho}\ \mathsf{id}^-}{\Gamma, c^- \longrightarrow \rho}\ \mathsf{focus}L \qquad \text{as a derived rule :} \quad \dfrac{}{\Gamma, c^- \longrightarrow c^-}\ R_2$$

And finally our original second antecedent:

$$\dfrac{\dfrac{b^+ \in \Gamma}{\Gamma \longrightarrow [b^+]}\ \mathsf{id}^+ \quad \dfrac{\rho = c^-}{\Gamma, [c^-] \longrightarrow \rho}\ \mathsf{id}^-}{\Gamma, [b^+ \supset c^-] \longrightarrow \rho}\ {\supset}L \qquad \dfrac{}{\Gamma, b^+ \longrightarrow c^-}\ R_3$$

Here is the summary of the three derived rules:

$$\dfrac{\Gamma, a^+, b^+ \longrightarrow \rho \quad \Gamma, a^+, c^- \longrightarrow \rho}{\Gamma, a^+ \longrightarrow \rho}\ R_1 \qquad \dfrac{}{\Gamma, c^- \longrightarrow c^-}\ R_2 \qquad \dfrac{}{\Gamma, b^+ \longrightarrow c^-}\ R_3$$

This antecedents are persistent, we *replace* the two propositions which yielded R_1 and R_3 with the rules and we have to prove

$$a^+ \longrightarrow c^-$$

which works as follows (where we are now only allowed to use derived rules):

$$\dfrac{\dfrac{}{a^+, b^+ \longrightarrow c^-}\ R_3 \quad \dfrac{}{a^+, c^- \longrightarrow c^-}\ R_2}{a^+ \longrightarrow c^-}\ R_1$$

In this technique of deriving rules, each derived rules will only have stable sequents in the conclusion and premises. The rule generation will start with a negative antecedent or positive succedent, break it down until it encounters an atom, or an up or down shift, respectively, then proceed by inversion until another stable sequent is reached. Andreoli called propositions of this form *bipoles* because they traverse *negative* to *positive* or *positive* to *negative*, and back [And01].

References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

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[Sim14] Robert J. Simmons. Structural focalization. *Transactions on Computational Logic*, 15(3):21:1–21:33, July 2014.